L_p-NORM OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. In this paper we show among others that

$$\max\left\{\left\|f((1-\lambda)x+\lambda y)-\int_0^1 f((1-t)x+ty)\,dt\right\|,\right.$$
$$\left.\left\|(1-\lambda)f(y)+\lambda f(x)-\int_0^1 f((1-t)x+ty)\,dt\right\|\right\}$$
$$\leq \frac{1}{2^{1+1/p}\pi}\|y-x\|\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left(\int_\gamma |f(\xi)|^q\,|d\xi|\right)^{1/q}$$
$$\times\left[\int_\gamma\left[(|\xi|-\|x\|)^{-2p}+(|\xi|-\|y\|)^{-2p}\right]|d\xi|\right]^{1/p}$$

for $\lambda \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$ with $p, q > 1$. Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

In 1998, Dragomir and Wang proved the following Ostrowski type inequality in terms of the p-norm of derivative, see [14]:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{u-a}{b-a}\right)^{q+1}+\left(\frac{b-u}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.1}$$

for all $u \in [a,b]$, where $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

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The corresponding version for trapezoid was obtained by Cerone and Dragomir in 2000 and is as follows, see [3]:

Theorem 2. *With the assumptions of Theorem 1, we have*

$$\left|\frac{(u-a)f(a)+(b-u)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{u-a}{b-a}\right)^{q+1}+\left(\frac{b-u}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, \tag{1.2}$$

for all $u\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

In order to extend these inequalities for functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{b\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi) (\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [5, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [15] and [19].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [6]-[12].

2. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f_{x,y}^{'}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = D(f)(x)(y-x) \ \text{ and } f_{x,y}^{'}(1-) = D(f)(y)(y-x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f_{x,y}^{'}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.5).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v(\xi - x)^{-1} d\xi, \tag{2.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} &f(x+hv) - f(x) \\ &= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v(\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y - x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{2.5}$$

for all $t \in (0, 1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y - x)(\xi - x)^{-1} d\xi, \tag{2.6}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y - x)(\xi - y)^{-1} d\xi. \tag{2.7}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *With the assumptions of Lemma 3, we have the bounds*

$$\begin{aligned}
(2.8)\qquad & \left\| f_{x,y}^{'}(t) \right\| \\
&\le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \\
&\le \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
&\times \left(\int_\gamma \left\| (\xi - (1-t)x - ty)^{-1} \right\|^{2p} |d\xi| \right)^{1/p} \\
&\le \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
&\times \left(\int_\gamma (|\xi| - \|(1-t)x + ty\|)^{-2p} |d\xi| \right)^{1/p} \\
&\le \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
&\times \left(\int_\gamma [(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2p} |d\xi| \right)^{1/p} \\
&\le \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
&\times \left(\int_\gamma \left[(1-t)(|\xi| - \|x\|)^{-2p} + t(|\xi| - \|y\|)^{-2p} \right] |d\xi| \right)^{p}
\end{aligned}$$

for all $t \in [0,1]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$.

Proof. By taking the norm in (2.5) we get, by Hölder's inequality, that

$$\begin{aligned}
(2.9)\qquad & \left\| f_{x,y}^{'}(t) \right\| \\
&\le \frac{1}{2\pi} \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} (y-x) (\xi - (1-t)x - ty)^{-1} \right\| |d\xi| \\
&\le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \\
&= \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| |\xi|^{-2} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^2 |d\xi| \\
&\le \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
&\times \left(\int_\gamma |\xi|^{-2p} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^{2p} |d\xi| \right)^{1/p}
\end{aligned}$$

for all $t \in [0,1]$, which proves the first and second inequalities in (2.8).

Since

$$\left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \le (1-t)\left\| \frac{x}{\xi} \right\| + t\left\| \frac{y}{\xi} \right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^{k}.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k}\\ &=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2p}\leq|\xi|^{2p}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2p}$$

for $\xi \in \gamma$.

Therefore

$$\int_{\gamma}|\xi|^{-2p}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2p}|d\xi|\leq\int_{\gamma}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2p}|d\xi|$$

and we derive the third inequality in (2.8).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\geq|\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)>0\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\leq\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-2p}\leq\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}$$

for $\xi \in \gamma$ and $t\in[0,1]$. This proves the fourth inequality in (2.8).

