Approximations by Multivariate Generalized Trigonometric type Singular Integral operators

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

This research and survey work deals exclusively with the study of the approximation of generalized multivariate Trigonometric type singular integrals to the identity-unit operator. Here we study quantitatively most of their approximation properties. These operators are not in general positive linear operators. In particular we study the rate of convergence of these integral operators to the unit operator, as well as the related simultaneous approximation. These are given via Jackson type inequalities and by the use of multivariate high order modulus of smoothness of the high order partial derivatives of the involved function. Also we study the global smoothness preservation properties of these integral operators. These multivariate inequalities are nearly sharp and in one case the inequality is attained, that is sharp. Furthermore we give asymptotic expansions of Voronovskaya type for the error of approximation. The above properties are studied with respect to L_p norm, $1 \leq p \leq \infty$.

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1 Introduction

We start with our motivation for this work. The following come from [5].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{1}$$

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that is $\sum\limits_{j=0}^{r} \alpha_j = 1$. Here it is $\xi \in (0,1]$.

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}_+$, and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, $\beta \in \mathbb{N}$, we define for $x \in \mathbb{R}$, the trigonometric integral

$$T_{r,\xi}(f;x) := \frac{1}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt, \tag{2}$$

where

$$W = \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 2\xi^{1-2\beta} \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt \overset{(6)}{=}$$

$$2\xi^{1-2\beta}\pi(-1)^{\beta}\beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}. \tag{3}$$

$T_{r,\xi}$ operators are not positive operators, see [7].

We mention:

Let p and m be integers with $1 \leq p \leq m$. We define the integral

$$I(m,p) := \int_{-\infty}^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx = 2\int_{0}^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx. \tag{4}$$

That is an (absolutely) convergent integral.

According to [9], page 210, item 1033, we obtain

$$I(m,p) = \pi \frac{(-1)^p (2m)!}{4^{m-p}(2p-1)!} \sum_{k=1}^{m} (-1)^k \frac{k^{2p-1}}{(m-k)!\,(m+k)!}. \tag{5}$$

In particular, for $p = m$ the above formula becomes

$$\int_{0}^{\infty} \frac{(\sin x)^{2m}}{x^{2m}} dx = \pi(-1)^m m \sum_{k=1}^{m} (-1)^k \frac{k^{2m-1}}{(m-k)!\,(m+k)!}. \tag{6}$$

We need the rth L_p-modulus of smoothness

$$\omega_r\left(f^{(n)}, h\right)_p := \sup_{|t| \leq h} \left\| \Delta_t^r f^{(n)}(x) \right\|_{p,x}, \quad h > 0, \tag{7}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x+jt), \tag{8}$$

see [8], p. 44. Here we have that $\omega_r\left(f^{(n)}, h\right)_p < \infty$, $h > 0$.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, ..., n \in \mathbb{N}. \tag{9}$$

Call

$$\tau(w,x) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x+jw) - \delta_n f^{(n)}(x). \tag{10}$$

Notice also that

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

According to [2], p. 306, [1], we get

$$\tau(w,x) = \Delta_w^r f^{(n)}(x). \tag{11}$$

Thus

$$\|\tau(w,x)\|_{p,x} \le \omega_r \left(f^{(n)}, |w|\right)_p, \quad w \in \mathbb{R}. \tag{12}$$

Using Taylor's formula one has

$$\sum_{j=0}^{r} \alpha_j \left[f(x+jt) - f(x)\right] = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + R_n(0,t,x), \tag{13}$$

where

$$R_n(0,t,x) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w,x)\, dw, \quad n \in \mathbb{N}. \tag{14}$$

Assume

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t) \in \mathbb{R}, \quad k = 1, ..., n, \tag{15}$$

where

$$d\mu_\xi(t) := \frac{1}{W} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta} dt, \quad \forall\, t \in \mathbb{R}.$$

Using the above terminology we derive

$$\Delta(x) := T_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} = R_n^*(x), \tag{16}$$

where

$$R_n^*(x) := \int_{-\infty}^{\infty} R_n(0,t,x)\, d\mu_\xi(t), \quad n \in \mathbb{N}. \tag{17}$$

Let $\lceil \cdot \rceil$ denote the ceiling of a real number. We mention

Theorem 1 *([5]) Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n, \beta \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + np + 1}{2}$ and the rest as above. Then*

$$\|\Delta(x)\|_p \le \frac{1}{((n-1)!)\,(q(n-1)+1)^{\frac{1}{q}}\,(rp+1)^{\frac{1}{p}}} \tag{18}$$

$$\left[\frac{1}{\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\beta}dt}\sum_{j=1}^{\lceil rp\rceil+1}\int_0^\infty\left[t^{np-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right]dt\right]^{\frac{1}{p}}\xi^n\omega_r\left(f^{(n)},\xi\right)_p.$$

Moreover, as $\xi\to 0$ *we get that* $\|\Delta(x)\|_p\to 0$.

The counterpart of Theorem 1 follows, case of $p=1$.

Theorem 2 *([5]) Let* $f\in C^n(\mathbb{R})$ *and* $f^{(n)}\in L_1(\mathbb{R})$, $n\in\mathbb{N}$, $\beta\in\mathbb{N}$, $\beta>\frac{r+1+n}{2}$. *Then*

$$\|\Delta(x)\|_1\le\frac{1}{(r+1)(n-1)!\left[\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]} \tag{19}$$

$$\sum_{j=1}^{r+1}\left(\int_0^\infty\left[t^{n-1+j}\left(\frac{\sin t}{t}\right)^{2\beta}\right]dt\right)\xi^n\omega_r\left(f^{(n)},\xi\right)_1.$$

Hence as $\xi\to 0$ *we obtain* $\|\Delta(x)\|_1\to 0$.

The case $n=0$ is mentioned next.

Proposition 3 *([5]) Let* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, $\beta\in\mathbb{N}$, $\beta>\frac{\lceil rp\rceil+1}{2}$ *and the rest as above. Then*

$$\|T_{r,\xi}(f)-f\|_p\le\omega_r(f,\xi)_p\left(\frac{1}{\left[\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{\lceil rp\rceil}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]\right)^{\frac{1}{p}}. \tag{20}$$

Also as $\xi\to 0$ *we obtain* $T_{r,\xi}\to$ *unit operator* I *in the* L_p *norm,* $p>1$.

We also give

Proposition 4 *([5]) For* $\beta\in\mathbb{N}$, $\beta>\frac{r+1}{2}$, *we have*

$$\|T_{r,\xi}(f)-f\|_1\le\frac{\omega_r(f,\xi)_1}{\left[\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]}\sum_{j=0}^{r}\left[\int_0^\infty t^j\left(\frac{\sin t}{t}\right)^{2\beta}dt\right]. \tag{21}$$

Moreover as $\xi\to 0$ *we get that* $T_{r,\xi}\to I$ *in the* L_1 *norm.*

We also mention:
Case $\beta=2$.

Corollary 5 *([5]) Let* $f\in C^1(\mathbb{R})$ *and* $f'\in L_1(\mathbb{R})$. *Then*

$$\|T_{1,\xi}(f;x)-f(x)\|_1\le\frac{3}{2\pi}\left(\ln 2+\frac{\pi}{4}\right)\xi\omega_1(f',\xi)_1. \tag{22}$$

Corollary 6 *([5]) Let $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$. Then*

$$\|T_{2,\xi}(f;x) - f(x)\|_1 \leq \left(\frac{40}{33\pi} \ln\left(\frac{32^{\frac{27}{16}}}{4} \right) + \frac{5}{33} + \frac{5}{22\pi} \ln \frac{256}{27} \right) \xi \omega_2(f', \xi)_1 . \tag{23}$$

Corollary 7 *([5]) It holds*

$$\|T_{1,\xi}(f) - f\|_4 \leq \omega_1(f,\xi)_4 \sqrt[4]{\frac{40}{11\pi} \ln\left(\frac{3^{\frac{27}{16}}}{4} \right) + \frac{15}{22\pi} \ln \frac{256}{27} + \frac{47}{22}}. \tag{24}$$

Also as $\xi \to 0$ we obtain $T_{1,\xi} \to$ unit operator I in the L_4 norm.

Corollary 8 *([5]) We have*

$$\|T_{6,\xi}(f) - f\|_1 \leq \omega_6(f,\xi)_1 \left(\frac{630}{151\pi} \ln \frac{2^{\frac{251}{60}}}{3^{\frac{9}{5}}} + \frac{5671}{2416} \right). \tag{25}$$

Moreover as $\xi \to 0$ we get that $T_{6,\xi} \to I$ in the L_1 norm.

We will use a lot the following:

Remark 9 *([6]) Let $j, m \in \mathbb{Z}$, $m \geq 1$ such that $0 \leq j < 2m-1$. The integral*

$$\int_{-\infty}^{\infty} x^j \left(\frac{\sin x}{x} \right)^{2m} dx = \begin{cases} 2\int_0^{\infty} x^j \left(\frac{\sin x}{x} \right)^{2m} dx, & \text{if } j \text{ is even} \\ 0, & \text{if } j \text{ is odd} \end{cases}, \tag{26}$$

is an (absolutely) convergent integral.

