LOWER AND UPPER BOUNDS FOR SLATER'S GAP FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. We establish in this paper some lower and upper bounds for Slater's gap in the general setting of Hermitian unital Banach $*$-algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [10] (see also [1, Theorem 41.5]), then

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [9], Tanahashi and Uchiyama [11] proved the following fundamental properties (see also [7]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [9] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

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In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1}\, dz.$$

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1}\, dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [11, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [4].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

In the recent papers [6] and [5] we obtained between others the following results:

Theorem 1. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that f is convex (in the usual sense) on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A and a selfadjoint element $c \in A$ with $\sigma(c) \subset I$. If*

$$\psi(f'(c)) \neq 0 \text{ and } \frac{\psi(cf'(c))}{\psi(f'(c))} \in I,$$

then we have the Slater's type inequalities

$$0 \leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c)) \leq f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)\left(\frac{\psi(cf'(c))}{\psi(f'(c))} - \psi(c)\right). \tag{1.1}$$

Motivated by the above result, we establish in this paper some new lower and upper bounds for Slater's gap in the general setting of Hermitian unital Banach $*$-algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

2. Functional Properties

We denote by $\mathfrak{P}_1(A)$ the set of all linear, positive functionals defined on A with the property that, if $\varphi \in \mathfrak{P}_1(A)$, then $\varphi(1) > 0$. If φ, $\omega \in \mathfrak{P}_1(A)$ then $\varphi + \omega \in \mathfrak{P}_1(A)$ and for all $\alpha > 0$ we have $\alpha\varphi \in \mathfrak{P}_1(A)$.

For a convex function f as above and a selfadjoint element c with $\sigma(c) \subset I$ we consider the subset of Slater's functionals

$$\mathfrak{P}_{f,c,1,+}(A) := \left\{\varphi \in \mathfrak{P}_1(A) \mid \varphi(f'(c)) > 0 \text{ and } \frac{\varphi(cf'(c))}{\varphi(f'(c))} \in I\right\}.$$

Observe that, if φ, $\omega \in \mathfrak{P}_{f,c,1,+}(A)$, then $\varphi(f'(c)) > 0$, $\omega(f'(c)) > 0$, and $\frac{\varphi(cf'(c))}{\varphi(f'(c))}, \frac{\omega(cf'(c))}{\omega(f'(c))} \in I$. Therefore $(\varphi + \omega)(f'(c)) > 0$ and

$$\begin{aligned}\frac{(\varphi + \omega)(cf'(c))}{(\varphi + \omega)(f'(c))} &= \frac{\varphi(cf'(c)) + \omega(cf'(c))}{\varphi(f'(c)) + \omega(f'(c))} \\ &= \frac{\varphi(f'(c))\frac{\varphi(cf'(c))}{\varphi(f'(c))} + \omega(f'(c))\frac{\omega(cf'(c))}{\omega(f'(c))}}{\varphi(f'(c)) + \omega(f'(c))} \in I\end{aligned}$$

since the interval I is a convex set.

We conclude that $\varphi + \omega \in \mathfrak{P}_{f,c,1,+}(A)$ and also $\alpha\varphi \in \mathfrak{P}_{f,c,1,+}(A)$ for all $\alpha > 0$.

We define the functional $\mathcal{S} : \mathfrak{P}_{f,c,1,+}(A) \to \mathbb{R}$ by

$$\mathcal{S}(f, c, \varphi) := f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))}\right)\varphi(f'(c)). \tag{2.1}$$

Theorem 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that f is convex on the interval I and $c \in A$ is a selfadjoint element with $\sigma(c) \subset I$. If φ, $\omega \in \mathfrak{P}_{f,c,1,+}(A)$, then*

$$\mathcal{S}(f, c, \varphi + \omega) \leq \mathcal{S}(f, c, \varphi) + \mathcal{S}(f, c, \omega), \tag{2.2}$$

namely $\mathcal{S}(f, c, \cdot)$ is subadditive on $\mathfrak{P}_{f,c,1,+}(A)$. It is also positive homogeneous on $\mathfrak{P}_{f,c,1,+}(A)$.

