LOWER AND UPPER BOUNDS FOR JENSEN'S GAP FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. We establish in this paper some lower and upper bounds for Jensen's gap in the general setting of Hermitian unital Banach $*$-algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [9] (see also [1, Theorem 41.5]), then

$$a^* a \geq 0 \text{ for every } a \in A. \quad \text{(SF)}$$

Based on this fact, Okayasu [8], Tanahashi and Uchiyama [10] proved the following fundamental properties (see also [6]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [8] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

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In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} dz.$$

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [6], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [10, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;

(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [4].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

In the recent paper [5] we obtained between others the following results:

Theorem 1. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex (in the usual sense) on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$, we have the Jensen type inequalities*

$$0 \le \psi(f(c)) - f(\psi(c)) \le \psi(cf'(c)) - \psi(c)\psi(f'(c)). \tag{1.1}$$

Motivated by the above results, we establish in this paper some new lower and upper bounds for Jensen's gap in the general setting of Hermitian unital Banach $*$-algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

2. Functional Properties

We denote by $\mathfrak{P}_1(A)$ the set of all linear, positive functionals defined on A with the property that, if $\varphi \in \mathfrak{P}_1(A)$, then $\varphi(1) > 0$. If $\varphi, \omega \in \mathfrak{P}_1(A)$ then $\varphi + \omega \in \mathfrak{P}_1(A)$ and for all $\alpha > 0$ we have $\alpha\varphi \in \mathfrak{P}_1(A)$.

We define the order relation "$\succ$"on $\mathfrak{P}_1(A)$ by $\varphi \succ \omega$ iff $\varphi - \omega \in \mathfrak{P}_1(A)$.

Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex on the interval I and $\varphi \in \mathfrak{P}_1(A)$, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$, we have by (1.1) that

$$f\left(\frac{\varphi(c)}{\varphi(1)}\right) \le \frac{\varphi(f(c))}{\varphi(1)}. \tag{2.1}$$

With the above assumptions for f and c, we define the functional $\mathcal{J}(f, c, \cdot) : \mathfrak{P}_1(A) \to [0, \infty)$ by

$$\mathcal{J}(f, c, \varphi) := \varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right) \ge 0.$$

We have

Theorem 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that f is convex on the interval I and $c \in A$ is a selfadjoint element with $\sigma(c) \subset I$.*

(i) *If $\varphi, \omega \in \mathfrak{P}_1(A)$, then*

$$\mathcal{J}(f, c, \varphi + \omega) \ge \mathcal{J}(f, c, \varphi) + \mathcal{J}(f, c, \omega) \ge 0, \tag{2.2}$$

namely $\mathcal{J}(f, c, \cdot)$ is superadditive on $\mathfrak{P}_1(A)$.

(ii) *If $\varphi, \omega \in \mathfrak{P}_1(A)$ with $\varphi \succ \omega$, then*

$$\mathcal{J}(f, c, \varphi) \ge \mathcal{J}(f, c, \omega) \ge 0, \tag{2.3}$$

namely $\mathcal{J}(f, c, \cdot)$ is monotonic nondecreasing on $\mathfrak{P}_1(A)$.

Proof. (i) If $\varphi, \omega \in \mathfrak{P}_1(A)$, then by the convexity of f we have

$$\begin{aligned}&(\varphi + \omega)(1) f\left(\frac{(\varphi + \omega)(c)}{(\varphi + \omega)(1)}\right) \\ &= (\varphi(1) + \omega(1)) f\left(\frac{\varphi(c) + \omega(c)}{\varphi(1) + \omega(1)}\right)\end{aligned}$$

$$= (\varphi(1) + \omega(1)) f\left(\frac{\varphi(1)\frac{\varphi(c)}{\varphi(1)} + \omega(1)\frac{\omega(c)}{\omega(1)}}{\varphi(1) + \omega(1)}\right)$$
$$\leq (\varphi(1) + \omega(1)) \frac{\varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right) + \omega(1) f\left(\frac{\omega(c)}{\omega(1)}\right)}{\varphi(1) + \omega(1)}$$
$$= \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right) + \omega(1) f\left(\frac{\omega(c)}{\omega(1)}\right).$$

Therefore

$$\begin{aligned}\mathcal{J}(f, c, \varphi + \omega) &= (\varphi + \omega)(f(c)) - (\varphi + \omega)(1) f\left(\frac{(\varphi + \omega)(c)}{(\varphi + \omega)(1)}\right) \\ &\geq \varphi(f(c)) + w(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right) - \omega(1) f\left(\frac{\omega(c)}{\omega(1)}\right) \\ &= \varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right) + w(f(c)) - \omega(1) f\left(\frac{\omega(c)}{\omega(1)}\right) \\ &= \mathcal{J}(f, c, \varphi) + \mathcal{J}(f, c, \omega)\end{aligned}$$

and the inequality (2.2) is proved.