By the convexity of the power function $(\cdot)^{-2p}$ we also have

$$\begin{aligned}&\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}\\ &\leq(1-t)(|\xi|-\|x\|)^{-2p}+t(|\xi|-\|y\|)^{-2p}\end{aligned}$$

for $t\in[0,1]$, which proves the last inequality in (2.8). □

Corollary 1. *Assume that $f: D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and x, $y\in\mathcal{B}$ with $\sigma(x)$, $\sigma(y)\subset D(0,R)\subset D$, where $D(0,R)$ is an* *open disk centered in* 0 *and of radius* R, *then*

$$
\begin{aligned}
(2.10) \qquad & \left\| f'_{x,y}(t) \right\| \\
& \leq R \|y-x\| \int_0^1 \left| f\left(Re^{2\pi is}\right) \right| \left\| \left(Re^{2\pi is} - (1-t)x - ty\right)^{-1} \right\|^2 ds \\
& \leq R \|y-x\| \left(\int_0^1 \left| f\left(Re^{2\pi is}\right) \right|^q ds \right)^{1/q} \\
& \times \left(\int_0^1 \left\| \left(Re^{2\pi is} - (1-t)x - ty\right)^{-1} \right\|^{2p} ds \right)^{1/p} \\
& \leq R \|y-x\| \left(R - \|(1-t)x + ty\|\right)^{-2} \\
& \times \left(\int_0^1 \left| f\left(Re^{2\pi is}\right) \right|^q ds \right)^{1/q} \\
& \leq R \|y-x\| \left[(1-t)(R-\|x\|) + t(R-\|y\|)\right]^{-2} \\
& \times \left(\int_0^1 \left| f\left(Re^{2\pi is}\right) \right|^q ds \right)^{1/q} \\
& \leq R \|y-x\| \left[(1-t)(R-\|x\|)^{-2} + t(R-\|y\|)^{-2}\right] \\
& \times \left(\int_0^1 \left| f\left(Re^{2\pi is}\right) \right|^q ds \right)^{1/q}
\end{aligned}
$$

for all $t \in [0,1]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p,\ q > 1$.

It follows by taking γ parametrized by $\xi(s) = Re^{2\pi is}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi is} ds$, $|d\xi(s)| = 2\pi R ds$, $|\xi| = R$ and by (2.8) we get (2.10).

We have the following upper bounds for the p-norm of the derivative:

Corollary 2. *With the assumptions of Lemma 3 we have the bounds*

$$
\begin{aligned}
(2.11) \qquad & \left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\
& \leq \frac{1}{2\pi} \|y-x\| \left(\int_0^1 \left(\int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \right)^p dt \right)^{1/p} \\
& \leq \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\
& \times \left(\int_0^1 \left(\int_\gamma \left\| (\xi - (1-t)x - ty)^{-1} \right\|^{2p} |d\xi| \right) dt \right)^{1/p}
\end{aligned}
$$

$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left(\int_{0}^{1}\left(\int_{\gamma}\left(\left\vert \xi\right\vert -\left\Vert \left(1-t\right) x+ty\right\Vert \right) ^{-2p}\left\vert d\xi\right\vert \right) dt\right) ^{1/p}$$
$$\le \frac{1}{2\left(2p-1\right) ^{1/p}\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left[\frac{1}{\left(\left\Vert y\right\Vert -\left\Vert x\right\Vert \right) }\int_{\gamma}\frac{\left(\left\vert \xi\right\vert -\left\Vert x\right\Vert \right) ^{2p-1}-\left(\left\vert \xi\right\vert -\left\Vert y\right\Vert \right) ^{2p-1}}{\left(\left\vert \xi\right\vert -\left\Vert x\right\Vert \right) ^{2p-1}\left(\left\vert \xi\right\vert -\left\Vert y\right\Vert \right) ^{2p-1}}\left\vert d\xi\right\vert \right] ^{1/p}$$
$$\le \frac{1}{2^{1+1/p}\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left[\int_{\gamma}\left[\left(\left\vert \xi\right\vert -\left\Vert x\right\Vert \right) ^{-2p}+\left(\left\vert \xi\right\vert -\left\Vert y\right\Vert \right) ^{-2p}\right] \left\vert d\xi\right\vert \right] ^{1/p},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,\ q>1$.