Proof. If $\varphi,\,\omega\in\mathfrak{P}_{f,c,1,+}(A)$, then $\varphi+\omega\in\mathfrak{P}_{f,c,1,+}(A)$ and

$$\begin{aligned}
\mathcal{S}(f,c,\varphi+\omega) &= f\left(\frac{(\varphi+\omega)(cf'(c))}{(\varphi+\omega)(f'(c))}\right)(\varphi+\omega)(f'(c))\\
&= f\left(\frac{\varphi(f'(c))\frac{\varphi(cf'(c))}{\varphi(f'(c))}+\omega(f'(c))\frac{\omega(cf'(c))}{\omega(f'(c))}}{\varphi(f'(c))+\omega(f'(c))}\right)(\varphi+\omega)(f'(c))\\
&\le\left[\frac{\varphi(f'(c))\,f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))}\right)+\omega(f'(c))\,f\left(\frac{\omega(cf'(c))}{\omega(f'(c))}\right)}{\varphi(f'(c))+\omega(f'(c))}\right](\varphi+\omega)(f'(c))\\
&= \varphi(f'(c))\,f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))}\right)+\omega(f'(c))\,f\left(\frac{\omega(cf'(c))}{\omega(f'(c))}\right)\\
&= \mathcal{S}(f,c,\varphi)+\mathcal{S}(f,c,\omega),
\end{aligned}$$

which proves (2.2).

If $\alpha>0$ and $\varphi\in\mathfrak{P}_{f,c,1,+}(A)$, then $\alpha\varphi\in\mathfrak{P}_{f,c,1,+}(A)$ and

$$\mathcal{S}(f,c,\alpha\varphi):=f\left(\frac{(\alpha\varphi)(cf'(c))}{(\alpha\varphi)(f'(c))}\right)(\alpha\varphi)(f'(c))=\alpha\mathcal{S}(f,c,\varphi).$$

□

Let X be a linear space. A subset $C\subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x,y\in C$ imply $x+y\in C$;

(ii) $x\in C$, $\alpha\ge 0$ imply $\alpha x\in C$.

A functional $h:C\to\mathbb{R}$ is called *subadditive* on C if

(iii) $h(x+y)\le h(x)+h(y)$ for any $x,y\in C$

and *nonnegative (strictly positive)* on C if, it satisfies

(iv) $h(x)\ge(>)\,0$ for each $x\in C$.

The functional h is *s-positive homogeneous* on C, for a given $s>0$, if

(v) $h(\alpha x)=\alpha^s h(x)$ for any $\alpha\ge 0$ and $x\in C$.

In the paper [3] we obtained the following results for superadditive functions:

Lemma 2. *Let C be a convex cone in the linear space X and $v:C\to(0,\infty)$ an additive functional on C. If $h:C\to[0,\infty)$ is a subadditive functional on C and $0<p<1$ then the functional*

$$\Psi_p:C\to[0,\infty),\,\Psi_p(x)=v^{1-\frac{1}{p}}(x)\,h(x) \tag{2.3}$$

is subadditive on C.

Corollary 1. *Assume that X,C and v are as in Theorem 2. If $h:C\to[0,\infty)$ is a subadditive functional on C and $0<p,\,q<1$ then the two parameters functional*

$$\Psi_{p,q}:C\to[0,\infty),\,\Psi_{p,q}(x)=v^{q\left(1-\frac{1}{p}\right)}(x)\,h^q(x) \tag{2.4}$$

is subadditive on C.

Remark 1. *If we consider the functional $\psi_p(x):=v^{p-1}(x)\,h^p(x)$ then for $0<p<1$ and $h:C\to[0,\infty)$ a subadditive functional on C, the functional ψ_p is also subadditive on C.*

Corollary 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that f is convex and positive on the interval I and $c \in A$ is a selfadjoint element with $\sigma(c) \subset I$. For $0 < p,\ q < 1$ we define the functional $\mathcal{S}_{p,q}(f,c,\cdot) : \mathfrak{P}_{f,c,1,+}(A) \to [0,\infty)$ by*

$$\mathcal{S}_{p,q}(f,c,\varphi) := \left[f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^q [\varphi(f'(c))]^{q\left(2-\frac{1}{p}\right)}.$$

Then $\mathcal{S}_{p,q}(f,c,\cdot)$ is subadditive on $\mathfrak{P}_{f,c,1,+}(A)$.