(ii) If $\varphi \succ \omega$, then $\chi := \varphi - \omega \in \mathfrak{P}_1(A)$. By (2.2) we get

$$\mathcal{J}(f, c, \varphi) = \mathcal{J}(f, c, \chi + \omega) \geq \mathcal{J}(f, c, \chi) + \mathcal{J}(f, c, \omega),$$

which implies that

$$\mathcal{J}(f, c, \varphi) - \mathcal{J}(f, c, \omega) \geq \mathcal{J}(f, c, \chi) = \mathcal{J}(f, c, \varphi - \omega) \geq 0$$

and the inequality (2.3) is proved. □

Corollary 1. *With the assumptions of Theorem 2 for f and c and if there exists the constants $0 < m < M < \infty$ such that $M\omega \succ \varphi \succ m\omega$, then*

$$M\mathcal{J}(f, c, \omega) \geq \mathcal{J}(f, c, \varphi) \geq m\mathcal{J}(f, c, \omega) \geq 0. \tag{2.4}$$

Proof. From (2.3) we get

$$\mathcal{J}(f, c, M\omega) \geq \mathcal{J}(f, c, \varphi) \geq \mathcal{J}(f, c, m\omega)$$

and since $\mathcal{J}(f, c, M\omega) = M\mathcal{J}(f, c, \omega)$ and $\mathcal{J}(f, c, m\omega) = m\mathcal{J}(f, c, \omega)$, hence we obtain (2.4). □

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;
(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is called *superadditive* on C if

(iii) $h(x + y) \geq h(x) + h(y)$ for any $x, y \in C$

and *nonnegative (strictly positive)* on C if, it satisfies

(iv) $h(x) \geq (>) 0$ for each $x \in C$.

The functional h is *s-positive homogeneous* on C, for a given $s > 0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

In the paper [3] we obtained the following result for superadditive functions:

Lemma 2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $p \geq 1$ then the composite functional*

$$\Psi_p : C \to [0, \infty), \Psi_p(x) = v^{1-\frac{1}{p}}(x) h(x) \tag{2.5}$$

is superadditive on C.

Corollary 2. *Assume that X, C and v are as in Theorem 2. If $h : C \to [0, \infty)$ is a superadditive functional on C and p, $q \geq 1$ then the two parameters functional*

$$\Psi_{p,q} : C \to [0, \infty), \Psi_{p,q}(x) = v^{q\left(1-\frac{1}{p}\right)}(x) h^q(x) \tag{2.6}$$

is superadditive on C.

Remark 1. *If we consider the functional $\psi_p(x) := v^{p-1}(x) h^p(x)$ then for $p \geq 1$ and $h : C \to [0, \infty)$ a superadditive functional on C, the functional ψ_p is also superadditive on C.*

Corollary 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that f is convex on the interval I and $c \in A$ is a selfadjoint element with $\sigma(c) \subset I$. For p, $q \geq 1$ we define the functional $\mathcal{J}_{p,q}(f, c, \cdot) : \mathfrak{P}_1(A) \to [0, \infty)$ by*

$$\mathcal{J}_{p,q}(f, c, \varphi) := [\varphi(1)]^{q\left(1-\frac{1}{p}\right)} \left[\varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right)\right]^q. \tag{2.7}$$

Then $\mathcal{J}_{p,q}(f, c, \cdot)$ is superadditive on $\mathfrak{P}_1(A)$.

The proof follows by Corollary 2 by choosing

$$v(\varphi) = \varphi(1) \text{ and } h(\varphi) = \mathcal{J}(f, c, \varphi) = \varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right).$$

We also observe that for $p \geq 1$

$$\mathcal{J}_p(f, c, \varphi) := [\varphi(1)]^{p-1} \left[\varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right)\right]^p \tag{2.8}$$

and

$$\mathcal{J}_{p,1}(f, c, \varphi) := [\varphi(1)]^{1-\frac{1}{p}} \left[\varphi(f(c)) - \varphi(1) f\left(\frac{\varphi(c)}{\varphi(1)}\right)\right] \tag{2.9}$$

are *superadditive* on $\mathfrak{P}_1(A)$.

3. Lower and Upper Bounds

We have:

Theorem 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$ and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A. For any selfadjoint element $c \in A$ with $\sigma(c) \subset I$,*

$$\begin{aligned} 0 &\leq \left(\inf_{z \in \mathring{I}} \int_0^1 f''((1-s)t + sz)(1-s) ds\right) \psi\left[(c-t)^2\right] \\ &\leq \psi(f(c)) - f'(t)(\psi(c) - t) - f(t) \\ &\leq \left(\sup_{z \in \mathring{I}} \int_0^1 f''((1-s)t + sz)(1-s) ds\right) \psi\left[(c-t)^2\right] \end{aligned} \tag{3.1}$$

for all $t \in \mathring{I}$.