Proof. From (2.8), by taking the power p, the integral and then the power $1/p$ we have,

$$\left(\int_{0}^{1}\left\Vert f_{x,y}^{\prime}\left(t\right) \right\Vert ^{p}dt\right) ^{1/p}$$
$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{0}^{1}\left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert \left\Vert \left(\xi-\left(1-t\right) x-ty\right) ^{-1}\right\Vert ^{2}\left\vert d\xi\right\vert \right) ^{p}dt\right) ^{1/p}$$
$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left(\int_{0}^{1}\left(\int_{\gamma}\left\Vert \left(\xi-\left(1-t\right) x-ty\right) ^{-1}\right\Vert ^{2p}\left\vert d\xi\right\vert \right) dt\right) ^{1/p}$$

$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left(\int_{0}^{1}\left(\int_{\gamma}\left(\left\vert \xi\right\vert -\left\Vert \left(1-t\right) x+ty\right\Vert \right) ^{-2p}\left\vert d\xi\right\vert \right) dt\right) ^{1/p}$$
$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left(\int_{0}^{1}\left(\int_{\gamma}\left[\left(1-t\right) \left(\left\vert \xi\right\vert -\left\Vert x\right\Vert \right) +t\left(\left\vert \xi\right\vert -\left\Vert y\right\Vert \right) \right] ^{-2p}\left\vert d\xi\right\vert \right) dt\right) ^{1/p}$$
$$\le \frac{1}{2\pi}\left\Vert y-x\right\Vert \left(\int_{\gamma}\left\vert f\left(\xi\right) \right\vert ^{q}\left\vert d\xi\right\vert \right) ^{1/q}$$
$$\times \left(\int_{0}^{1}\left(\int_{\gamma}\left[\left(1-t\right) \left(\left\vert \xi\right\vert -\left\Vert x\right\Vert \right) ^{-2p}+t\left(\left\vert \xi\right\vert -\left\Vert y\right\Vert \right) ^{-2p}\right] \left\vert d\xi\right\vert \right) dt\right) ^{1/p}.$$

By Fubini's theorem,

$$\begin{aligned}
&\int_0^1 \left(\int_\gamma [(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2p} |d\xi| \right) dt \\
&= \int_\gamma \left(\int_0^1 [(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2p} dt \right) |d\xi| \\
&= \int_\gamma \left(\left. \frac{[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2p+1}}{(1-2p)(\|x\| - \|y\|)} \right|_0^1 \right) |d\xi| \\
&= \int_\gamma \frac{(|\xi| - \|y\|)^{-2p+1} - (|\xi| - \|x\|)^{-2p+1}}{(1-2p)(\|x\| - \|y\|)} |d\xi| \\
&= \int_\gamma \frac{\frac{1}{(|\xi| - \|y\|)^{2p-1}} - \frac{1}{(|\xi| - \|x\|)^{2p-1}}}{(1-2p)(\|x\| - \|y\|)} |d\xi| \\
&= \frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1}(|\xi| - \|y\|)^{2p-1}} |d\xi|,
\end{aligned}$$

which proves the third inequality in (2.11).

The last inequality follows by the fact that

$$\begin{aligned}
&\int_0^1 \left(\int_\gamma \left[(1-t)(|\xi| - \|x\|)^{-2p} + t(|\xi| - \|y\|)^{-2p} \right] |d\xi| \right) dt \\
&= \left(\int_\gamma \int_0^1 \left[(1-t)(|\xi| - \|x\|)^{-2p} + t(|\xi| - \|y\|)^{-2p} \right] dt \right) |d\xi| \\
&= \frac{1}{2} \int_\gamma \left[(|\xi| - \|x\|)^{-2p} + (|\xi| - \|y\|)^{-2p} \right] |d\xi|.
\end{aligned}$$