The proof follows by Corollary 1 on choosing

$$v(\varphi) = \varphi(f'(c)) \text{ and } h(\varphi) = \mathcal{S}(f,c,\varphi) := f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \varphi(f'(c)).$$

We also observe that for $0 < p < 1$,

$$\mathcal{S}_{p,p}(f,c,\varphi) := \left[f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^p [\varphi(f'(c))]^{p\left(2-\frac{1}{p}\right)}$$

and for $0 < q < 1$,

$$\mathcal{S}_{1/2,q}(f,c,\varphi) := \left[f\left(\frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^q$$

are subadditive on $\mathfrak{P}_{f,c,1,+}(A)$.

3. Lower and Upper Bounds

We have:

Theorem 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$ and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A. For any selfadjoint element $c \in A$ with $\sigma(c) \subset I$,*

$$\begin{aligned} 0 &\leq \left(\inf_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \psi\left[(c-z)^2 \right] \\ &\leq f(z) - \psi(f(c)) - z\psi(f'(c)) + \psi(cf'(c)) \\ &\leq \left(\sup_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \psi\left[(c-z)^2 \right], \end{aligned} \tag{3.1}$$

for all $z \in \mathring{I}$.

In particular, we have

$$\begin{aligned} 0 &\leq \left(\inf_{w \in \mathring{I}} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\,ds \right) \left[\psi\left(c^2\right) - (\psi(c))^2 \right] \\ &\leq f(\psi(c)) - \psi(f(c)) - \psi(c)\psi(f'(c)) + \psi(cf'(c)) \\ &\leq \left(\sup_{w \in \mathring{I}} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\,ds \right) \left[\psi\left(c^2\right) - (\psi(c))^2 \right]. \end{aligned} \tag{3.2}$$

Proof. Using Taylor's representation with the integral remainder we can write the following identity

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(z-t)^k + \frac{1}{n!} \int_t^z f^{(n+1)}(s)(z-s)^n\,ds \tag{3.3}$$

for any $z,\ t \in \mathring{I}$, the interior of I.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $s = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(s)\,ds = (d-c)\int_0^1 h((1-s)c+sd)\,ds.$$

Therefore,

$$\begin{aligned}&\int_t^z f^{(n+1)}(s)(z-s)^n\,ds\\&=(z-t)\int_0^1 f^{(n+1)}((1-s)t+sz)(z-(1-s)t-sz)^n\,ds\\&=(z-t)^{n+1}\int_0^1 f^{(n+1)}((1-s)t+sz)(1-s)^n\,ds.\end{aligned}$$

The identity (3.3) can then be written as

$$\begin{aligned}f(z)&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(t)(z-t)^k\\&\quad+\frac{1}{n!}(z-t)^{n+1}\int_0^1 f^{(n+1)}((1-s)t+sz)(1-s)^n\,ds.\end{aligned}\tag{3.4}$$

For $n=1$ we get

$$f(z)=f(w)+(z-w)f'(w)+(z-w)^2\int_0^1 f''((1-s)w+sz)(1-s)\,ds\tag{3.5}$$

for any z, $w \in \mathring{I}$.

Since

$$\begin{aligned}0&\le\inf_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds\le\int_0^1 f''((1-s)w+sz)(1-s)\,ds\\&\le\sup_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds,\end{aligned}$$

hence

$$\begin{aligned}0&\le\left(\inf_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds\right)(z-w)^2\\&\le f(z)-f(w)-(z-w)f'(w)\\&\le\left(\sup_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds\right)(z-w)^2,\end{aligned}\tag{3.6}$$

for any z, $w \in \mathring{I}$.