In particular, we have

$$0 \leq \left(\inf_{z\in\mathring{I}} \int_0^1 f''\left((1-s)\psi(c)+sz\right)(1-s)\,ds\right)\left[\psi\left(c^2\right)-\left(\psi(c)\right)^2\right] \tag{3.2}$$
$$\leq \psi\left(f(c)\right)-f\left(\psi(c)\right)$$
$$\leq \left(\sup_{z\in\mathring{I}} \int_0^1 f''\left((1-s)\psi(c)+sz\right)(1-s)\,ds\right)\left[\psi\left(c^2\right)-\left(\psi(c)\right)^2\right].$$

Proof. Using Taylor's representation with the integral remainder we can write the following identity

$$f(z)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(t)(z-t)^k+\frac{1}{n!}\int_t^z f^{(n+1)}(s)(z-s)^n\,ds \tag{3.3}$$

for any z, $t \in \mathring{I}$, the interior of I.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $s=(1-s)c+sd$, $s\in[0,1]$ that

$$\int_c^d h(s)\,ds=(d-c)\int_0^1 h\left((1-s)c+sd\right)ds.$$

Therefore,

$$\int_t^z f^{(n+1)}(s)(z-s)^n\,ds$$
$$=(z-t)\int_0^1 f^{(n+1)}\left((1-s)t+sz\right)\left(z-(1-s)t-sz\right)^n ds$$
$$=(z-t)^{n+1}\int_0^1 f^{(n+1)}\left((1-s)t+sz\right)(1-s)^n\,ds.$$

The identity (3.3) can then be written as

$$f(z)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(t)(z-t)^k \tag{3.4}$$
$$+\frac{1}{n!}(z-t)^{n+1}\int_0^1 f^{(n+1)}\left((1-s)t+sz\right)(1-s)^n\,ds.$$

For $n=1$ we get

$$f(z)=f(t)+(z-t)f'(t)+(z-t)^2\int_0^1 f''\left((1-s)t+sz\right)(1-s)\,ds \tag{3.5}$$

for any z, $t\in\mathring{I}$.

Since

$$0\leq \inf_{z\in\mathring{I}}\int_0^1 f''\left((1-s)t+sz\right)(1-s)\,ds\leq\int_0^1 f''\left((1-s)t+sz\right)(1-s)\,ds$$
$$\leq \sup_{z\in\mathring{I}}\int_0^1 f''\left((1-s)t+sz\right)(1-s)\,ds,$$

hence

$$
\begin{aligned}
0 &\leq \left(\inf_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)t+sz\right)\left(1-s\right)ds \right)\left(z-t\right)^2 \\
&\leq f\left(z\right)-f\left(t\right)-\left(z-t\right)f'\left(t\right) \\
&\leq \left(\sup_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)t+sz\right)\left(1-s\right)ds \right)\left(z-t\right)^2 ,
\end{aligned} \tag{3.6}
$$

for any $z,\ t\in \mathring{I}$.

Fix $t\in I$. Using Lemma 1 and the inequality (3.7) we obtain for the element $c\in A$ with $\sigma\left(c\right)\subset I$ the following inequality in the order of A

$$
\begin{aligned}
0 &\leq \left(\inf_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)t+sz\right)\left(1-s\right)ds \right)\left(c-t\right)^2 \\
&\leq f\left(c\right)-f\left(t\right)-\left(c-t\right)f'\left(t\right) \\
&\leq \left(\sup_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)t+sz\right)\left(1-s\right)ds \right)\left(c-t\right)^2 ,
\end{aligned}
$$

for any $t\in \mathring{I}$.

If we take in this inequality the functional ψ we get (3.1).

If we take in (3.1) $t=\psi\left(c\right)$, then we get

$$
\begin{aligned}
0 &\leq \left(\inf_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)\psi\left(c\right)+sz\right)\left(1-s\right)ds \right)\psi\left[\left(c-\psi\left(c\right)\right)^2\right] \\
&\leq \psi\left(f\left(c\right)\right)-f\left(\psi\left(c\right)\right) \\
&\leq \left(\sup_{z\in \mathring{I}} \int_0^1 f''\left(\left(1-s\right)\psi\left(c\right)+sz\right)\left(1-s\right)ds \right)\psi\left[\left(c-\psi\left(c\right)\right)^2\right].
\end{aligned} \tag{3.7}
$$

Since

$$
\begin{aligned}
\psi\left[\left(c-\psi\left(c\right)\right)^2\right] &= \psi\left(c^2-2\psi\left(c\right)c+\left(\psi\left(c\right)\right)^2\right) \\
&= \psi\left(c^2\right)-2\left(\psi\left(c\right)\right)^2+\left(\psi\left(c\right)\right)^2 = \psi\left(c^2\right)-\left(\psi\left(c\right)\right)^2 ,
\end{aligned}
$$

hence by (3.7) we get (3.2). □

Corollary 4. *With the assumptions of Theorem 3 and, if, in addition,* $\psi(f'(c)) \neq 0$ *with* $t = \frac{\psi(cf'(c))}{\psi(f'(c))} \in \mathring{I}$, *then we have*

$$0 \leq \left(\inf_{z \in \mathring{I}} \int_0^1 f''\left((1-s) \frac{\psi(cf'(c))}{\psi(f'(c))} + sz \right) (1-s)\, ds \right) \times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right] \leq \psi(f(c)) - f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))} \right) \left(\psi(c) - \frac{\psi(cf'(c))}{\psi(f'(c))} \right) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))} \right) \leq \left(\sup_{z \in \mathring{I}} \int_0^1 f''\left((1-s) \frac{\psi(cf'(c))}{\psi(f'(c))} + sz \right) (1-s)\, ds \right) \times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right]. \tag{3.8}$$