□

Corollary 3. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0, R) \subset D$ where $D(0, R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}
&\left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\
&\leq R\|y - x\| \\
&\times \left(\int_0^1 \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| \left\| \left(Re^{2\pi i s} - (1-t)x - ty\right)^{-1} \right\|^2 ds \right)^p dt \right)^{1/p} \\
&\leq R\|y - x\| \left(\int_\gamma \left| f\left(Re^{2\pi i s}\right) \right|^q ds \right)^{1/q} \\
&\times \left(\int_0^1 \left(\int_0^1 \left\| \left(Re^{2\pi i s} - (1-t)x - ty\right)^{-1} \right\|^{2p} ds \right) dt \right)^{1/p}
\end{aligned} \tag{2.12}$$

$$\leq \frac{1}{2\pi}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}$$
$$\times\left(\int_0^1 (R-\|(1-t)x+ty\|)^{-2p}dt\right)^{1/p}$$
$$\leq \frac{R}{(2p-1)^{1/p}}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}$$
$$\times\left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p}$$
$$\leq \frac{R}{2^{1/p}\pi}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}$$
$$\times\left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,\ q>1$.

3. Main Results

We have the following Ostrowski and midpoint type inequalities in terms of L_p-norm:

Theorem 3. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$. *Then we have the norm inequality*

$$\left\|f((1-\lambda)x+\lambda y)-\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.1}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\lambda^{1+1/q}\left\|f'_{x,y}\right\|_{[0,\lambda],p}+(1-\lambda)^{1+1/q}\left\|f'_{x,y}\right\|_{[\lambda,1],p}\right]$$
$$\leq\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left\|f'_{x,y}\right\|_{[0,1],p}$$

for all $\lambda\in[0,1]$, *where* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,\ q>1$.

In particular, we have the midpoint inequality

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.2}$$
$$\leq\frac{1}{2^{1+1/q}(q+1)^{1/q}}\left[\left\|f'_{x,y}\right\|_{[0,\lambda],p}+\left\|f'_{x,y}\right\|_{[\lambda,1],p}\right]$$
$$\leq\frac{1}{2(q+1)^{1/q}}\left\|f'_{x,y}\right\|_{[0,1],p}.$$

Proof. If we use the integration by parts formula for the Bochner's integral, then we have

$$\int_0^\lambda tf'_{x,y}(t)\,dt=\lambda f_{x,y}(\lambda)-\int_0^\lambda f_{x,y}(t)\,dt$$

and

$$\int_\lambda^1 (t-1)f'_{x,y}(t)\,dt=(1-\lambda)f_{x,y}(\lambda)-\int_\lambda^1 f_{x,y}(t)\,dt,$$

for all $\lambda \in [0,1]$.

Adding these two equalities, we obtain the Montgomery type identity (see for example [17, p. 565]):

$$f_{x,y}(\lambda) - \int_0^1 f_{x,y}(t)\,dt = \int_0^\lambda t f'_{x,y}(t)\,dt + \int_\lambda^1 (t-1) f'_{x,y}(t)\,dt, \tag{3.3}$$

namely, by (2.2), the equality of interest in terms of Fréchet derivative

$$\begin{aligned} f((1-\lambda)x + \lambda y) - \int_0^1 f((1-t)x + ty)\,dt \\ = \int_a^\lambda tD(f)((1-t)x + ty)(y-x)\,dt \\ + \int_\lambda^1 (t-1) D(f)((1-t)x + ty)(y-x)\,dt \end{aligned} \tag{3.4}$$

for all $\lambda \in [0,1]$.

If we take the norm in (3.3), use the integral's properties and Hölder's inequality, we have

$$\begin{aligned} &\left\| f_{x,y}(\lambda) - \int_0^1 f_{x,y}(t)\,dt \right\| \\ &\le \left\| \int_0^\lambda t f'_{x,y}(t)\,dt \right\| + \left\| \int_\lambda^1 (t-1) f'_{x,y}(t)\,dt \right\| \\ &\le \int_0^\lambda t \left\| f'_{x,y}(t) \right\| dt + \int_\lambda^1 (1-t) \left\| f'_{x,y}(t) \right\| dt \\ &\le \left(\int_0^\lambda t^q dt \right)^{1/q} \left(\int_0^\lambda \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\ &+ \left(\int_\lambda^1 (1-t)^q\,dt \right)^{1/q} \left(\int_\lambda^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\ &= \frac{\lambda^{1+1/q}}{(q+1)^{1/q}} \left\| f'_{x,y} \right\|_{[0,\lambda],p} + \frac{(1-\lambda)^{1+1/q}}{(q+1)^{1/q}} \left\| f'_{x,y} \right\|_{[\lambda,1],p}, \end{aligned} \tag{3.5}$$

which proves the first part of (3.1).