According to [9], page 210, item 1033, we obtain

case 1: j is even, $j < 2m-1$

$$\int_0^{\infty} x^j \left(\frac{\sin x}{x} \right)^{2m} dx = \frac{\pi (-1)^{\frac{2m-j}{2}} (2m)!}{2^{j+1} (2m-j-1)!} \sum_{k=1}^{m} (-1)^k \frac{k^{2m-j-1}}{(m-k)!(m+k)!}, \tag{27}$$

and

case 2: j is odd, $j < 2m-1$

$$\int_0^{\infty} x^j \left(\frac{\sin x}{x} \right)^{2m} dx = \frac{(-1)^{\frac{j-1}{2}} (2m)!}{2^j (2m-j-1)!} \sum_{k=1}^{m} (-1)^{m-k} \frac{k^{2m-j-1} [\ln(2k)]}{(m-k)!(m+k)!}. \tag{28}$$

In particular, for $j = 0$ the formula (27) becomes

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^{2m} dx = \pi (-1)^m m \sum_{k=1}^{m} (-1)^k \frac{k^{2m-1}}{(m-k)!(m+k)!}. \tag{29}$$

In this work we study the approximation properties of general multivariate smooth Trigonometric singular integral operators:

$$T_{r,n}^{[m]}\left(f;x_1,...,x_N\right):=$$

$$\lambda_n^{-N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}f\left(x_1+s_1j,...,x_N+s_Nj\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N, \quad (30)$$

with $\beta\in N$, and

$$\lambda_n:=2\xi_n^{1-2\beta}\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}, \quad (31)$$

see [7], [9], p. 210, item 1033.

Notice that

$$\lambda_n^{-N}\int_{\mathbb{R}^N}\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N=1, \quad (32)$$

see also [7], [9], p. 210, item 1033, and [3], p. 16.

We call

$$\gamma:=2\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}, \quad (33)$$

that is

$$\lambda_n=\gamma\xi_n^{1-2\beta}. \quad (34)$$

Here $r\in\mathbb{N}$, $m\in Z_+$, and

$$\alpha_{j,r}^{[m]}:=\begin{cases}\left(-1\right)^{r-j}\binom{r}{j}j^{-m}, & \text{if } j=1,2,...,r,\\ 1-\sum_{i=1}^{r}\left(-1\right)^{r-i}\binom{r}{i}i^{-m}, & \text{if } j=0,\end{cases} \quad (35)$$

and

$$\delta_{k,r}^{[m]}:=\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^k, \quad k=1,2,...,m\in\mathbb{N}. \quad (36)$$

See that $\sum_{j=0}^{r}\alpha_{j,r}^{[m]}=1$.

Also here $\xi_n\in(0,1]$, $n\in\mathbb{N}$, and $f:\mathbb{R}^N\to\mathbb{R}$ is a Borel measurable function. The above operator $T_{r,n}^{[m]}$ is a special case of a more general operator $\theta_{r,n}^{[m]}$ studied in general in [3] by the author.

We mention next about $\theta_{r,n}^{[m]}$.

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N\geq 1$.

We define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]}\left(f;x_1,...,x_N\right) := \sum_{j=0}^{r}\alpha_{j.r}^{[m]}\int_{\mathbb{R}^N} f\left(x_1+s_1j,x_2+s_2j,...,x_N+s_Nj\right)d\mu_{\xi_n}\left(s\right), \tag{37}$$

where $s:=(s_1,...,s_N)$, $x:=(x_1,...,x_N)\in\mathbb{R}^N$.

The operators $\theta_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi\left(u_1,...,u_N\right)=\sum\limits_{i=1}^{N}u_i^2$ and also take $r=2$, $m=3$; $x_i=0$, $i=1,...,N$. See that $\varphi\geq 0$, however

$$\theta_{2,n}^{[3]}\left(\varphi;0,0,...,0\right)=\left(\sum_{j=1}^{2}j^2\alpha_{j,2}^{[3]}\right)\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N}s_i^2\right)d\mu_{\xi_n}\left(s\right)=$$

$$\left(\alpha_{1,2}^{[3]}+4\alpha_{2,2}^{[3]}\right)\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N}s_i^2\right)d\mu_{\xi_n}\left(s\right)=\left(-2+\frac{1}{2}\right)\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N}s_i^2\right)d\mu_{\xi_n}\left(s\right)<0. \tag{38}$$

assuming that $\int_{\mathbb{R}^N}\left(\sum\limits_{i=1}^{N}s_i^2\right)d\mu_{\xi_n}\left(s\right)<\infty$.

Clearly in the case of $T_{r,n}^{[m]}$ we have that

$$d\mu_{\xi_n}\left(s\right)=\lambda_n^{-N}\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i=:d\varphi_{\xi_n}\left(s\right),\quad s\in\mathbb{R}^N. \tag{39}$$

Lemma 10 *The operator $\theta_{r,n}^{[m]}$ preserves the constant functions in N variables.*

We need the following definition.

Definition 11 *Let $f\in C_B\left(\mathbb{R}^N\right)$, the space of all bounded and continuous functions or uniformly continuous on $\mathbb{R}^N$. Then, the rth multivariate modulus of smoothness of f is given by (see, e.g. [4])*

$$\omega_r\left(f;h\right):=\sup_{\sqrt{u_1^2+...+u_N^2}\leq h}\left\|\Delta_{u_1,u_2,...,u_N}^r\left(f\right)\right\|_\infty<\infty,\quad h>0, \tag{40}$$

where $\left\|\cdot\right\|_\infty$ is the sup-norm and

$$\Delta_u^r f\left(x\right):=\Delta_{u_1,u_2,...,u_N}^r f\left(x_1,...,x_N\right)=$$

$$\sum_{j=0}^{r}\left(-1\right)^{r-j}\binom{r}{j}f\left(x_1+ju_1,x_2+ju_2,...,x_N+ju_N\right). \tag{41}$$

Let $m\in\mathbb{N}$ and let $f\in C^m\left(\mathbb{R}^N\right)$.

Suppose that all partial derivatives of f of order m are bounded, i.e.

$$\left\| \frac{\partial^m f(\cdot,\cdot,...,\cdot)}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}} \right\|_\infty < \infty, \tag{42}$$

for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N$; $\sum_{j=1}^{N} \alpha_j = m$.

In this work we apply the general theory developed in [3] about $\theta_{r,n}^{[m]}$ to the operators $T_{r,n}^{[m]}$, so we can obtain computationaly specific results and show that the general theory has applications and it is a valid theory.

So for the very important in various branches of mathematics operators $T_{r,n}^{[m]}$ we prove the very essential properties of uniform approximation, L_p approximation, global smoothness preservation and simultaneously approximation, Voronovskaya asymptotic expansions and complex simultaneous approximation.

2 Auxilliary Essential Results

We will use

Lemma 12 *Let $N \in \mathbb{N}$, $r > 0$, $z_i \in \mathbb{R}_+$, $i = 1, ..., N$. Then*

$$\left(1 + \sum_{i=1}^{N} z_i\right)^r \leq \prod_{i=1}^{N} (1 + z_i)^r. \tag{43}$$

Proof. We have

$$\left(1 + \sum_{i=1}^{N} z_i\right)^r \leq \left(N + \sum_{i=1}^{N} z_i\right)^r = [(1+z_1) + (1+z_2) + ... + (1+z_N)]^r =$$

$$\left(\sum_{i=1}^{N} (1+z_i)\right)^r \leq \prod_{i=1}^{N} (1+z_i)^r, \quad \text{by } 1 + z_i \geq 1,\ i = 1, ..., N.$$

■

We give

Theorem 13 *Let $r, N, \beta \in \mathbb{N}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{r+m+1}{2}$, and γ, λ_n are as in (33) and (34), respectively. Also we take $\lambda = 0, 1, ..., r$. When λ is even we define*

$$\psi_{1\lambda} := \frac{\pi (-1)^{\frac{2\beta-\lambda}{2}} (2\beta)!}{2^{\lambda+1} (2\beta - \lambda - 1)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-1}}{(\beta-k)!(\beta+k)!}\right), \tag{44}$$

and when λ is odd we define

$$\psi_{2\lambda} := \frac{(-1)^{\frac{\lambda-1}{2}}(2\beta)!}{2^{\lambda}(2\beta-\lambda-1)!}\left(\sum_{k=1}^{\beta}(-1)^{\beta-k}\frac{k^{2\beta-\lambda-1}[\ln(2k)]}{(\beta-k)!(\beta+k)!}\right), \tag{45}$$

and we set

$$\psi_{\lambda} := \begin{cases}\psi_{1\lambda}, & \text{if } \lambda \text{ is even,}\\ \psi_{2\lambda}, & \text{if } \lambda \text{ is odd.}\end{cases} \tag{46}$$

Similarly, it is defined $\psi_{\lambda+m}$, just set in (44), (45), (46), $\lambda+m$ in place of λ. Then

$$A_{\xi_n}(\overline{\alpha}) := \lambda_n^{-N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i \le$$

$$\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{r}\binom{r}{\lambda}\left[\psi_{\lambda}+\psi_{\lambda+m}\right]\right\}^N \le$$

$$2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{r}\binom{r}{\lambda}\left[\psi_{\lambda}+\psi_{\lambda+m}\right]\right\}^N < +\infty, \tag{47}$$

uniformly bounded, and convergent to zero as $\xi_n \to 0$, when $n \to +\infty$.