Corollary 5. *For any selfadjoint element* $c \in A$ *with* $\sigma(c) \subseteq [m, M] \subset I$, *we have*

$$0 \leq \left(\inf_{z \in \mathring{I}} \int_0^1 f''\left((1-s) \frac{m+M}{2} + sz \right) (1-s)\, ds \right) \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \leq \psi(f(c)) - f'\left(\frac{m+M}{2} \right) \left(\psi(c) - \frac{m+M}{2} \right) - f\left(\frac{m+M}{2} \right) \leq \left(\sup_{z \in \mathring{I}} \int_0^1 f''\left((1-s) \frac{m+M}{2} + sz \right) (1-s)\, ds \right) \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \leq \frac{1}{4}(M-m)^2 \left(\sup_{z \in \mathring{I}} \int_0^1 f''\left((1-s) \frac{m+M}{2} + sz \right) (1-s)\, ds \right). \tag{3.9}$$

We also have:

Theorem 4. *Let* $f(z)$ *be analytic in* G, *an open subset of* $\mathbb{C}$ *and convex on the real interval* $I \subset G$ *and* $\psi : A \to \mathbb{C}$ *is a positive normalized linear functional on* A. *For any selfadjoint element* $c \in A$ *with* $\sigma(c) \subseteq [m, M] \subset I$, *we have*

$$0 \leq \frac{1}{2} \inf_{(z,t) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)t + sz)(1-s)\, ds \times \left\{ \frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\} \leq \frac{1}{2}\left[\psi(f(c)) + \frac{(M - \psi(c)) f(M) + \psi(c) f(m)}{M-m} \right] - \frac{1}{M-m} \int_m^M f(t)\, dt \leq \frac{1}{2} \sup_{(z,t) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)t + sz)(1-s)\, ds \times \left\{ \frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\}. \tag{3.10}$$

Proof. We have, by (3.6), on taking the integral mean over $t \in [m, M]$ that

$$0 \leq \frac{1}{M-m}\int_m^M \left(\inf_{z\in \mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)(z-t)^2\,dt \tag{3.11}$$
$$\leq f(z) - \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{1}{M-m}\int_m^M (z-t)f'(t)\,dt$$
$$\leq \frac{1}{M-m}\int_m^M \left(\sup_{z\in \mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)(z-t)^2\,dt$$

for all $z \in \mathring{I}$.

Now, observe that

$$\left(\inf_{(z,t)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)\frac{1}{M-m}\int_m^M (z-t)^2\,dt$$
$$\leq \frac{1}{M-m}\int_m^M \left(\inf_{z\in \mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)(z-t)^2\,dt$$

and

$$\frac{1}{M-m}\int_m^M \left(\sup_{z\in \mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)(z-t)^2\,dt$$
$$\leq \left(\sup_{(z,t)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds\right)\frac{1}{M-m}\int_m^M (z-t)^2\,dt$$

for all $z \in \mathring{I}$.

Also,

$$\frac{1}{M-m}\int_m^M (z-t)^2\,dt = \frac{(M-z)^3+(z-m)^3}{3(M-m)}$$
$$= \frac{1}{3}\left[(z-m)^2+(M-z)^2-(z-m)(M-z)\right]$$
$$= \frac{1}{3}\left[\frac{1}{4}(M-m)^2+3\left(z-\frac{m+M}{2}\right)^2\right]$$
$$= \frac{1}{12}(M-m)^2+\left(z-\frac{m+M}{2}\right)^2$$

and

$$\frac{1}{M-m}\int_m^M (z-t)f'(t)\,dt$$
$$= \frac{1}{M-m}\left[(z-t)f(t)\big|_m^M+\int_m^M f(t)\,dt\right]$$
$$= \frac{1}{M-m}\left[\int_m^M f(t)\,dt-(M-z)f(M)-(z-m)f(m)\right]$$
$$= \frac{1}{M-m}\int_m^M f(t)\,dt-\frac{(M-z)f(M)+(z-m)f(m)}{M-m}$$

for all $z \in [m, M]$.

By (3.11) we then get

$$\begin{aligned}
0 &\leq \inf_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(z-\frac{m+M}{2}\right)^2\right] \\
&\leq f(z) - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&- \frac{1}{M-m}\int_m^M f(t)\,dt + \frac{(M-z)f(M)+(z-m)f(m)}{M-m} \\
&\leq \sup_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(z-\frac{m+M}{2}\right)^2\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &\leq \frac{1}{2}\inf_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(z-\frac{m+M}{2}\right)^2\right] \\
&\leq \frac{1}{2}\left[f(z) + \frac{(M-z)f(M)+(z-m)f(m)}{M-m}\right] - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&\leq \frac{1}{2}\sup_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(z-\frac{m+M}{2}\right)^2\right],
\end{aligned} \tag{3.12}$$

for $z \in [m, M]$.