Using the elementary inequality

$$ac + bd \le (a^q + b^q)^{1/q} (c^p + d^p)^{1/p}$$

for a, b, c, $d \ge 0$, where $\frac{1}{p} + \frac{1}{q} = 1$ with p, $q > 1$, then we get

$$\begin{aligned} &\lambda^{1+1/q} \left\| f'_{x,y} \right\|_{[0,\lambda],p} + (1-\lambda)^{1+1/q} \left\| f'_{x,y} \right\|_{[\lambda,1],p} \\ &\le \left[\lambda^{(1+1/q)q} + (1-\lambda)^{(1+1/q)q} \right]^{1/q} \left[\left\| f'_{x,y} \right\|^p_{[0,\lambda],p} + \left\| f'_{x,y} \right\|^p_{[\lambda,1],p} \right]^{1/p} \\ &= \left[\lambda^{q+1} + (1-\lambda)^{q+1} \right]^{1/q} \left[\int_0^\lambda \left\| f'_{x,y}(t) \right\|^p dt + \int_\lambda^1 \left\| f'_{x,y}(t) \right\|^p dt \right]^{1/p} \\ &= \left[\lambda^{q+1} + (1-\lambda)^{q+1} \right]^{1/q} \left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p}, \end{aligned}$$

which proves the desired result (3.1). □

Corollary 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}(3.6)\quad & \left\| f((1-\lambda)x + \lambda y) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ & \leq \frac{1}{2(2p-1)^{1/p}\pi} \|y - x\| \left[\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1}\right]^{1/q} \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\ & \times \left[\frac{1}{(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1}(|\xi| - \|y\|)^{2p-1}} |d\xi|\right]^{1/p} \\ & \leq \frac{1}{2^{1+1/p}\pi} \|y - x\| \left[\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1}\right]^{1/q} \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\ & \times \left[\int_\gamma \left[(|\xi| - \|x\|)^{-2p} + (|\xi| - \|y\|)^{-2p}\right] |d\xi|\right]^{1/p}\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular, we have

$$\begin{aligned}(3.7)\quad & \left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ & \leq \frac{1}{4(q+1)^{1/q}(2p-1)^{1/p}\pi} \|y - x\| \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\ & \times \left[\frac{1}{(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1}(|\xi| - \|y\|)^{2p-1}} |d\xi|\right]^{1/p} \\ & \leq \frac{1}{2^{2+1/p}(q+1)^{1/q}\pi} \|y - x\| \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\ & \times \left[\int_\gamma \left[(|\xi| - \|x\|)^{-2p} + (|\xi| - \|y\|)^{-2p}\right] |d\xi|\right]^{1/p}.\end{aligned}$$

A simpler case that may be useful in applications is as follows:

Corollary 5. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open* *disk centered in* 0 *and of radius* R*, then*

$$\left\|f\left((1-\lambda)x+\lambda y\right)-\int_0^1 f\left((1-t)x+ty\right)dt\right\| \tag{3.8}$$
$$\leq \frac{R}{(2p-1)^{1/p}}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}$$
$$\times\left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p}$$
$$\leq \frac{R}{2^{1/p}\pi}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}$$
$$\times\left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p}$$

for all $\lambda\in[0,1]$.

In particular, we have

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f\left((1-t)x+ty\right)dt\right\| \tag{3.9}$$
$$\leq \frac{R}{2(q+1)^{1/q}(2p-1)^{1/p}}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}$$
$$\times\left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p}$$
$$\leq \frac{R}{2^{1+1/p}(q+1)^{1/q}\pi}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}$$
$$\times\left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p}.$$

We have the trapezoid inequality:

Theorem 4. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$. *Then we have the norm inequality*

$$\left\|(1-\lambda)f(y)+\lambda f(x)-\int_0^1 f\left((1-t)x+ty\right)dt\right\| \tag{3.10}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\lambda^{1+1/q}\left\|f'_{x,y}\right\|_{[0,\lambda],p}+(1-\lambda)^{1+1/q}\left\|f'_{x,y}\right\|_{[\lambda,1],p}\right]$$
$$\leq\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left\|f'_{x,y}\right\|_{[0,1],p}$$

for all $\lambda\in[0,1]$.