Proof. We estimate

$$A_{\xi_n}(\overline{\alpha}) = \lambda_n^{-N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i =$$

$$2^N\lambda_n^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i \le \tag{48}$$

$$2^N\lambda_n^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(1+\sum_{i=1}^{N}\left(\frac{s_i}{\xi_n}\right)\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i =$$

$$\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}z_i^{\alpha_i}\right)\left(1+\sum_{i=1}^{N}z_i\right)^r\prod_{i=1}^{N}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}\prod_{i=1}^{N}dz_i \overset{(43)}{\le}$$

$$\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}z_i^{\alpha_i}\right)\left(\prod_{i=1}^{N}(1+z_i)^r\right)\prod_{i=1}^{N}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}\prod_{i=1}^{N}dz_i =$$

$$\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\prod_{i=1}^{N}\left(\int_0^{\infty}z^{\alpha_i}(1+z)^r\left(\frac{\sin z}{z}\right)^{2\beta}dz\right) =$$

$$\xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \prod_{i=1}^{N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \left(\int_0^\infty z^{\lambda+\alpha_i} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right) \right] =$$

$$\xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \prod_{i=1}^{N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\int_0^1 z^{\lambda+\alpha_i} \left(\frac{\sin z}{z} \right)^{2\beta} dz + \right.\right.$$

$$\left.\left. \int_1^\infty z^{\lambda+\alpha_i} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\} \leq \tag{49}$$

$$\xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\int_0^1 z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz + \right.\right.$$

$$\left.\left. \int_1^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}^N \leq$$

$$\xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\int_0^\infty z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz + \right.\right.$$

$$\left.\left. \int_0^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}^N =: I. \tag{50}$$

Based on [9], p. 210, item 1033 and [6], see (27), (28), and by assuming $\mathbb{N} \ni \beta > \frac{r+m+1}{2}$, i.e. $\lambda < \lambda + m < 2\beta - 1$, for all $\lambda = 0, 1, ..., r$, we have the following calculations:

Let λ be even, then

$$\int_0^\infty z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\pi \, (-1)^{\frac{2\beta-\lambda}{2}} \, (2\beta)!}{2^{\lambda+1} \, (2\beta-\lambda-1)!} \sum_{k=1}^{\beta} (-1)^k \, \frac{k^{2\beta-\lambda-1}}{(\beta-k)! \, (\beta+k)!} = \psi_{1\lambda}. \tag{51}$$

Let λ be odd, then

$$\int_0^\infty z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{(-1)^{\frac{\lambda-1}{2}} \, (2\beta)!}{2^{\lambda} \, (2\beta-\lambda-1)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \, \frac{k^{2\beta-\lambda-1} \, [\ln(2k)]}{(\beta-k)! \, (\beta+k)!} = \psi_{2\lambda}. \tag{52}$$

Therefore

$$\int_0^\infty z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \psi_\lambda = \begin{cases} \psi_{1\lambda}, & \text{when } \lambda \text{ is even}, \\ \psi_{2\lambda}, & \text{when } \lambda \text{ is odd}. \end{cases} \tag{53}$$

Similarly, for $\lambda + m$ being even, we get

$$\int_0^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\pi \, (-1)^{\frac{2\beta-\lambda-m}{2}} \, (2\beta)!}{2^{\lambda+m+1} \, (2\beta-\lambda-m-1)!} \tag{54}$$

$$\sum_{k=1}^{\beta}(-1)^k \frac{k^{2\beta-\lambda-m-1}}{(\beta-k)!\,(\beta+k)!}=\psi_{1(\lambda+m)}.$$

And when $\lambda+m$ is odd we get that

$$\int_0^{\infty} z^{\lambda+m}\left(\frac{\sin z}{z}\right)^{2\beta} dz=\frac{(-1)^{\frac{\lambda+m-1}{2}}\,(2\beta)!}{2^{\lambda+m}\,(2\beta-\lambda-m-1)!} \tag{55}$$

$$\sum_{k=1}^{\beta}(-1)^{\beta-k} \frac{k^{2\beta-\lambda-m-1}\,[\ln(2k)]}{(\beta-k)!\,(\beta+k)!}=\psi_{2(\lambda+m)}.$$

Therefore, it holds

$$\int_0^{\infty} z^{\lambda+m}\left(\frac{\sin z}{z}\right)^{2\beta} dz=\psi_{\lambda+m}=\begin{cases}\psi_{1(\lambda+m)}, & \text{when } \lambda+m \text{ is even,}\\ \psi_{2(\lambda+m)}, & \text{when } \lambda+m \text{ is odd.}\end{cases} \tag{56}$$

That is

$$I=\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{r}\binom{r}{\lambda}\left[\psi_\lambda+\psi_{\lambda+m}\right]\right\}^N \leq$$

$$2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{r}\binom{r}{\lambda}\left[\psi_\lambda+\psi_{\lambda+m}\right]\right\}^N<+\infty. \tag{57}$$

I.e. $A_{\xi_n}(\overline{\alpha})$ is uniformly bounded. The theorem is proved. ■

We continue with

Theorem 14 *Let $r,n\in\mathbb{N}$, $\xi_n\in(0,1]$, $\beta\in\mathbb{N}:\beta>\frac{r+1}{2}$, $N\in\mathbb{N}-\{1\}$. Here γ,λ_n are as in (33) and (34), respectively, and ψ_λ is defined by (44), (45) and (46), $\lambda=0,1,...,r$. Then*

$$B_{\xi_n}:=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i\leq$$

$$\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N\leq$$

$$2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N<+\infty, \tag{58}$$

uniformly bounded, and convergent to zero as $\xi_n\to 0$, when $n\to+\infty$.

Proof. We estimate

$$B_{\xi_n}=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i=$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^r \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \le \tag{59}$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1+\sum_{i=1}^{N}\left(\frac{s_i}{\xi_n}\right)\right)^r \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(1+\sum_{i=1}^{N} z_i\right)^r \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i \overset{(43)}{\le}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} (1+z_i)^r\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^{\infty} (1+z)^r \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)^N =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \left(\int_0^{\infty} z^{\lambda} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)\right]^N \overset{(53)}{=}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda\right]^N \le \tag{60}$$

$$2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda\right]^N < +\infty,$$

under $\beta > \frac{r+1}{2}$. The theorem is proved. ■

We also give

Theorem 15 *Let $p > 1$; $r, \beta, N \in \mathbb{N}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N$: $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{\lceil rp \rceil + m + 1}{2}$, and γ, λ_n are as in (33) and (34), respectively, and λ runs as $\lambda = 0, 1, ..., \lceil rp \rceil$. Furthermore ψ_λ is defined as in (44), (45) and (46). Similarly, it is defined $\psi_{\lambda+mp}$, just set in (44), (45), (46), $(\lambda + mp)$ in place of λ. Then*

$$C_{\xi_n}(\overline{\alpha}) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \le \tag{61}$$

$$\xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\psi_\lambda + \psi_{\lambda+mp}\right]\right\}^N \le$$

$$2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\psi_\lambda+\psi_{\lambda+mp}\right]\right\}^N<+\infty,$$

uniformly bounded, and convergent to zero as $\xi_n\to 0$, when $n\to+\infty$.

Above $\lceil\cdot\rceil$ is the ceiling of the number.

Proof. We estimate

$$C_{\xi_n}(\overline{\alpha})=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(\left(\prod_{i=1}^N|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i=$$

$$2^N\lambda_n^{-N}\int_{\mathbb{R}_+^N}\left(\left(\prod_{i=1}^N s_i^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i\le$$

$$2^N\lambda_n^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^N s_i^{\alpha_i p}\right)\left(1+\sum_{i=1}^N\left(\frac{s_i}{\xi_n}\right)\right)^{rp}\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i= \tag{62}$$

$$2^N\gamma^{-N}\xi_n^{2\beta(N-1)+mp}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^N z_i^{\alpha_i p}\right)\left(1+\sum_{i=1}^N z_i\right)^{rp}\prod_{i=1}^N\left(\frac{\sin z_i}{z_i}\right)^{2\beta}\prod_{i=1}^N dz_i\overset{(43)}{\le}$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^N z_i^{\alpha_i p}\right)\left(\prod_{i=1}^N(1+z_i)^{rp}\right)\prod_{i=1}^N\left(\frac{\sin z_i}{z_i}\right)^{2\beta}\prod_{i=1}^N dz_i=$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\prod_{i=1}^N\left(\int_0^\infty(1+z)^{rp}z^{\alpha_i p}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right)\le$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\prod_{i=1}^N\left(\int_0^\infty(1+z)^{\lceil rp\rceil}z^{\alpha_i p}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right)= \tag{63}$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\prod_{i=1}^N\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left(\int_0^\infty z^{\lambda+\alpha_i p}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right)\right\}=$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\prod_{i=1}^N\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\int_0^1 z^{\lambda+\alpha_i p}\left(\frac{\sin z}{z}\right)^{2\beta}dz+\right.\right.$$

$$\left.\left.\int_1^\infty z^{\lambda+\alpha_i p}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]\right\}\le$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\int_0^1 z^{\lambda}\left(\frac{\sin z}{z}\right)^{2\beta}dz+\right.\right.$$

$$\int_1^\infty z^{\lambda+mp}\left(\frac{\sin z}{z}\right)^{2\beta}dz\Bigg]\Bigg\}^N \leq$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\int_0^\infty z^{\lambda}\left(\frac{\sin z}{z}\right)^{2\beta}dz+\right.\right.$$

$$\left.\left.\int_0^\infty z^{\lambda+mp}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]\right\}^N \overset{((53),(56))}{=}$$

$$\xi_n^{2\beta(N-1)+mp}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\psi_\lambda+\psi_{\lambda+mp}\right]\right\}^N \leq \tag{64}$$

$$2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left[\psi_\lambda+\psi_{\lambda+mp}\right]\right\}^N < +\infty.$$

I.e. $C_{\xi_n}(\overline{\alpha})$ is uniformly bounded.

Above we assumed that $\mathbb{N}\ni\beta>\frac{\lceil rp\rceil+m+1}{2}$, i.e. $\lambda<\lambda+m<2\beta-1$, for all $\lambda=0,1,...,\lceil rp\rceil$.

The theorem is proved. ■

We also present

Theorem 16 *Let $p>1$; $r,\beta\in\mathbb{N}$, $N\in\mathbb{N}-\{1\}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Here we take $\beta>\frac{\lceil rp\rceil+1}{2}$, and γ,λ_n are as in (33) and (34), respectively, and λ runs as $\lambda=0,1,...,\lceil rp\rceil$. Furthermore ψ_λ is defined as in (44), (45) and (46). Then*

$$D_{\xi_n}:=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp}\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i \leq$$

$$\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\left[\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\psi_\lambda\right]^N \leq \tag{65}$$

$$2^N\gamma^{-N}\left[\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\psi_\lambda\right]^N < +\infty,$$

uniformly bounded, and convergent to zero as $\xi_n\to 0$, when $n\to+\infty$.