Using Lemma 1 and the inequality (3.12) we obtain for the element $c \in A$ with $\sigma(c) \subset [m, M]$ the following inequality in the order of A

$$\begin{aligned}
0 &\leq \frac{1}{2}\inf_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(c-\frac{m+M}{2}\right)^2\right] \\
&\leq \frac{1}{2}\left[f(c) + \frac{(M-c)f(M)+(c-m)f(m)}{M-m}\right] - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&\leq \frac{1}{2}\sup_{(z,t)\in\mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \\
&\times \left[\frac{1}{12}(M-m)^2 + \left(c-\frac{m+M}{2}\right)^2\right].
\end{aligned} \tag{3.13}$$

If we take the functional ψ on (3.13), then we get (3.10). □

Corollary 6. *With the assumptions of Theorem 4 we have*

$$0 \le \frac{1}{24}(M-m)^2 \inf_{(z,t)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds \tag{3.14}$$
$$\le \frac{1}{2}\left[\psi(f(c)) + \frac{(M-\psi(c))f(M)+\psi(c)f(m)}{M-m}\right]$$
$$-\frac{1}{M-m}\int_m^M f(t)\,dt$$
$$\le \frac{1}{6}(M-m)^2 \sup_{(z,t)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)t+sz)(1-s)\,ds.$$

The proof follows by (3.10) on observing that

$$0 \le \psi\left[\left(c-\frac{m+M}{2}\right)^2\right] \le \frac{1}{4}(M-m)^2.$$

Remark 2. *If there exists the constants $0<\gamma<\Gamma<\infty$ such that $\gamma \le f''(x) \le \Gamma$ for almost every $x \in \mathring{I}$, then by (3.1) and (3.2) we get*

$$0 \le \frac{1}{2}\gamma\psi\left[(c-t)^2\right] \le \psi(f(c)) - f'(t)(\psi(c)-t) - f(t) \le \frac{1}{2}\Gamma\psi\left[(c-t)^2\right] \tag{3.15}$$

for all $t \in \mathring{I}$.

In particular, we have

$$0 \le \frac{1}{2}\gamma\left[\psi\left(c^2\right)-(\psi(c))^2\right] \le \psi(f(c)) - f(\psi(c)) \le \frac{1}{2}\Gamma\left[\psi\left(c^2\right)-(\psi(c))^2\right]. \tag{3.16}$$

From (3.8) we get

$$0 \le \frac{1}{2}\gamma\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \tag{3.17}$$
$$\le \psi(f(c)) - f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)\left(\psi(c)-\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)$$
$$\le \frac{1}{2}\Gamma\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right],$$

while from (3.9) we get

$$0 \le \frac{1}{2}\gamma\psi\left[\left(c-\frac{m+M}{2}\right)^2\right] \tag{3.18}$$
$$\le \psi(f(c)) - f'\left(\frac{m+M}{2}\right)\left(\psi(c)-\frac{m+M}{2}\right) - f\left(\frac{m+M}{2}\right)$$
$$\le \frac{1}{2}\Gamma\psi\left[\left(c-\frac{m+M}{2}\right)^2\right] \le \frac{1}{8}(M-m)^2\Gamma.$$

From (3.10) we obtain

$$0 \leq \frac{1}{4}\gamma\left\{\frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2}\right)^2\right]\right\} \tag{3.19}$$
$$\leq \frac{1}{2}\left[\psi(f(c)) + \frac{(M-\psi(c))f(M) + \psi(c)f(m)}{M-m}\right]$$
$$- \frac{1}{M-m}\int_m^M f(t)\,dt$$
$$\leq \frac{1}{4}\Gamma\left\{\frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2}\right)^2\right]\right\},$$

while from (3.14) we have

$$0 \leq \frac{1}{48}\gamma(M-m)^2 \tag{3.20}$$
$$\leq \frac{1}{2}\left[\psi(f(c)) + \frac{(M-\psi(c))f(M) + \psi(c)f(m)}{M-m}\right]$$
$$- \frac{1}{M-m}\int_m^M f(t)\,dt$$
$$\leq \frac{1}{12}\Gamma(M-m)^2.$$

We observe that from (3.17) we get the simpler reverse of Slater's inequality

$$(0 \leq)\, f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c)) \leq f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)\left(\frac{\psi(cf'(c))}{\psi(f'(c))} - \psi(c)\right). \tag{3.21}$$

Theorem 5. *Let f be analytic in G, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$, $\psi : A \to \mathbb{C}$ be a positive normalized linear functional on A and $c \in A$ a selfadjoint element with $\sigma(c) \subset I$. If f'' is monotonic nondecreasing on $[m, M] \subset \mathring{I}$, then*

$$0 \leq \frac{1}{t-m}\left\{f'(t) - \frac{f(t)-f(m)}{t-m}\right\}\psi\left[(c-t)^2\right] \tag{3.22}$$
$$\leq \psi(f(c)) - f(t) - (\psi(c)-t)f'(t)$$
$$\leq \frac{1}{M-t}\left\{\frac{f(M)-f(t)}{M-t} - f'(t)\right\}\psi\left[(c-t)^2\right]$$

for $t \in (m, M)$.