In particular, we have the trapezoid inequality

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\| \leq \frac{1}{2^{1+1/q}(q+1)^{1/q}}\left[\left\|f'_{x,y}\right\|_{[0,\lambda],p}+\left\|f'_{x,y}\right\|_{[\lambda,1],p}\right] \leq \frac{1}{2(q+1)^{1/q}}\left\|f'_{x,y}\right\|_{[0,1],p}. \tag{3.11}$$

Proof. If we use the integration by parts formula for the Bochner's integral, we have

$$\int_0^1 (t-\lambda) f'_{x,y}(t)\,dt = (1-\lambda) f_{x,y}(1)+\lambda f_{x,y}(0)-\int_0^1 f_{x,y}(t)\,dt,$$

for $\lambda\in[0,1]$, namely we have the following equality of interest in terms of Fréchet derivative

$$(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t)x+ty)\,dt = \int_0^1 (t-\lambda) f'_{x,y}(t)\,dt = \int_0^1 (t-\lambda) D(f)((1-t)x+ty)(y-x)\,dt \tag{3.12}$$

for $\lambda\in[0,1]$.

If we take the norm in (3.12), then we get

$$\begin{aligned}&\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t)x+ty)\,dt\right\|\\ &\leq \int_0^1 |t-\lambda|\left\|f'_{x,y}(t)\right\|dt\\ &= \int_0^\lambda (\lambda-t)\left\|f'_{x,y}(t)\right\|dt+\int_\lambda^1 (t-\lambda)\left\|f'_{x,y}(t)\right\|dt\\ &\leq \frac{1}{(q+1)^{1/q}}\left[\lambda^{1+1/q}\left\|f'_{x,y}\right\|_{[0,\lambda],p}+(1-\lambda)^{1+1/q}\left\|f'_{x,y}\right\|_{[\lambda,1],p}\right]\\ &\leq \left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left\|f'_{x,y}\right\|_{[0,1],p},\end{aligned}$$

which proves the inequality (3.10). □

Corollary 6. *Assume that $f: G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve* *in G and such that* $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. *Then*

$$\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) d t\right\| \tag{3.13}$$
$$\leq \frac{1}{2(2p-1)^{1/p} \pi}\|y-x\|\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}$$
$$\times\left[\frac{1}{(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right]^{1/p}$$
$$\leq \frac{1}{2^{1+1/p} \pi}\|y-x\|\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}$$
$$\times\left[\int_\gamma\left[(|\xi|-\|x\|)^{-2p}+(|\xi|-\|y\|)^{-2p}\right]|d\xi|\right]^{1/p}$$

for all $\lambda \in [0,1]$.

In particular, we have

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) d t\right\| \tag{3.14}$$
$$\leq \frac{1}{4(q+1)^{1/q}(2p-1)^{1/p} \pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}$$
$$\times\left[\frac{1}{(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right]^{1/p}$$
$$\leq \frac{1}{2^{2+1/p}(q+1)^{1/q} \pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}$$
$$\times\left[\int_\gamma\left[(|\xi|-\|x\|)^{-2p}+(|\xi|-\|y\|)^{-2p}\right]|d\xi|\right]^{1/p}.$$

Corollary 7. *Assume that* $f: D \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* D *and* $x, y \in \mathcal{B}$ *with* $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R, *then*

$$\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) d t\right\| \tag{3.15}$$
$$\leq \frac{R}{(2p-1)^{1/p}}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi i s}\right)\right|^q ds\right)^{1/q}\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}$$
$$\times\left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p}$$
$$\leq \frac{R}{2^{1/p} \pi}\|y-x\|\left(\int_0^1\left|f\left(Re^{2\pi i s}\right)\right|^q ds\right)^{1/q}\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q}$$
$$\times\left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p}$$

for all $\lambda \in [0,1]$.