Proof. We estimate

$$D_{\xi_n}=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp}\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i = \tag{66}$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \le$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$2^N \gamma^{-N} \xi_n^{2\beta(N-1)} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^{N} z_i\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i \overset{(43)}{\le}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} (1 + z_i)^{rp}\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^{\infty} (1+z)^{rp} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)^N \le$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^{\infty} (1+z)^{\lceil rp \rceil} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)^N =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\int_0^{\infty} z^{\lambda} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)\right]^N \overset{(53)}{=} \tag{67}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_{\lambda}\right]^N \le$$

$$2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_{\lambda}\right]^N < +\infty,$$

under $\beta > \frac{\lceil rp \rceil + 1}{2}$. The theorem is proved. ∎

We proceed it

Theorem 17 *Let $n, N \in \mathbb{N}$, $\xi_n \in (0,1]$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$. Here $\beta \in \mathbb{N} : \beta > \frac{m+1}{2}$, and γ, λ_n are as in (33) and (34), respectively. Furthermore ψ_{α_i} is defined as in (44), (45) and (46), just replace λ by α_i, $i = 1, ..., N$. Then*

$$F_{\xi_n}(\overline{\alpha}) := \xi_n^{-m} \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \le$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) \leq \tag{68}$$

$$2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) =: \varphi < +\infty.$$

Proof. We estimate

$$F_{\xi_n}(\overline{\alpha}) = \xi_n^{-m} \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$\xi_n^{-m} \lambda_n^{-N} 2^N \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i = \tag{69}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \prod_{i=1}^{N} \left(\int_0^{\infty} z^{\alpha_i} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right) \overset{(53)}{=}$$

$$\xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) \leq \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) \leq$$

$$2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) < +\infty,$$

under $\beta > \frac{m+1}{2}$, i.e. $\alpha_i < 2\beta - 1$, $i = 1, ..., N$. The theorem is proved. ■

We make

Remark 18 *All as in Theorem 17. We denote*

$$\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}} := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i, \tag{70}$$

where $\widetilde{j} = 1, ..., m,$ *and* $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$.

By (68) we obtain

$$|\overline{c}_{\overline{\alpha},n}| = \left| \overline{c}_{\overline{\alpha},n,\widetilde{j}} \right| \leq \varphi \xi_n^m \leq \varphi. \tag{71}$$

3 Main Results for $T_{r,n}^{[m]}$

3.1 Uniform Approximation

We start with an application to $T_{r,n}^{[m]}$ of the following theorem.

Theorem 19 *([3], p. 11) Let $m \in \mathbb{N}$, $f \in C^m\left(\mathbb{R}^N\right)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f(\cdot,\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \right\|_\infty < \infty$, for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^N \alpha_i = m$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence.*

Suppose that for all $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^N \alpha_i = m$ we have that

$$u_{\xi_n}(\overline{\alpha}) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty. \tag{72}$$

For $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{j=1}^N \alpha_j = \widetilde{j}$, call

$$c_{\overline{\alpha},n} := c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s_1, ..., s_N). \tag{73}$$

Then

i)

$$E_{r,n}^{[m]}(x) := \left| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right|$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{(\omega_r(f_{\overline{\alpha}}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i\right)} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) \right). \tag{74}$$

$\forall\ x \in \mathbb{R}^N$.

ii)

$$\left\| E_{r,n}^{[m]} \right\|_\infty \leq R.H.S.(74). \tag{75}$$

Given that $\xi_n \to 0$, as $n \to \infty$, and u_{ξ_n} is uniformly bounded, then we derive that $\left\| E_{r,n}^{[m]} \right\| \to 0$ with rates.

iii) It holds also that

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_\infty \leq \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\left| c_{\overline{\alpha},n,\widetilde{j}} \right| \|f_{\overline{\alpha}}\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) + R.H.S.(74), \tag{76}$$

given that $\|f_{\overline{\alpha}}\|_\infty < \infty$, *for all* $\overline{\alpha} : |\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$. *Furthermore, as* $\xi_n \to 0$ *when* $n \to \infty$, *assuming that* $c_{\overline{\alpha},n,\widetilde{j}} \to 0$, *while* u_{ξ_n} *is uniformly bounded, we conclude that*

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_\infty \to 0 \tag{77}$$

with rates.

A uniform approximation result for $T_{r,n}^{[m]}$ follows:

Theorem 20 *Let* $r, N, \beta, m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. *Assume* $\left\| \frac{\partial^m f(\cdot,\cdot,...,\cdot)}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}} \right\|_\infty < \infty$, *for all* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^N \alpha_i = m$. *Let* φ_{ξ_n} *be the Borel probability measure on* $\mathbb{R}^N$, *see (39), where* $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. *Here* $\beta > \frac{m+r+1}{2}$, *and* $A_{\xi_n}(\overline{\alpha})$ *as in (47), and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

i)

$$\overline{E}_{r,n}^{[m]}(x) := \left| T_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,...,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right|$$

$$\leq \sum_{\substack{\alpha_1,...,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{(\omega_r(f_{\overline{\alpha}}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i!\right)} A_{\xi_n}(\overline{\alpha}), \tag{78}$$

$\forall\ x \in \mathbb{R}^N$.

ii)

$$\left\| \overline{E}_{r,n}^{[m]} \right\|_\infty \leq R.H.S.(78). \tag{79}$$

Given that $\xi_n \to 0$, *as* $n \to +\infty$, *we have that* $A_{\xi_n}(\overline{\alpha}) \to 0$ *and are uniformly bounded, and then we derive that* $\left\| \overline{E}_{r,n}^{[m]} \right\|_\infty \to 0$ *with rates.*

iii) It holds also that

$$\left\| T_{r,n}^{[m]}(f) - f \right\|_\infty \leq \sum_{\widetilde{j}=1}^m \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{\substack{\alpha_1,...,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\left| \overline{c}_{\overline{\alpha},n,\widetilde{j}} \right| \|f_{\overline{\alpha}}\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) + R.H.S.(78), \tag{80}$$

given that $\|f_{\overline{\alpha}}\|_\infty < +\infty$, *for all* $\overline{\alpha} : |\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$. *Furthermore, as* $\xi_n \to 0$ *when* $n \to +\infty$, *we have that* $\overline{c}_{\overline{\alpha},n,\widetilde{j}} \to 0$ *and* $A_{\xi_n}(\overline{\alpha}) \to 0$, *and both are uniformly bounded, and we conclude that*

$$\left\| T_{r,n}^{[m]}(f) - f \right\|_\infty \to 0 \tag{81}$$

with rates.

Proof. Mainly by applying Theorem 19. By Theorem 13 we get that $A_{\xi_n}(\overline{\alpha}) \to 0$ and $A_{\xi_n}(\overline{\alpha})$ are uniformly bounded. By Theorem 17 and Remark 18 we get $\overline{c}_{\overline{\alpha},n} \to 0$ and $\overline{c}_{\overline{\alpha},n}$ are uniformly bounded. ■

We mention

Theorem 21 *([3], p. 14) Let $f \in C_B(\mathbb{R}^N)$, uniformly continuous, $N \geq 1$, $\xi_n \in (0,1]$. Then*

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_{\infty} \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r(f, \xi_n), \tag{82}$$

under the assumption

$$\Phi_{\xi_n} := \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{83}$$

As $n \to \infty$ and $\xi_n \to 0$, given that Φ_{ξ_n} are uniformly bounded, we derive

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_{\infty} \to 0 \tag{84}$$

with rates.

We give

Theorem 22 *Let $f \in C_B(\mathbb{R}^N)$, uniformly continuous, $\beta, r \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$, $\beta > \frac{r+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Then*

$$\left\| T_{r,n}^{[0]} f - f \right\|_{\infty} \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r(f, \xi_n). \tag{85}$$

As $n \to \infty$ and $\xi_n \to 0$, we derive

$$\left\| T_{r,n}^{[0]} f - f \right\|_{\infty} \to 0 \tag{86}$$

with rates.

Proof. By Theorems 14 and 21. ■

3.2 L_p Approximation for $T_{r,n}^{[m]}$

We need

Definition 23 *([4], [8]) We call*

$$\Delta_u^r f(x) := \Delta_{u_1,u_2,\ldots,u_N}^r f(x_1,\ldots,x_N) := \tag{87}$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, \ldots, x_N + ju_N).$$

Let $p \geq 1$, the modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{88}$$

$h > 0$.

We will apply

Theorem 24 *([3], p. 24) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_{\overline{\alpha}} \in L_p(\mathbb{R}^N)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^N \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{89}$$

For $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{j=1}^N \alpha_j = \widetilde{j}$, call

$$c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{90}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p = \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p,x} \tag{91}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right)$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_{\overline{\alpha}}, \xi_n)_p.$$

As $n \to \infty$ and $\xi_n \to 0$, by (91) we obtain that $\left\| E_{r,n}^{[m]} \right\| \to 0$ with rates.