Fix $z \in I$. Using Lemma 1 and the inequality (3.7) we obtain for the element $c \in A$ with $\sigma(c) \subset I$ the following inequality in the order of A

$$\begin{aligned}0&\le\left(\inf_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds\right)(z-c)^2\\&\le f(z)-f(c)-(z-c)f'(c)\\&\le\left(\sup_{w\in\mathring{I}}\int_0^1 f''((1-s)w+sz)(1-s)\,ds\right)(z-c)^2,\end{aligned}$$

for any $z \in \mathring{I}$.

If we take in this inequality the functional ψ we get (3.1).

If we take in (3.1) $z = \psi(c)$, then we get

$$0 \leq \left(\inf_{w \in \mathring{I}} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\,ds \right) \psi\left[(z - \psi(c))^2 \right] \tag{3.7}$$
$$\leq f(z) - \psi(f(c)) - z\psi(f'(c)) + \psi(cf'(c))$$
$$\leq \left(\sup_{w \in \mathring{I}} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\,ds \right) \psi\left[(z - \psi(c))^2 \right].$$

Since

$$\psi\left[(c - \psi(c))^2 \right] = \psi\left(c^2 - 2\psi(c)c + (\psi(c))^2 \right)$$
$$= \psi(c^2) - 2(\psi(c))^2 + (\psi(c))^2 = \psi(c^2) - (\psi(c))^2,$$

hence by (3.7) we get (3.2). □

Corollary 3. *With the assumptions of Theorem 3 and, if, in addition, $\psi(f'(c)) \neq 0$ with $z = \frac{\psi(cf'(c))}{\psi(f'(c))} \in \mathring{I}$, then we have the refinement and reverse of Slater's inequality*

$$0 \leq \left(\inf_{w \in \mathring{I}} \int_0^1 f''\left((1-s)w + s\frac{\psi(cf'(c))}{\psi(f'(c))} \right)(1-s)\,ds \right) \tag{3.8}$$
$$\times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right]$$
$$\leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))} \right) - \psi(f(c))$$
$$\leq \left(\sup_{w \in \mathring{I}} \int_0^1 f''\left((1-s)w + s\frac{\psi(cf'(c))}{\psi(f'(c))} \right)(1-s)\,ds \right)$$
$$\times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right].$$

Corollary 4. *For any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m, M] \subset I$, we have*

$$0 \leq \left(\inf_{w \in \mathring{I}} \int_0^1 f''\left((1-s)w + s\frac{m+M}{2} \right)(1-s)\,ds \right) \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \tag{3.9}$$
$$\leq f\left(\frac{m+M}{2} \right) - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c))$$
$$\leq \left(\sup_{w \in \mathring{I}} \int_0^1 f''\left((1-s)w + s\frac{m+M}{2} \right)(1-s)\,ds \right) \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right].$$

We also have:

Theorem 4. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$ and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A.*

For any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m, M] \subset I$, we have

$$
\begin{aligned}
(3.10)\qquad 0 &\leq \left(\inf_{(w,z) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \\
&\quad \times \left\{ \frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\} \\
&\leq \frac{1}{M-m} \int_m^M f(z)\,dx - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c)) \\
&\leq \left(\sup_{(w,z) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \\
&\quad \times \left\{ \frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \right\}.
\end{aligned}
$$

Proof. If we take the integral mean in (3.6) over $z \in [m, M]$, we get

$$
\begin{aligned}
(3.11)\qquad 0 &\leq \frac{1}{M-m} \int_m^M \left(\inf_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right)(z-w)^2\,dz \\
&\leq \frac{1}{M-m} \int_m^M f(z)\,dx - f(w) - \left(\frac{m+M}{2} - w \right) f'(w) \\
&\leq \frac{1}{M-m} \int_m^M \left(\sup_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right)(z-w)^2\,dz,
\end{aligned}
$$

for all $w \in \mathring{I}$.