In particular, we have

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \tag{3.16}$$
$$\leq \frac{R}{2(q+1)^{1/q}(2p-1)^{1/p}} \|y-x\| \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right)\right|^q ds \right)^{1/q}$$
$$\times \left[\frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right]^{1/p}$$
$$\leq \frac{R}{2^{1+1/p}(q+1)^{1/q}\pi} \|y-x\| \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right)\right|^q ds \right)^{1/q}$$
$$\times \left[(R-\|x\|)^{-2p} + (R-\|y\|)^{-2p} \right]^{1/p}.$$

4. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi i t}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned}
(4.1)\qquad & \int_0^1 \exp\left[R\cos\left(2\pi t\right)\right] dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned}$$

Therefore

$$\left|\exp\left(Re^{2\pi it}\right)\right|^q = \exp\left[qR\cos\left(2\pi t\right)\right]$$

and

$$\left(\int_0^1 \left|\exp\left(Re^{2\pi it}\right)\right|^q dt\right)^{1/q} = \left(\int_0^1 \exp\left[qR\cos\left(2\pi t\right)\right] dt\right)^{1/q} = I_0^{1/q}(qR) \text{ for } q \geq 1.$$

By utilising Corollary 5 written for $f(t) = \exp t$ we get for $x,\ y \in \mathcal{B}$ and $\|x\|$, $\|y\| < R$ that

$$\begin{aligned}
(4.2)\qquad & \left\|\exp\left(\left(1-\lambda\right)x + \lambda y\right) - \int_0^1 \exp\left(\left(1-t\right)x + ty\right) dt\right\| \\
&\leq \frac{R}{\left(2p-1\right)^{1/p}} \left\|y - x\right\| I_0^{1/q}(qR) \left[\frac{\lambda^{q+1} + \left(1-\lambda\right)^{q+1}}{q+1}\right]^{1/q} \\
&\times \left[\frac{\left(R - \|x\|\right)^{2p-1} - \left(R - \|y\|\right)^{2p-1}}{\left(\|y\| - \|x\|\right)\left(R - \|x\|\right)^{2p-1}\left(R - \|y\|\right)^{2p-1}}\right]^{1/p} \\
&\leq \frac{R}{2^{1/p}\pi} \left\|y - x\right\| I_0^{1/q}(qR) \left[\frac{\lambda^{q+1} + \left(1-\lambda\right)^{q+1}}{q+1}\right]^{1/q} \\
&\times \left[\left(R - \|x\|\right)^{-2p} + \left(R - \|y\|\right)^{-2p}\right]^{1/p}
\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular, we have

$$\begin{aligned}
(4.3)\qquad & \left\|\exp\left(\frac{x+y}{2}\right) - \int_0^1 \exp\left(\left(1-t\right)x + ty\right) dt\right\| \\
&\leq \frac{R}{2\left(q+1\right)^{1/q}\left(2p-1\right)^{1/p}} \left\|y - x\right\| I_0^{1/q}(qR) \\
&\times \left[\frac{\left(R - \|x\|\right)^{2p-1} - \left(R - \|y\|\right)^{2p-1}}{\left(\|y\| - \|x\|\right)\left(R - \|x\|\right)^{2p-1}\left(R - \|y\|\right)^{2p-1}}\right]^{1/p} \\
&\leq \frac{R}{2^{1+1/p}\left(q+1\right)^{1/q}\pi} \left\|y - x\right\| I_0^{1/q}(qR) \\
&\times \left[\left(R - \|x\|\right)^{-2p} + \left(R - \|y\|\right)^{-2p}\right]^{1/p}.
\end{aligned}$$

By utilising Corollary 7 for the exponential function we also have the trapezoid inequalities

$$\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) d t\right\| \tag{4.4}$$
$$\leq \frac{R}{(2 p-1)^{1 / p}}\|y-x\| I_0^{1 / q}(q R)\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1 / q}$$
$$\times\left[\frac{(R-\|x\|)^{2 p-1}-(R-\|y\|)^{2 p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2 p-1}(R-\|y\|)^{2 p-1}}\right]^{1 / p}$$
$$\leq \frac{R}{2^{1 / p} \pi}\|y-x\| I_0^{1 / q}(q R)\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1 / q}$$
$$\times\left[(R-\|x\|)^{-2 p}+(R-\|y\|)^{-2 p}\right]^{1 / p}$$

for all $\lambda \in[0,1]$ and

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) d t\right\| \tag{4.5}$$
$$\leq \frac{R}{2(q+1)^{1 / q}(2 p-1)^{1 / p}}\|y-x\| I_0^{1 / q}(q R)$$
$$\times\left[\frac{(R-\|x\|)^{2 p-1}-(R-\|y\|)^{2 p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2 p-1}(R-\|y\|)^{2 p-1}}\right]^{1 / p}$$
$$\leq \frac{R}{2^{1+1 / p}(q+1)^{1 / q} \pi}\|y-x\| I_0^{1 / q}(q R)$$
$$\times\left[(R-\|x\|)^{-2 p}+(R-\|y\|)^{-2 p}\right]^{1 / p} .$$