One also finds by (91) that

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) \right\|_{p,x} \leq \sum_{\widetilde{j}=1}^m \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\left| c_{\overline{\alpha},n,\widetilde{j}} \right|}{\prod_{i=1}^N \alpha_i!} \|f_{\overline{\alpha}}\|_p \right) + R.H.S.(91), \tag{92}$$

given that $\|f_{\overline{\alpha}}\|_p < \infty$, $|\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Assuming that $c_{\overline{\alpha},n,\widetilde{j}} \to 0$, $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_p \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *the unit operator, in* L_p *norm, with rates.*

We present

Theorem 25 *Let* $f \in C^m\left(\mathbb{R}^N\right)$, $r, \beta, N, m \in \mathbb{N}$, *with* $f_{\overline{\alpha}} \in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here* φ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *as in (39), for* $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Let* $\beta > \frac{\lceil rp \rceil + m + 1}{2}$; $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$. *Here* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j} = 1, ..., m$, *and* $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. *Then*

$$\left\| \overline{E}_{r,n}^{[m]} \right\|_p = \left\| T_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}| = \widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\psi_\lambda + \psi_{\lambda+mp} \right) \right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \xi_n^{\left(\frac{2\beta(N-1)}{p} + m \right)} \omega_r\left(f_{\overline{\alpha}}, \xi_n\right)_p. \tag{93}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (93) we obtain that* $\left\| \overline{E}_{r,n}^{[m]} \right\| \to 0$ *with rates.*

One also finds by (93) that

$$\left\| T_{r,n}^{[m]}(f; x) - f(x) \right\|_{p,x} \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\left| \overline{c}_{\overline{\alpha},n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_{\overline{\alpha}}\|_p \right) + R.H.S.(93), \tag{94}$$

given that $\|f_{\overline{\alpha}}\|_p < \infty$, $|\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Assuming that $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| T_{r,n}^{[m]}(f) - f \right\|_p \to 0$, *that is* $T_{r,n}^{[m]} \to I$ *the unit operator, in* L_p *norm, with rates.*

Proof. By Theorem 24. From Theorem 15 we get that $C_{\xi_n}(\overline{\alpha})$ is uniformly bounded, see (61) and $C_{\xi_n}(\overline{\alpha}) \to 0$, as $\xi_n \to 0$, when $n \to \infty$. Also by Theorem 17 and Remark 18 we get that $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ are uniformly bounded and $\overline{c}_{\overline{\alpha},n} \to 0$, as $\xi_n \to 0$, when $n \to \infty$. ■

We continue with an application of

Theorem 26 *([3], p. 26) Let* $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$*;* $N \geq 1$*;* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Assume* μ_{ξ_n} *probability Borel measure on* $\mathbb{R}^N$*,* $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s) < \infty. \tag{95}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \leq \tag{96}$$

$$\left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p.$$

As $\xi_n \to 0$*, when* $n \to \infty$*, we derive* $\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \to 0$*, i.e.* $\theta_{r,n}^{[0]} \to I$*, the unit operator, in* L_p *norm.*

We give

Theorem 27 *Let* $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$*;* $N \in \mathbb{N} - \{1\}$*,* $\beta, r \in \mathbb{N}$*;* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*;* $\beta > \frac{\lceil rp \rceil + 1}{2}$*,* $\xi_n \in (0, 1]$*,* $n \in \mathbb{N}$*. Then*

$$\left\|T_{r,n}^{[0]}(f) - f\right\|_p \leq 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda\right]^{\frac{N}{p}} \xi_n^{\frac{2\beta(N-1)}{p}} \omega_r(f, \xi_n)_p. \tag{97}$$

As $\xi_n \to 0$*, when* $n \to \infty$*, we derive* $\left\|T_{r,n}^{[0]}(f) - f\right\|_p \to 0$*, i.e.* $T_{r,n}^{[0]} \to I$*, the unit operator, in* L_p *norm.*

Proof. By Theorems 26, 16. ■
We mention

Theorem 28 *([3], p. 27) Let* $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$*;* $N \geq 1$*. Assume* μ_{ξ_n} *probability Borel measure on* $\mathbb{R}^N$*,* $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{r} d\mu_{\xi_n}(s) < \infty. \tag{98}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f) - f\right\|_1 \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{r} d\mu_{\xi_n}(s)\right) \omega_r(f, \xi_n)_1. \tag{99}$$

As $\xi_n \to 0$*, we get* $\theta_{r,n}^{[0]} \to I$*, in* L_1 *norm.*

We give

Theorem 29 *Let $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$; $N \in \mathbb{N} - \{1\}$, $r, \beta \in \mathbb{N}$, $\beta > \frac{r+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Then*

$$\left\| T_{r,n}^{[0]}(f) - f \right\|_1 \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r(f, \xi_n)_1 . \tag{100}$$

As $\xi_n \to 0$, we get $T_{r,n}^{[0]} \to I$, in L_1 norm.

Proof. By Theorems 14, 28. ∎

We mention

Theorem 30 *([3], p. 29) Let $f \in C^m\left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$, with $f_{\overline{\alpha}} \in L_1\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Suppose for all $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \tag{101}$$

For $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, call

$$c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s) . \tag{102}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_1 = \left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \tag{103}$$

$$\leq \sum_{|\overline{\alpha}|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r(f_{\overline{\alpha}}, \xi_n)_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) .$$

As $\xi_n \to 0$, we get $\left\| E_{r,n}^{[m]} \right\|_1 \to 0$ with rates.

From (103) we get

$$\left\| \theta_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\left| c_{\overline{\alpha},n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_{\overline{\alpha}}\|_1 \right) + R.H.S.(103), \tag{104}$$

given that $\|f_{\overline{\alpha}}\|_1 < \infty$, $|\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

As $n \to \infty$, *assuming* $\xi_n \to 0$ *and* $c_{\overline{\alpha},n,\widetilde{j}} \to 0$, *we obtain* $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_1 \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *in* L_1 *norm, with rates.*

We give

Theorem 31 *Let* $f \in C^m\left(\mathbb{R}^N\right)$, $r, N, \beta, m \in \mathbb{N}$, *with* $f_{\overline{\alpha}} \in L_1\left(\mathbb{R}^N\right)$, *where* $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$, $x \in \mathbb{R}^N$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$, *and* $\beta > \frac{m+r+1}{2}$. *Here* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j} = 1, ..., m$, *and* $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. *Also here* φ_{ξ_n} *is the Borel probability measure on* $\mathbb{R}^N$, *see (39). Then*

$$\left\| \overline{E}_{r,n}^{[m]} \right\|_1 = \left\| T_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \tag{105}$$

$$\leq \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r\left(f_{\overline{\alpha}}, \xi_n\right)_1 \right) \xi_n^{2\beta(N-1)+m}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m}\right] \right\}^N.$$

As $\xi_n \to 0$, *we get* $\left\| \overline{E}_{r,n}^{[m]} \right\|_1 \to 0$ *with rates.*

From (105) we get

$$\left\| T_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\left| \overline{c}_{\overline{\alpha},n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_{\overline{\alpha}}\|_1 \right) + R.H.S.(105), \tag{106}$$

given that $\|f_{\overline{\alpha}}\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

As $n \to \infty$, *assuming* $\xi_n \to 0$, *we get* $\overline{c}_{\overline{\alpha},n,\widetilde{j}} \to 0$ *and* $\left\| T_{r,n}^{[m]}(f) - f \right\|_1 \to 0$, *that is* $T_{r,n}^{[m]} \to I$ *in* L_1 *norm, with rates.*

Proof. By Theorem 30, also by Theorem 13, see (47) and by Theorem 17 and Remark 18. ■

3.3 Global Smoothness Preservation and Simultaneous Approximation of $T_{r,n}^{[m]}$

We need

Definition 32 *([3], p. 34) Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by*

$$\omega_m\left(f; h\right)_p := \sup_{\|t\|_2 \leq h} \left\|\Delta_t^m\left(f\right)\right\|_{p,x}, \tag{107}$$

$h > 0$, where

$$\Delta_t^m f\left(x\right) := \sum_{j=0}^{m} \left(-1\right)^{m-j} \binom{m}{j} f\left(x + jt\right). \tag{108}$$

Denote

$$\omega_m\left(f; h\right)_\infty = \omega_m\left(f, h\right). \tag{109}$$

Above, $x, t \in \mathbb{R}^N$.

We present the related global smoothness preservation result

Theorem 33 *We assume $T_{r,n}^{[\widetilde{m}]}\left(f; x\right) \in \mathbb{R}$, $\widetilde{m} \in \mathbb{Z}_+$, $\forall\ x \in \mathbb{R}$. Let $h > 0$, $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$.*

i) Assume $\omega_m\left(f, h\right) < \infty$. Then

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right) \leq \left(\sum_{\widetilde{j}=0}^{r} \left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right) \omega_m\left(f, h\right). \tag{110}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$. Then

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right)_1 \leq \left(\sum_{\widetilde{j}=0}^{r} \left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right) \omega_m\left(f, h\right)_1. \tag{111}$$

iii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p > 1$. Then

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right)_p \leq \left(\sum_{\widetilde{j}=0}^{r} \left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right) \omega_m\left(f, h\right)_p. \tag{112}$$

Proof. Direct application of ([3]) Theorem 3.2, p. 35. ■

We make

Remark 34 *Let $r = 1$, $\widetilde{m} \in \mathbb{Z}_+$, then $\alpha_{0,1}^{[\widetilde{m}]} = 0$, $\alpha_{1,1}^{[\widetilde{m}]} = 1$. Hence*

$$T_{1,n}^{[\widetilde{m}]}\left(f; x\right) = \lambda_n^{-N} \int_{\mathbb{R}^N} f\left(x + s\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 ... ds_N =: T_n\left(f; x\right). \tag{113}$$

By Theorem 33, we get

Theorem 35 *We suppose* $T_n(f;x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}$. *Let* $h > 0$, $f \in C(\mathbb{R}^N)$, $N \geq 1$.