Since

$$
\begin{aligned}
&\left(\inf_{(w,z) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \frac{1}{M-m} \int_m^M (z-w)^2\,dz \\
&\leq \frac{1}{M-m} \int_m^M \left(\inf_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right)(z-w)^2\,dz
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{M-m} \int_m^M \left(\sup_{w \in \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right)(z-w)^2\,dz \\
&\leq \left(\sup_{(w,z) \in \mathring{I} \times \mathring{I}} \int_0^1 f''((1-s)w + sz)(1-s)\,ds \right) \frac{1}{M-m} \int_m^M (z-w)^2\,dz
\end{aligned}
$$

for all $w \in \mathring{I}$.

Hence

$$\begin{aligned}(3.12)\quad &\left(\inf_{(w,z)\in\mathring{I}\times\mathring{I}}\int_0^1 f''\left((1-s)w+sz\right)(1-s)\,ds\right)\frac{1}{M-m}\int_m^M (z-w)^2\,dz \\ &\le \frac{1}{M-m}\int_m^M f(z)\,dx - f(w) - \left(\frac{m+M}{2}-w\right)f'(w) \\ &\le \left(\sup_{(w,z)\in\mathring{I}\times\mathring{I}}\int_0^1 f''\left((1-s)w+sz\right)(1-s)\,ds\right)\frac{1}{M-m}\int_m^M (z-w)^2\,dz\end{aligned}$$

for all $w \in \mathring{I}$.

Observe that

$$\begin{aligned}\frac{1}{M-m}\int_m^M (z-w)^2\,dz &= \frac{(M-w)^3+(w-m)^3}{3(M-m)} \\ &= \frac{1}{3}\left[(w-m)^2+(M-w)^2-(w-m)(M-w)\right] \\ &= \frac{1}{3}\left[\frac{1}{4}(M-m)^2+3\left(w-\frac{m+M}{2}\right)^2\right] \\ &= \frac{1}{12}(M-m)^2+\left(w-\frac{m+M}{2}\right)^2\end{aligned}$$

for all $w \in [m, M]$.

From (3.12) we then get

$$\begin{aligned}(3.13)\quad &\left(\inf_{(w,z)\in\mathring{I}\times\mathring{I}}\int_0^1 f''\left((1-s)w+sz\right)(1-s)\,ds\right) \\ &\times\left[\frac{1}{12}(M-m)^2+\left(w-\frac{m+M}{2}\right)^2\right] \\ &\le \frac{1}{M-m}\int_m^M f(z)\,dx - f(w) - \left(\frac{m+M}{2}-w\right)f'(w) \\ &\le \left(\sup_{(w,z)\in\mathring{I}\times\mathring{I}}\int_0^1 f''\left((1-s)w+sz\right)(1-s)\,ds\right) \\ &\times\left[\frac{1}{12}(M-m)^2+\left(w-\frac{m+M}{2}\right)^2\right]\end{aligned}$$

for all $w \in [m, M]$.

Using Lemma 1 and the inequality (3.13) we obtain for the element $c \in A$ with $\sigma(c) \subset I$ the following inequality in the order of A

$$
\begin{aligned}
(3.14)\quad & \left(\inf_{(w,z)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)w+sz)(1-s)\,ds \right) \\
& \times \left[\frac{1}{12}(M-m)^2 + \left(c - \frac{m+M}{2} \right)^2 \right] \\
& \leq \frac{1}{M-m}\int_m^M f(z)\,dx - f(c) - \frac{m+M}{2} f'(c) + cf'(c) \\
& \leq \left(\sup_{(w,z)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)w+sz)(1-s)\,ds \right) \\
& \times \left[\frac{1}{12}(M-m)^2 + \left(c - \frac{m+M}{2} \right)^2 \right].
\end{aligned}
$$

If we take the functional ψ on (3.14), then we get (3.10). $\square$

Corollary 5. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
(3.15)\quad 0 & \leq \frac{1}{12}(M-m)^2 \left(\inf_{(w,z)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)w+sz)(1-s)\,ds \right) \\
& \leq \frac{1}{M-m}\int_m^M f(z)\,dx - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c)) \\
& \leq \frac{1}{3}(M-m)^2 \left(\sup_{(w,z)\in \mathring{I}\times\mathring{I}} \int_0^1 f''((1-s)w+sz)(1-s)\,ds \right).
\end{aligned}
$$