Let f be an analytic functions on the open disk $D(0, \rho)$ given by the *power series* $f(\lambda):=\sum_{j=0}^{\infty} \alpha_j \lambda^j \ (|\lambda|<\rho)$. If $\nu(a)<\rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty}|\alpha_j|\left\|a^j\right\|<\infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda): D(0, \rho) \rightarrow \mathbb{C}, f_A(\lambda):=\sum_{n=0}^{\infty}|\alpha_n| \lambda^n .$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n=\ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1) ; \tag{4.6}$$
$$g(\lambda)=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2 n)!} \lambda^{2 n}=\cos \lambda, \ \lambda \in \mathbb{C} ;$$

$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$

$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{4.7}$$

$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$

$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{4.8}$$

$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

Lemma 5. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. Then*

$$|f(\lambda)| \le f_A(|\lambda|) \ \textit{for}\ \lambda \in D(0,\rho). \tag{4.9}$$

Also, we have for $0 < R < \rho$ that

$$\left(\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right|^q ds\right)^{1/q} \le f_A(R) \ \textit{for}\ q \ge 1. \tag{4.10}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty}\left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \le \lim_{n\to\infty}\sum_{j=0}^{n}\left|\alpha_j \lambda^j\right| \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}|\alpha_j|\,|\lambda|^j = \sum_{j=0}^{\infty}|\alpha_j|\,|\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the inequality (4.9).

The inequality (4.10) follows by (4.9). □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty}\alpha_j\lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned} &\left\| f((1-\lambda)x+\lambda y) - \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\le \frac{R}{(2p-1)^{1/p}}\|y-x\|\, f_A(R)\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q} \\ &\times \left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p} \\ &\le \frac{R}{2^{1/p}\pi}\|y-x\|\, f_A(R)\left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right]^{1/q} \\ &\times \left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p} \end{aligned} \tag{4.11}$$

for all $\lambda \in [0,1]$.

Corollary 8. *In particular, we have*

$$\begin{aligned} &\left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\le \frac{R}{2(q+1)^{1/q}(2p-1)^{1/p}}\|y-x\|\, f_A(R) \\ &\times \left[\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right]^{1/p} \\ &\le \frac{R}{2^{1+1/p}(q+1)^{1/q}\pi}\|y-x\|\, f_A(R) \\ &\times \left[(R-\|x\|)^{-2p}+(R-\|y\|)^{-2p}\right]^{1/p}. \end{aligned} \tag{4.12}$$

We also have

$$\left\| (1-\lambda) f(y) + \lambda f(x) - \int_0^1 f((1-t)x + ty)\, dt \right\| \tag{4.13}$$

$$\leq \frac{R}{(2p-1)^{1/p}} \|y-x\| f_A(R) \left[\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1} \right]^{1/q}$$

$$\times \left[\frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right]^{1/p}$$

$$\leq \frac{R}{2^{1/p}\pi} \|y-x\| f_A(R) \left[\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1} \right]^{1/q}$$

$$\times \left[(R-\|x\|)^{-2p} + (R-\|y\|)^{-2p} \right]^{1/p}$$

for all $\lambda \in [0,1]$ *and*

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\, dt \right\| \tag{4.14}$$

$$\leq \frac{R}{2(q+1)^{1/q}(2p-1)^{1/p}} \|y-x\| f_A(R)$$

$$\times \left[\frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(\|y\|-\|x\|)(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right]^{1/p}$$

$$\leq \frac{R}{2^{1+1/p}(q+1)^{1/q}\pi} \|y-x\| f_A(R)$$

$$\times \left[(R-\|x\|)^{-2p} + (R-\|y\|)^{-2p} \right]^{1/p}.$$

The proof follows by Corollaries 5, 7 and Lemma 5. As examples, one can consider the functions f and f_A listed above.

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