If f'' is monotonic nonincreasing on $[m, M] \subset \mathring{I}$, then

$$0 \leq \frac{1}{M-t}\left\{\frac{f(M)-f(t)}{M-t} - f'(t)\right\}\psi\left[(c-t)^2\right] \tag{3.23}$$
$$\leq \psi(f(c)) - f(t) - (\psi(c)-t)f'(t)$$
$$\leq \frac{1}{t-m}\left\{f'(t) - \frac{f(t)-f(m)}{t-m}\right\}\psi\left[(c-t)^2\right]$$

for $t \in (m, M)$.

Proof. If f'' is monotonic nondecreasing on $[m, M] \subset \mathring{I}$, then

$$f(z) - f(t) - (z-t) f'(t) = (z-t)^2 \int_0^1 f''((1-s)t + sz)(1-s)\,ds \geq (z-t)^2 \int_0^1 f''((1-s)t + sm)(1-s)\,ds \tag{3.24}$$

and

$$f(z) - f(t) - (z-t) f'(t) = (z-t)^2 \int_0^1 f''((1-s)t + sz)(1-s)\,ds \leq (z-t)^2 \int_0^1 f''((1-s)t + sM)(1-s)\,ds. \tag{3.25}$$

First, observe that for u, $v \in [m, M]$ with $u \neq v$ we have

$$\begin{aligned}
&\int_0^1 f''((1-s)v + su)(1-s)\,ds \\
&= \frac{1}{u-v} \int_0^1 (1-s)\,d\left(f'((1-s)v + su)\right) \\
&= \frac{1}{u-v} \left[(1-s) f'((1-s)v + su)\big|_0^1 + \int_0^1 f'((1-s)v + su)\,ds \right] \\
&= \frac{1}{u-v} \left\{ -f'(v) + \int_0^1 f'((1-s)v + su)\,ds \right\} \\
&= \frac{1}{v-u} \left\{ f'(v) - \int_0^1 f'((1-s)v + su)\,ds \right\} \\
&= \frac{1}{v-u} \left\{ f'(v) - \frac{f(v) - f(u)}{v-u} \right\}.
\end{aligned}$$

Using this equality, we have

$$\int_0^1 f''((1-s)t + sm)(1-s)\,ds = \frac{1}{t-m} \left\{ f'(t) - \frac{f(t) - f(m)}{t-m} \right\}$$

and

$$\begin{aligned}
\int_0^1 f''((1-s)t + sM)(1-s)\,ds &= \frac{1}{t-M} \left\{ f'(t) - \frac{f(t) - f(M)}{t-M} \right\} \\
&= \frac{1}{M-t} \left\{ \frac{f(M) - f(t)}{M-t} - f'(t) \right\}.
\end{aligned}$$

Then by (3.24) and (3.25) we get

$$\begin{aligned}
&\frac{1}{t-m} \left\{ f'(t) - \frac{f(t) - f(m)}{t-m} \right\} (z-t)^2 \\
&\leq f(z) - f(t) - (z-t) f'(t) \\
&\leq \frac{1}{M-t} \left\{ \frac{f(M) - f(t)}{M-t} - f'(t) \right\} (z-t)^2
\end{aligned} \tag{3.26}$$

for all $t \in (m, M)$ and $z \in I$.

Fix $t \in (m, M)$. Using Lemma 1 and the inequality (3.26) we obtain for the element $c \in A$ with $\sigma(c) \subset I$ the following inequality in the order of A

$$\begin{aligned}\frac{1}{t-m}\left\{f'(t)-\frac{f(t)-f(m)}{t-m}\right\}(c-t)^2 \\ \leq f(c)-f(t)-(c-t)f'(t) \\ \leq \frac{1}{M-t}\left\{\frac{f(M)-f(t)}{M-t}-f'(t)\right\}(c-t)^2\end{aligned} \tag{3.27}$$

for all $t \in (m, M)$.

Taking the functional ψ in the inequality (3.27) we get (3.22). □

Corollary 7. *With the assumptions of Theorem 5 and if $\psi(c) \in (m, M)$, then*

$$\begin{aligned}0 &\leq \frac{1}{\psi(c)-m}\left\{f'(\psi(c))-\frac{f(\psi(c))-f(m)}{\psi(c)-m}\right\}\left[\psi(c^2)-(\psi(c))^2\right] \\ &\leq \psi(f(c))-f(\psi(c)) \\ &\leq \frac{1}{M-\psi(c)}\left\{\frac{f(M)-f(\psi(c))}{M-\psi(c)}-f'(\psi(c))\right\}\left[\psi(c^2)-(\psi(c))^2\right]\end{aligned} \tag{3.28}$$

if f'' is monotonic nondecreasing on $[m, M]$ and

$$\begin{aligned}0 &\leq \frac{1}{M-\psi(c)}\left\{\frac{f(M)-f(\psi(c))}{M-\psi(c)}-f'(\psi(c))\right\}\left[\psi(c^2)-(\psi(c))^2\right] \\ &\leq \psi(f(c))-f(\psi(c)) \\ &\leq \frac{1}{\psi(c)-m}\left\{f'(\psi(c))-\frac{f(\psi(c))-f(m)}{\psi(c)-m}\right\}\left[\psi(c^2)-(\psi(c))^2\right]\end{aligned} \tag{3.29}$$

if f'' is monotonic nonincreasing on $[m, M]$.