i) Assume $\omega_m(f,h) < \infty$. *Then*

$$\omega_m(T_n f, h) \leq \omega_m(f,h). \tag{114}$$

ii) Assume $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$. *Then*

$$\omega_m(T_n f, h)_1 \leq \omega_m(f,h)_1. \tag{115}$$

iii) Assume $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$, $p > 1$. *Then*

$$\omega_m(T_n f, h)_p \leq \omega_m(f,h)_p. \tag{116}$$

Next, we get an optimality result

Proposition 36 *Above inequality (114):*

$$\omega_m(T_n f, h) \leq \omega_m(f,h)$$

is sharp, namely it is attained by any

$$f_j^*(x) = x_j^m, \quad j = 1,...,N, \quad x = (x_1,...,x_j,...,x_N) \in \mathbb{R}^N. \tag{117}$$

Proof. Apply Proposition 3.5, p. 38, of [3]. ∎

We need

Theorem 37 *([3], p. 39) Let* $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\overline{\beta} := (\beta_1,...,\beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1,...,N$; $|\overline{\beta}| := \sum_{i=1}^N \beta_i = l$. *Here* $f(x+sj)$, $x, s \in \mathbb{R}^N$, *is* μ_{ξ_n}*-integrable wrt* s, *for* $j = 1,...,r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}$, ..., $h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j} \geq 0$ $(j = 1,...,r)$ *on* $\mathbb{R}^N$ *such that*

$$\left| \frac{\partial^{i_1} f(x+sj)}{\partial x_1^{i_1}} \right| \leq h_{i_1,j}(s), \quad i_1 = 1,...,\beta_1, \tag{118}$$

$$\left| \frac{\partial^{\beta_1+i_2} f(x+sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1,...,\beta_2,$$

$$\vdots$$

$$\left| \frac{\partial^{\beta_1+\beta_2+...+\beta_{N-1}+i_N} f(x+sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} ... \partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j}(s), \quad i_N = 1,...,\beta_N,$$

$\forall\ x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[\widetilde{m}]}\left(f;x\right)\right)_{\overline{\beta}}=\theta_{r,n}^{[\widetilde{m}]}\left(f_{\overline{\beta}};x\right), \quad \widetilde{m}\in\mathbb{Z}_{+}. \tag{119}$$

In particular it holds

$$\left(T_{r,n}^{[\widetilde{m}]}\left(f;x\right)\right)_{\overline{\beta}}=T_{r,n}^{[\widetilde{m}]}\left(f_{\overline{\beta}};x\right), \tag{120}$$

when

$$d\mu_{\xi_n}=d\varphi_{\xi_n}\left(s\right), \quad s\in\mathbb{R}^N,$$

see (39).

Corollary 38 *(by Theorem 37, $r=1$) It holds*

$$\left(T_n\left(f;x\right)\right)_{\overline{\beta}}=T_n\left(f_{\overline{\beta}};x\right). \tag{121}$$

We present simultaneous global smoothness results.

Theorem 39 *Let $h>0$ and the assumptions of Theorem 37 are valid for $d\mu_{\xi_n}=d\varphi_{\xi_n}$. Here $\overline{\gamma}=0,\overline{\beta}$ $(0=(0,...,0))$, $\widetilde{m}\in\mathbb{Z}_{+}$.*

i) Assume $\omega_m\left(f_{\overline{\gamma}},h\right)<\infty$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}},h\right)\leq\left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}},h\right). \tag{122}$$

ii) Additionally suppose $f_{\overline{\gamma}}\in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}},h\right)_1\leq\left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}},h\right)_1. \tag{123}$$

iii) Additionally suppose $f_{\overline{\gamma}}\in L_p\left(\mathbb{R}^N\right)$, $p>1$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}},h\right)_p\leq\left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}},h\right)_p. \tag{124}$$

It follows

Corollary 40 *(to Theorem 39) Let $h>0$, $r=1$ and $\overline{\gamma}=0,\overline{\beta}$.*

i) Assume $\omega_m\left(f_{\overline{\gamma}},h\right)<\infty$. Then

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}},h\right)\leq\omega_m\left(f_{\overline{\gamma}},h\right). \tag{125}$$

ii) Additionally suppose $f_{\overline{\gamma}}\in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}},h\right)_1\leq\omega_m\left(f_{\overline{\gamma}},h\right)_1. \tag{126}$$

iii) Additionally suppose $f_{\overline{\gamma}} \in L_p\left(\mathbb{R}^N\right)$, $p>1$. *Then*

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}}, h\right)_p \leq \omega_m\left(f_{\overline{\gamma}}, h\right)_p . \tag{127}$$

Next comes multi-simultaneous approximation. We give

Theorem 41 *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. *The assumptions of Theorem 37 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Assume* $\left\|f_{\overline{\gamma}+\overline{\alpha}}\right\|_\infty < \infty$, *and let* $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Here* $\beta, r \in \mathbb{N}$, $\beta > \frac{m+r+1}{2}$, *and* $A_{\xi_n}\left(\overline{\alpha}\right)$ *as in (47), and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

$$\left\|\left(T_{r,n}^{[m]}\left(f;\cdot\right)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}\left(\cdot\right) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}\left(\cdot\right)}{\prod_{i=1}^{N} \alpha_i!}\right)\right\|_\infty \tag{128}$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{\left(\omega_r\left(f_{\overline{\gamma}+\overline{\alpha}}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} A_{\xi_n}\left(\overline{\alpha}\right).$$

Proof. Based on Theorems 20, 37. ■

We continue with

Theorem 42 *Let* $f \in C_B^l\left(\mathbb{R}^N\right)$, $r, l, \beta \in \mathbb{N}$ *(functions l-times continuously differentiable and bounded),* $N \in \mathbb{N} - \{1\}$. *The assumptions of Theorem 37 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$. *Call* $\overline{\gamma} = 0, \overline{\beta}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Let also* $\beta > \frac{r+1}{2}$. *Then*

$$\left\|\left(T_{r,n}^{[0]} f\right)_{\overline{\gamma}} - f_{\overline{\gamma}}\right\|_\infty \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda\right]^N \xi_n^{2\beta(N-1)} \omega_r\left(f_{\overline{\gamma}}, \xi_n\right). \tag{129}$$

If $\xi_n \to 0$, *as* $n \to \infty$, *then* $\left(T_{r,n}^{[0]} f\right)_{\overline{\gamma}} \to f_{\overline{\gamma}}$ *uniformly.*

Proof. By Theorems 22, 37. ■

We present

Theorem 43 *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $r, \beta, N, m, l \in \mathbb{N}$. *The assumptions of Theorem 37 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{(\overline{\gamma}+\overline{\alpha})} \in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Let also* $\beta > \frac{\lceil rp \rceil + m + 1}{2}$; $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N$, $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$. *Here* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j} = 1, \ldots, m$, *and* $\overline{\alpha} := \left(\alpha_1, \ldots, \alpha_N\right)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$,

$|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. *Then*

$$\left\| \left(T_{r,n}^{[m]}(f;x) \right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} (\psi_\lambda + \psi_{\lambda+mp}) \right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \xi_n^{\left(\frac{2\beta(N-1)}{p} + m \right)} \omega_r \left(f_{\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_p . \quad (130)$$

Proof. Theorems 25 and 37. ■
We continue with

Theorem 44 *Let* $f \in C^l(\mathbb{R}^N)$, $\beta, r, l \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$. *The assumptions of Theorem 37 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{\overline{\gamma}} \in L_p(\mathbb{R}^N)$ *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here* $\beta > \frac{\lceil rp \rceil + 1}{2}$. *Then*

$$\left\| \left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_p \leq$$

$$2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^{\frac{N}{p}} \xi_n^{\frac{2\beta(N-1)}{p}} \omega_r \left(f_{\overline{\gamma}}, \xi_n \right)_p . \quad (131)$$

As $n \to +\infty$ *and* $\xi_n \to 0$, *then* $\left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} \overset{\|\cdot\|_p}{\to} f_{\overline{\gamma}}$.

Proof. By Theorems 27 and 37. ■
We continue with

Theorem 45 *Let* $f \in C^l(\mathbb{R}^N)$, $l \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$. *The assumptions of Theorem 37 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{\overline{\gamma}} \in L_1(\mathbb{R}^N)$ *and* $\beta, r \in \mathbb{N}$, $\beta > \frac{r+1}{2}$. *Then*

$$\left\| \left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_1 \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r \left(f_{\overline{\gamma}}, \xi_n \right)_1 . \quad (132)$$

As $n \to +\infty$ *and* $\xi_n \to 0$, *then* $\left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} \overset{\|\cdot\|_1}{\to} f_{\overline{\gamma}}$.

Proof. By Theorems 29, 37. ■

We continue with

Theorem 46 *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $r, N, \beta, m, l \in \mathbb{N}$. The assumptions of Theorem 37 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{(\overline{\gamma}+\overline{\alpha})} \in L_1\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, $\beta > \frac{m+r+1}{2}$. Here $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. Then*

$$\left\| \left(T_{r,n}^{[m]}(f;x)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \leq$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r \left(f_{\overline{\gamma}+\overline{\alpha}}, \xi_n\right)_1 \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m}\right] \right\}^N. \tag{133}$$

Proof. By Theorems 31, 37. ■

3.4 Voronovskaya Asymptotic Expansions for $T_{r,n}^{[m]}$

We will apply

Theorem 47 *([3], p. 53) Let $f \in C^m\left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$, with all $\|f_{\overline{\alpha}}\|_\infty \leq M$, $M > 0$, all $\overline{\alpha} : |\overline{\alpha}| = m$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$.*

Call $c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, all $|\overline{\alpha}| = \widetilde{j} = 1, ..., m-1$. Suppose $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \leq \rho$, all $\overline{\alpha} : |\overline{\alpha}| = m$, $\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma^* \leq 1$, $x \in \mathbb{R}^N$. Then

$$\theta_{r,n}^{[m]}(f;x) - f(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} \right) + 0\left(\xi_n^{m-\gamma^*}\right). \tag{134}$$

When $m = 1$, the sum collapses.