The proof follows by (3.10) on observing that

$$
0 \leq \psi\left[\left(c - \frac{m+M}{2} \right)^2 \right] \leq \frac{1}{4}(M-m)^2 .
$$

Remark 2. *If there exists the constants $0 < \gamma < \Gamma < \infty$ such that $\gamma \leq f''(x) \leq \Gamma$ for almost every $x \in \mathring{I}$, then by (3.1) and (3.2) we get*

$$
\begin{aligned}
(3.16)\quad 0 & \leq \frac{1}{2}\gamma\psi\left[(c-z)^2 \right] \leq f(z) - \psi(f(c)) - z\psi(f'(c)) + \psi(cf'(c)) \\
& \leq \frac{1}{2}\Gamma\psi\left[(c-z)^2 \right],
\end{aligned}
$$

for all $z \in \mathring{I}$.

In particular, we have

$$
\begin{aligned}
(3.17)\quad 0 & \leq \frac{1}{2}\gamma\left[\psi(c^2) - (\psi(c))^2 \right] \\
& \leq f(\psi(c)) - \psi(f(c)) - \psi(c)\psi(f'(c)) + \psi(cf'(c)) \\
& \leq \frac{1}{2}\Gamma\left[\psi(c^2) - (\psi(c))^2 \right].
\end{aligned}
$$

From (3.8) and (3.9) we have

$$0 \le \frac{1}{2}\gamma\psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \le f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c)) \le \frac{1}{2}\Gamma\psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \tag{3.18}$$

and

$$0 \le \frac{1}{2}\gamma\psi\left[\left(c - \frac{m+M}{2}\right)^2\right] \le f\left(\frac{m+M}{2}\right) - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c)) \le \frac{1}{2}\Gamma\psi\left[\left(c - \frac{m+M}{2}\right)^2\right]. \tag{3.19}$$

Also, by (3.10) and (3.15) we get

$$0 \le \frac{1}{2}\gamma\left\{\frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2}\right)^2\right]\right\} \le \frac{1}{M-m}\int_m^M f(z)\,dx - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c)) \le \frac{1}{2}\Gamma\left\{\frac{1}{12}(M-m)^2 + \psi\left[\left(c - \frac{m+M}{2}\right)^2\right]\right\} \tag{3.20}$$

and

$$0 \le \frac{1}{24}\gamma(M-m)^2 \le \frac{1}{M-m}\int_m^M f(z)\,dx - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c)) \le \frac{1}{6}\Gamma(M-m)^2. \tag{3.21}$$

We also have:

Theorem 5. *Let f be analytic in G, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$, $\psi : A \to \mathbb{C}$ be a positive normalized linear functional on A and $c \in A$ a selfadjoint element with $\sigma(c) \subset I$. If f'' is monotonic nondecreasing on $[m, M] \subset \mathring{I}$, then*

$$\frac{1}{z-m}\left\{\frac{f(z) - f(m)}{z-m} - f'(m)\right\}\psi\left[(c-z)^2\right] \le f(z) - f(c) - (z-c)f'(c) \le \frac{1}{M-z}\left\{\frac{f(M) - f(z)}{M-z} - f'(z)\right\}\psi\left[(c-z)^2\right] \tag{3.22}$$

for $z \in (m, M)$.

If f'' is monotonic nonincreasing on $[m, M] \subset \mathring{I}$, then

$$0 \leq \frac{1}{M-z}\left\{\frac{f(M)-f(z)}{M-z}-f'(z)\right\}\psi\left[(c-z)^2\right] \tag{3.23}$$
$$\leq f(z)-f(c)-(z-c)f'(c)$$
$$\leq \frac{1}{z-m}\left\{\frac{f(z)-f(m)}{z-m}-f'(m)\right\}\psi\left[(c-z)^2\right]$$

for $z \in (m, M)$.