Corollary 8. *With the assumptions of Theorem 5 and if $\psi(f'(c)) \neq 0$ with $\frac{\psi(cf'(c))}{\psi(f'(c))} \in (m, M)$, then*

$$\begin{aligned}0 &\leq \frac{1}{\frac{\psi(cf'(c))}{\psi(f'(c))}-m}\left\{f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)-\frac{f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)-f(m)}{\frac{\psi(cf'(c))}{\psi(f'(c))}-m}\right\} \\ &\quad\times\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \\ &\leq \psi(f(c))-f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)-\left(\psi(c)-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) \\ &\leq \frac{1}{M-\frac{\psi(cf'(c))}{\psi(f'(c))}}\left\{\frac{f(M)-f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{M-\frac{\psi(cf'(c))}{\psi(f'(c))}}-f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)\right\} \\ &\quad\times\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right]\end{aligned} \tag{3.30}$$

if f'' is monotonic nondecreasing on $[m, M]$.

The case of monotonic nonincreasing functions is similar.

4. Some Examples

In this section we provide some simple inequalities that can be derived from the above results by taking particular examples of convex functions such as: power function, exponential and logarithm. They generalize several known results obtained for selfadjoint operators in Hilbert spaces.

For $p \geq 1$, consider the power function $f_p : (0, \infty) \to (0, \infty)$, $f_p(x) = x^p$ which is analytic and convex on $(0, \infty)$ and $0 < c \in A$.

We define $\mathcal{J}_p(f, c, \cdot) : \mathfrak{P}_1(A) \to [0, \infty)$ by

$$\mathcal{J}_p(f, c, \varphi) := \varphi(c^p) - [\varphi(1)]^{p-1} [\varphi(c)]^p .$$

By Theorem 2, the functional $\mathcal{J}_p(f, c, \cdot)$ is superadditive on $\mathfrak{P}_1(A)$ and if there exists the constants $0 < m < M < \infty$ such that $M\omega - \varphi$ and $\varphi - m\omega \in \mathfrak{P}_1(A)$ with $\omega, \varphi \in \mathfrak{P}_1(A)$ then by Corollary 1

$$\begin{aligned} M\left\{\omega(c^p) - [\omega(1)]^{p-1}[\omega(c)]^p\right\} &\geq \varphi(c^p) - [\varphi(1)]^{p-1}[\varphi(c)]^p \\ &\geq m\left\{\omega(c^p) - [\omega(1)]^{p-1}[\omega(c)]^p\right\} \geq 0. \end{aligned} \tag{4.1}$$

Since $f_p''(t) = p(p-1)t^{p-2}$, $t > 0$ then

$$k_p := p(p-1)\begin{cases} M^{p-2} \text{ for } p \in (1,2) \\ \\ m^{p-2} \text{ for } p \in [2,\infty) \end{cases} \tag{4.2}$$

$$\leq f_p''(t) \leq K_p := p(p-1)\begin{cases} m^{p-2} \text{ for } p \in (1,2) \\ \\ M^{p-2} \text{ for } p \in [2,\infty) \end{cases}$$

for any $t \in [m, M]$.

Assume that $0 < c \in A$ and there exist the constants $0 < m < M$ such that $\sigma(c) \subseteq [m, M]$. By Remark 2 we get

$$0 \leq \frac{1}{2}k_p\psi\left[(c-t)^2\right] \leq \psi(c^p) - pt^{p-1}(\psi(c) - t) - t^p \leq \frac{1}{2}K_p\psi\left[(c-t)^2\right] \tag{4.3}$$

for all $t \in (m, M)$.

In particular, we have

$$0 \leq \frac{1}{2}k_p\left[\psi(c^2) - (\psi(c))^2\right] \leq \psi(c^p) - (\psi(c))^p \leq \frac{1}{2}K_p\left[\psi(c^2) - (\psi(c))^2\right] \tag{4.4}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}k_p\psi\left[\left(c - \frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right] \\ &\leq \psi(c^p) - p\left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^{p-1}\left(\psi(c) - \frac{\psi(c^p)}{\psi(c^{p-1})}\right) - \left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^p \\ &\leq \frac{1}{2}K_p\psi\left[\left(c - \frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right]. \end{aligned} \tag{4.5}$$

Also

$$0 \leq \frac{1}{2} k_p \psi \left[\left(c - \frac{m+M}{2} \right)^2 \right] \tag{4.6}$$
$$\leq \psi (c^p) - p \left(\frac{m+M}{2} \right)^{p-1} \left(\psi (c) - \frac{m+M}{2} \right) - \left(\frac{m+M}{2} \right)^p$$
$$\leq \frac{1}{2} K_p \psi \left[\left(c - \frac{m+M}{2} \right)^2 \right] \leq \frac{1}{8} (M-m)^2 K_p.$$