Above we assume $\theta_{r,n}^{[m]}(f;x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

We give

Theorem 48 *Let $r, m, \beta, N \in \mathbb{N}$, $\beta > \frac{m+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Also $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$. Here $f \in C^m(\mathbb{R}^N)$, with all $\|f_{\overline{\alpha}}\|_\infty \leq M$, $M > 0$, for all $\overline{\alpha} : |\overline{\alpha}| = m$; and $d\mu_{\xi_n}(s) = d\varphi_{\xi_n}(s)$, as in (39), $\forall\ s \in \mathbb{R}^N$. Assume $T_{r,n}^{[m]}(f; x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$. Here $\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), all $|\overline{\alpha}| = \widetilde{j} = 1, ..., m-1$. Let $0 < \gamma^* \leq 1$, $x \in \mathbb{R}^N$. Then*

$$T_{r,n}^{[m]}(f; x) - f(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) + 0\left(\xi_n^{m-\gamma^*}\right). \tag{135}$$

When $m = 1$, the sum collapses.

Proof. By Theorems 17, 47. Here $\rho = \varphi$, see (68). ■

We give

Corollary 49 *(to Theorem 48) Let $f \in C^1(\mathbb{R}^N)$, $N \in \mathbb{N}$, with all $\left\| \frac{\partial f}{\partial x_i} \right\|_\infty \leq M$, $M > 0$, $i = 1, ..., N$. Let $0 < \gamma^* \leq 1$. Assume $T_{r,n}^{[1]}(f; x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$. Here $r \in \mathbb{N}$ and $\beta \in \mathbb{N} - \{1\}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Then*

$$T_{r,n}^{[1]}(f; x) - f(x) = 0\left(\xi_n^{1-\gamma^*}\right). \tag{136}$$

Proof. By Theorems 17, 48. Here it is $\rho = \varphi$, apply (68) for $m = 1$. ■

We continue with

Corollary 50 *(to Theorem 48) Let $f \in C^2(\mathbb{R}^2)$, with all $\left\| \frac{\partial^2 f}{\partial x_1^2} \right\|_\infty$, $\left\| \frac{\partial^2 f}{\partial x_2^2} \right\|_\infty$, $\left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|_\infty \leq M$, $M > 0$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call*

$$c_1 = \int_{\mathbb{R}^2} s_1 d\varphi_{\xi_n}^*(s), \quad c_2 = \int_{\mathbb{R}^2} s_2 d\varphi_{\xi_n}^*(s), \tag{137}$$

where

$$d\varphi_{\xi_n}^* = \lambda_n^{-2} \prod_{i=1}^{2} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ds_2, \quad s = (s_1, s_2) \in \mathbb{R}^2.$$

Let $0 < \gamma^* \leq 1$ and assume $T_{r,n}^{[2]}(f; x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^2$. Here $r, \beta \in \mathbb{N}$ and $\beta > \frac{3}{2}$. Then

$$T_{r,n}^{[2]}(f; x) - f(x) = \left(\sum_{j=1}^{r} \alpha_{j,r}^{[2]} j \right) \left(c_1 \frac{\partial f}{\partial x_1}(x) + c_2 \frac{\partial f}{\partial x_2}(x) \right) + 0\left(\xi_n^{2-\gamma^*}\right). \tag{138}$$

Proof. By Theorems 17, 48. ■

We also give

Theorem 51 *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. Assumptions of Theorem 37 are valid for $d\varphi_{\xi_n}(s)$, $s \in \mathbb{R}^N$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Suppose $\|f_{\overline{\gamma}+\overline{\alpha}}\|_\infty \leq M$, $M > 0$, for all $\overline{\alpha} : |\overline{\alpha}| = m$. Here $\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ is as in (70), all $|\overline{\alpha}| = \widetilde{j} = 1, ..., m-1$; $0 < \gamma^* \leq 1$. Assume $T_{r,n}^{[m]}(f_{\overline{\gamma}}; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$. Let also $r, \beta \in \mathbb{N}$ and $\beta > \frac{m+1}{2}$. Then*

$$\left(T_{r,n}^{[m]}(f; x)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} \right) + 0\left(\xi_n^{m-\gamma^*}\right). \quad (139)$$

When $m = 1$, the sum collapses.

Proof. Use of Theorem 17 and Theorem 4.6, p. 54 of [3]. Here it is $\rho = \varphi$, see (68). ■

3.5 Simultaneous Approximation by multivariate complex $T_{r,n}^{[m]}$

We make

Remark 52 *We consider here complex valued Borel measurable functions $f : \mathbb{R}^N \to \mathbb{C}$ such that $f = f_1 + if_2$, $i = \sqrt{-1}$, where $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are implied to be real valued Borel measurable functions.*

We define the multivariate complex Trigonometric singular operators

$$T_{r,n}^{[m]}(f; x) := T_{r,n}^{[m]}(f_1; x) + iT_{r,n}^{[m]}(f_2; x), \quad x \in \mathbb{R}^N. \quad (140)$$

We assume that $T_{r,n}^{[m]}(f_j; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$, $j = 1, 2$.

One notices easily that

$$\left|T_{r,n}^{[m]}(f; x) - f(x)\right| \leq \left|T_{r,n}^{[m]}(f_1; x) - f_1(x)\right| + \left|T_{r,n}^{[m]}(f_2; x) - f_2(x)\right| \quad (141)$$

also

$$\left\|T_{r,n}^{[m]}(f; x) - f(x)\right\|_{\infty,x} \leq \left\|T_{r,n}^{[m]}(f_1; x) - f_1(x)\right\|_{\infty,x} + \left\|T_{r,n}^{[m]}(f_2; x) - f_2(x)\right\|_{\infty,x} \quad (142)$$

and

$$\left\|T_{r,n}^{[m]}(f) - f\right\|_p \leq \left\|T_{r,n}^{[m]}(f_1) - f_1\right\|_p + \left\|T_{r,n}^{[m]}(f_2) - f_2\right\|_p, \quad p \geq 1. \quad (143)$$

Furthermore, it holds

$$f_{\overline{\alpha}}(x) = f_{1,\overline{\alpha}}(x) + if_{2,\overline{\alpha}}(x), \quad (144)$$

where $\overline{\alpha}$ denotes a partial derivative of any order and arrangement.

We give

Theorem 53 *Let* $f : \mathbb{R}^N \rightarrow \mathbb{C}$*, such that* $f = f_1 + if_2$*,* $j = 1, 2$*. Here* $r, N, \beta, m \in \mathbb{N}$*,* $f_j \in C^m\left(\mathbb{R}^N\right)$*,* $x \in \mathbb{R}^N$*. Assume* $\left\|\frac{\partial^m f_j(\cdot,\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}\right\|_\infty < \infty$*, for all* $\alpha_i \in \mathbb{Z}^+$*,* $i = 1, \ldots, N : |\overline{\alpha}| := \sum_{i=1}^N \alpha_i = m$*. Let* φ_{ξ_n} *be the Borel probability measure on* $\mathbb{R}^N$*, see (39), where* $\xi_n \in (0, 1]$*,* $n \in \mathbb{N}$*. Here* $\beta > \frac{m+r+1}{2}$*, and* $A_{\xi_n}(\overline{\alpha})$ *as in (47), and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

$$\left\| T_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{\infty,x}$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{\left(\omega_r\left(f_{1,\overline{\alpha}}, \xi_n\right) + \omega_r\left(f_{2,\overline{\alpha}}, \xi_n\right)\right)}{\left(\prod_{i=1}^N \alpha_i!\right)} A_{\xi_n}(\overline{\alpha}). \tag{145}$$

Proof. By Theorem 20. ■

We proceed with

Theorem 54 *Let* $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$*,* $N \in \mathbb{N} - \{1\}$*,* $j = 1, 2$*. Here* $f_j \in C_B\left(\mathbb{R}^N\right)$ *uniformly continuous,* $\beta, r \in \mathbb{N}$*,* $\beta > \frac{r+1}{2}$*,* $\xi_n \in (0, 1]$*,* $n \in \mathbb{N}$*. Then*

$$\left\| T_{r,n}^{[0]} f - f \right\|_\infty \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \tag{146}$$

$$\left(\omega_r\left(f_1, \xi_n\right) + \omega_r\left(f_2, \xi_n\right)\right),$$

As $n \rightarrow \infty$ *and* $\xi_n \rightarrow 0$*, we derive*

$$\left\| T_{r,n}^{[0]} f - f \right\|_\infty \rightarrow 0 \tag{147}$$

with rates.

Proof. By Theorem 22. ■

Next comes multi-simultaneous approximation.

Theorem 55 *Let* $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$*,* $j = 1, 2$*. Here* $f_j \in C^{m+l}\left(\mathbb{R}^N\right)$*,* $N, m, l \in \mathbb{N}$*. The assumptions of Theorem 37 are valid for* f_j *and* $d\mu_{\xi_n} = d\varphi_{\xi_n}$*. Call* $\overline{\gamma} = 0, \overline{\beta}$*. Assume* $\left\|f_{j,\overline{\gamma}+\overline{\alpha}}\right\|_\infty < \infty$*, and let* $\xi_n \in (0, 1]$*,* $n \in \mathbb{N}$*. Here* $\beta, r \in \mathbb{N}$*,* $\beta > \frac{m+r+1}{2}$*, and* $A_{\xi_n}(\overline{\alpha})$ *as in (47), and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

$$\left\| \left(T_{r,n}^{[m]}(f; \cdot)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(\cdot) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(\cdot)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_\infty \tag{148}$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N\geq 0;\\ |\overline{\alpha}|=m}} \frac{\left(\omega_r\left(f_{1,\overline{\gamma}+\overline{\alpha}},\xi_n\right)+\omega_r\left(f_{2,\overline{\gamma}+\overline{\alpha}},\xi_n\right)\right)}{\left(\prod_{i=1}^{N}\alpha_i!\right)} A_{\xi_n}\left(\overline{\alpha}\right).$$

Proof. Based on Theorems 37, 41. ■

We continue with

Theorem 56 *Let $f:\mathbb{R}^N\to\mathbb{C}: f=f_1+if_2$, $j=1,2$. Here $f_j\in C_B^l\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}-\{1\}$, $l\in\mathbb{N}$ (functions l-times continuously differentiable and bounded). The assumptions of Theorem 37 are valid for f_j and $d\mu_{\xi_n}=d\varphi_{\xi_n}$. Call $\overline{\gamma}=0,\overline{\beta}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Let also $\beta,r\in\mathbb{N}$, $\beta>\frac{r+1}{2}$. Then*

$$\left\|\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}}-f_{\overline{\gamma}}\right\|_{\infty}\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N$$

$$\xi_n^{2\beta(N-1)}\left(\omega_r\left(f_{1,\overline{\gamma}},\xi_n\right)+\omega_r\left(f_{2,\overline{\gamma}},\xi_n\right)\right). \quad (149)$$

If $\xi_n\to 0$, as $n\to\infty$, then $\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}}\to f_{\overline{\gamma}}$ uniformly.

Proof. By Theorems 42 and 37. ■

We proceed with L_p approximations

Theorem 57 *Let $f:\mathbb{R}^N\to\mathbb{C}: f=f_1+if_2$, $j=1,2$. Here $f_j\in C^m\left(\mathbb{R}^N\right)$, $r,\beta,N,m\in\mathbb{N}$, with $f_{j,\overline{\alpha}}\in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Here φ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ as in (39), for $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Let $\beta>\frac{\lceil rp\rceil+m+1}{2}$; $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N:|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=m$. Here $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$. Then*

$$\left\|T_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right)\right\|_{p,x}$$

$$\leq\left(\frac{m}{\left(q(m-1)+1\right)^{\frac{1}{q}}}\right)2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left(\psi_\lambda+\psi_{\lambda+mp}\right)\right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\xi_n^{\left(\frac{2\beta(N-1)}{p}+m\right)}\left[\omega_r\left(f_{1,\overline{\alpha}},\xi_n\right)_p+\omega_r\left(f_{2,\overline{\alpha}},\xi_n\right)_p\right]. \quad (150)$$

Proof. By Theorem 25. ■

We continue with

Theorem 58 *Let* $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. *Here* $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $\beta,r\in\mathbb{N}$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $\beta>\frac{\lceil rp\rceil+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. *Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_p\leq 2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left[\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\psi_\lambda\right]^{\frac{N}{p}}$$

$$\xi_n^{\frac{2\beta(N-1)}{p}}\left[\omega_r\left(f_1,\xi_n\right)_p+\omega_r\left(f_2,\xi_n\right)_p\right]. \tag{151}$$

As $\xi_n\to0$, *when* $n\to\infty$, *we derive* $\left\|T_{r,n}^{[0]}f-f\right\|_p\to0$, *i.e.* $T_{r,n}^{[0]}\to I$, *the unit operator, in* L_p *norm.*

Proof. By Theorem 27. ■

We also give

Theorem 59 *Let* $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. *Here* $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $r,\beta\in\mathbb{N}$, $\beta>\frac{r+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. *Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_1\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N$$

$$\xi_n^{2\beta(N-1)}\left(\omega_r\left(f_1,\xi_n\right)_1+\omega_r\left(f_2,\xi_n\right)_1\right). \tag{152}$$

As $\xi_n\to0$, *we get* $T_{r,n}^{[0]}\to I$, *in* L_1 *norm.*

Proof. By Theorem 29. ■

We further present

Theorem 60 *Let* $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. *Here* $f_j\in C^m\left(\mathbb{R}^N\right)$, $N,\beta,m,r\in\mathbb{N}$, *with* $f_{j,\overline{\alpha}}\in L_1\left(\mathbb{R}^N\right)$, *where* $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^N\alpha_i=m$, $x\in\mathbb{R}^N$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$ *and* $\beta>\frac{m+r+1}{2}$. *Here* $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j}=1,...,m$, *and* $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^N\alpha_i=\widetilde{j}$. *Also here* φ_{ξ_n} *is the Borel probability measure on* $\mathbb{R}^N$, *see (39). Then*

$$\left\|T_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\prod_{i=1}^N\alpha_i!}\right)\right\|_{1,x} \tag{153}$$

$$\leq \left\{ \sum_{|\overline{\alpha}|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left[\omega_r \left(f_{1,\overline{\alpha}}, \xi_n \right)_1 + \omega_r \left(f_{2,\overline{\alpha}}, \xi_n \right)_1 \right] \right\} \xi_n^{2\beta(N-1)+m}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m} \right] \right\}^N .$$

Proof. By Theorem 31. ■

We continue with simultaneous L_p approximations.

Theorem 61 *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + i f_2$, $j = 1, 2$. Here $f_j \in C^{m+l}\left(\mathbb{R}^N\right)$, $r, \beta, N, m, l \in \mathbb{N}$. The assumptions of Theorem 37 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$ and f_j. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{j,(\overline{\gamma}+\overline{\alpha})} \in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\beta > \frac{\lceil rp \rceil + m + 1}{2}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N :$ $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$. Here $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. Then*

$$\left\| \left(T_{r,n}^{[m]}(f;x) \right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{\left(q(m-1)+1\right)^{\frac{1}{q}}} \right) 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\psi_\lambda + \psi_{\lambda+mp} \right) \right\}^{\frac{N}{p}} \tag{154}$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \xi_n^{\left(\frac{2\beta(N-1)}{p} + m \right)} \left[\omega_r \left(f_{1,\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_p + \omega_r \left(f_{2,\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_p \right].$$

Proof. By Theorems 37 and 43. ■

We give also

Theorem 62 *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + i f_2$, $j = 1, 2$. Here $f_j \in C^{l}\left(\mathbb{R}^N\right)$, $N \in \mathbb{N} - \{1\}$, $l \in \mathbb{N}$. The assumptions of Theorem 37 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$ and f_j. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{j,\overline{\gamma}} \in L_p\left(\mathbb{R}^N\right)$ and $p, q > 1 :$ $\frac{1}{p} + \frac{1}{q} = 1$. Here $\beta, r \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + 1}{2}$. Then*

$$\left\| \left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_p \leq 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^{\frac{N}{p}}$$

$$\xi_n^{\frac{2\beta(N-1)}{p}}\left[\omega_r\left(f_{1,\overline{\gamma}},\xi_n\right)_p+\omega_r\left(f_{2,\overline{\gamma}},\xi_n\right)_p\right]. \tag{155}$$

As $n\to+\infty$ and $\xi_n\to 0$, then $\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}}\overset{\|\cdot\|_p}{\to} f_{\overline{\gamma}}$.

Proof. By Theorems 37 and 44. ■

We continue with

Theorem 63 *Let $f:\mathbb{R}^N\to\mathbb{C}: f=f_1+if_2$, $j=1,2$. Here $f_j\in C^l\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}-\{1\}$, $l\in\mathbb{N}$. The assumptions of Theorem 37 are valid for $d\mu_{\xi_n}=d\varphi_{\xi_n}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$ and f_j. Call $\overline{\gamma}=0,\overline{\beta}$. Let $f_{j,\overline{\gamma}}\in L_1\left(\mathbb{R}^N\right)$ and $\beta,r\in\mathbb{N}$, $\beta>\frac{r+1}{2}$. Then*

$$\left\|\left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}}-f_{\overline{\gamma}}\right\|_1\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N$$

$$\xi_n^{2\beta(N-1)}\left[\omega_r\left(f_{1,\overline{\gamma}},\xi_n\right)_1+\omega_r\left(f_{2,\overline{\gamma}},\xi_n\right)_1\right]. \tag{156}$$

As $n\to+\infty$ and $\xi_n\to 0$, then $\left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}}\overset{\|\cdot\|_1}{\to} f_{\overline{\gamma}}$.

Proof. By Theorems 37, 45. ■

We finish with

Theorem 64 *Let $f:\mathbb{R}^N\to\mathbb{C}: f=f_1+if_2$, $j=1,2$. Here $f_j\in C^{m+l}\left(\mathbb{R}^N\right)$, $N,\beta.r,m,l\in\mathbb{N}$. The assumptions of Theorem 37 are valid for $d\mu_{\xi_n}=d\varphi_{\xi_n}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$ and f_j. Call $\overline{\gamma}=0,\overline{\beta}$. Let $f_{j,(\overline{\gamma}+\overline{\alpha})}\in L_1\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. Here $\beta>\frac{m+r+1}{2}$ and $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$. Then*

$$\left\|\left(T_{r,n}^{[m]}(f;x)\right)_{\overline{\gamma}}-f_{\overline{\gamma}}(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N}\alpha_i!}\right)\right\|_{1,x}$$

$$\leq\left(\sum_{|\overline{\alpha}|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\omega_r\left(f_{1,\overline{\gamma}+\overline{\alpha}},\xi_n\right)_1+\omega_r\left(f_{2,\overline{\gamma}+\overline{\alpha}},\xi_n\right)_1\right]\right)$$

$$\xi_n^{2\beta(N-1)+m}2^N\gamma^{-N}\left\{\sum_{\lambda=0}^{r}\binom{r}{\lambda}\left[\psi_\lambda+\psi_{\lambda+m}\right]\right\}^N. \tag{157}$$

Proof. By Theorems 37, 46. ■

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