We have

$$0 \leq \frac{1}{4} k_p \left\{ \frac{1}{12} (M-m)^2 + \psi \left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\} \tag{4.7}$$
$$\leq \frac{1}{2} \left[\psi (c^p) + \frac{(M - \psi (c)) M^p + \psi (c) m^p}{M-m} \right]$$
$$- \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)}$$
$$\leq \frac{1}{4} K_p \left\{ \frac{1}{12} (M-m)^2 + \psi \left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\},$$

and

$$0 \leq \frac{1}{48} k_p (M-m)^2 \tag{4.8}$$
$$\leq \frac{1}{2} \left[\psi (c^p) + \frac{(M - \psi (c)) M^p + \psi (c) m^p}{M-m} \right] - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)}$$
$$\leq \frac{1}{12} K_p (M-m)^2 .$$

The case of logarithmic function is also of interest. We take the function $f(t) = -\ln t$ and $0 < c \in A$. Define $\mathcal{J}_{\ln}(f, c, \cdot) : \mathfrak{P}_1(A) \to [0, \infty)$ by

$$\mathcal{J}_{\ln}(f, c, \varphi) := \varphi (1) \ln \left(\frac{\varphi (c)}{\varphi (1)} \right) - \varphi (\ln c) \geq 0. \tag{4.9}$$

By Theorem 2, the functional $\mathcal{J}_{\ln}(f, c, \cdot)$ is superadditive on $\mathfrak{P}_1(A)$ and if there exists the constants $0 < m < M < \infty$ such that $M\omega - \varphi$ and $\varphi - m\omega \in \mathfrak{P}_1(A)$ with ω, $\varphi \in \mathfrak{P}_1(A)$ then by Corollary 1 we obtain

$$M \left\{ \omega (1) \ln \left(\frac{\omega (c)}{\omega (1)} \right) - \omega (\ln c)^p \right\} \geq \varphi (1) \ln \left(\frac{\varphi (c)}{\varphi (1)} \right) - \varphi (\ln c) \tag{4.10}$$
$$\geq m \left\{ \omega (1) \ln \left(\frac{\omega (c)}{\omega (1)} \right) - \omega (\ln c)^p \right\} \geq 0.$$

Moreover, we have $f''(t) = \frac{1}{t^2}$. Therefore for $t \in [m, M] \subset (0, \infty)$ we get

$$\frac{1}{M^2} \leq f''(t) \leq \frac{1}{m^2}.$$

By using Remark 2 we get

$$0 \leq \frac{1}{2M^2} \psi \left[(c-t)^2 \right] \leq \ln (t) - \psi (\ln c) + \frac{1}{t} (\psi (c) - t) \leq \frac{1}{2m^2} \psi \left[(c-t)^2 \right] \tag{4.11}$$

for all $t \in (m, M)$.

In particular, we obtain

$$\begin{aligned}(4.12)\qquad 0 &\le \frac{1}{2M^2}\left[\psi\left(c^2\right)-\left(\psi\left(c\right)\right)^2\right] \le \ln\left(\psi\left(c\right)\right)-\psi\left(\ln c\right)\\ &\le \frac{1}{2m^2}\left[\psi\left(c^2\right)-\left(\psi\left(c\right)\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}(4.13)\qquad 0 &\le \frac{1}{2M^2}\psi\left[\left(c-\frac{1}{\psi\left(c^{-1}\right)}\right)^2\right]\\ &\le \psi\left(c^{-1}\right)\left(\psi\left(c\right)-\frac{1}{\psi\left(c^{-1}\right)}\right)-\ln\left(\psi\left(c^{-1}\right)\right)-\psi\left(\ln c\right)\\ &\le \frac{1}{2m^2}\psi\left[\left(c-\frac{1}{\psi\left(c^{-1}\right)}\right)^2\right].\end{aligned}$$

Also

$$\begin{aligned}(4.14)\qquad 0 &\le \frac{1}{2M^2}\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\\ &\le \left(\frac{m+M}{2}\right)^{-1}\left(\psi\left(c\right)-\frac{m+M}{2}\right)+\ln\left(\frac{m+M}{2}\right)-\psi\left(\ln c\right)\\ &\le \frac{1}{2m^2}\psi\left[\left(c-\frac{m+M}{2}\right)^2\right] \le \frac{1}{8}\left(\frac{M}{m}-1\right)^2.\end{aligned}$$

We finally have

$$\begin{aligned}(4.15)\qquad 0 &\le \frac{1}{4M^2}\left\{\frac{1}{12}\left(M-m\right)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right\}\\ &\le \frac{M\ln M-m\ln m-M+m}{M-m}\\ &-\frac{1}{2}\left[\psi\left(\ln c\right)+\frac{\left(M-\psi\left(c\right)\right)\ln M+\psi\left(c\right)\ln m}{M-m}\right]\\ &\le \frac{1}{4m^2}\left\{\frac{1}{12}\left(M-m\right)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right\},\end{aligned}$$

and

$$\begin{aligned}(4.16)\qquad 0 &\le \frac{1}{48}\left(1-\frac{m}{M}\right)^2\\ &\le \frac{M\ln M-m\ln m-M+m}{M-m}\\ &-\frac{1}{2}\left[\psi\left(\ln c\right)+\frac{\left(M-\psi\left(c\right)\right)\ln M+\psi\left(c\right)\ln m}{M-m}\right]\\ &\le \frac{1}{12}\left(\frac{M}{m}-1\right)^2.\end{aligned}$$

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