Proof. If f'' is monotonic nondecreasing on $[m, M] \subset \mathring{I}$, then

$$f(z) = f(w)+(z-w)f'(w)+(z-w)^2\int_0^1 f''((1-s)w+sz)(1-s)\,ds \tag{3.24}$$
$$\geq (z-w)^2\int_0^1 f''((1-s)m+sz)(1-s)\,ds$$

and

$$f(z) = f(w)+(z-w)f'(w)+(z-w)^2\int_0^1 f''((1-s)w+sz)(1-s)\,ds \tag{3.25}$$
$$\leq (z-w)^2\int_0^1 f''((1-s)M+sz)(1-s)\,ds.$$

First, observe that for u, $v \in [m, M]$ with $u \neq v$ we have

$$\int_0^1 f''((1-s)v+su)(1-s)\,ds$$
$$= \frac{1}{u-v}\int_0^1 (1-s)\,d\left(f'((1-s)v+su)\right)$$
$$= \frac{1}{u-v}\left[(1-s)f'((1-s)v+su)\Big|_0^1+\int_0^1 f'((1-s)v+su)\,ds\right]$$
$$= \frac{1}{u-v}\left\{-f'(v)+\int_0^1 f'((1-s)v+su)\,ds\right\}$$
$$= \frac{1}{v-u}\left\{f'(v)-\int_0^1 f'((1-s)v+su)\,ds\right\}$$
$$= \frac{1}{v-u}\left\{f'(v)-\frac{f(v)-f(u)}{v-u}\right\}.$$

Using this equality, we have

$$\int_0^1 f''((1-s)m+sz)(1-s)\,ds = \frac{1}{m-z}\left\{f'(m)-\frac{f(m)-f(z)}{w-z}\right\}$$
$$= \frac{1}{z-m}\left\{\frac{f(z)-f(m)}{z-m}-f'(m)\right\}$$

and

$$\int_0^1 f''((1-s)M+sz)(1-s)\,ds = \frac{1}{M-z}\left\{\frac{f(M)-f(z)}{M-z}-f'(z)\right\}.$$

Then by (3.24) and (3.25) we get

$$\begin{aligned}(3.26)\qquad &\frac{1}{z-m}\left\{\frac{f(z)-f(m)}{z-m}-f'(m)\right\}(z-w)^2\\ &\le f(z)-f(w)-(z-w)f'(w)\\ &\le \frac{1}{M-z}\left\{\frac{f(M)-f(z)}{M-z}-f'(z)\right\}(z-w)^2\end{aligned}$$

for all $w\in(m,M)$ and $z\in I$.

Fix $z\in(m,M)$. Using Lemma 1 and the inequality (3.26) we obtain for the element $c\in A$ with $\sigma(c)\subset(m,M)$ the following inequality in the order of A

$$\begin{aligned}(3.27)\qquad &\frac{1}{z-m}\left\{\frac{f(z)-f(m)}{z-m}-f'(m)\right\}(z-c)^2\\ &\le f(z)-f(c)-(z-c)f'(c)\\ &\le \frac{1}{M-z}\left\{\frac{f(M)-f(z)}{M-z}-f'(z)\right\}(z-c)^2\end{aligned}$$

for all $z\in(m,M)$.

Taking the functional ψ in the inequality (3.27) we get (3.22). □

Corollary 6. *With the assumptions of Theorem 5 and if $\psi(c)\in(m,M)$ then*

$$\begin{aligned}(3.28)\qquad &\frac{1}{\psi(c)-m}\left\{\frac{f(\psi(c))-f(m)}{\psi(c)-m}-f'(m)\right\}\left[\psi\left(c^2\right)-(\psi(c))^2\right]\\ &\le f(\psi(c))-f(c)-(\psi(c)-c)f'(c)\\ &\le \frac{1}{M-\psi(c)}\left\{\frac{f(M)-f(\psi(c))}{M-\psi(c)}-f'(\psi(c))\right\}\left[\psi\left(c^2\right)-(\psi(c))^2\right],\end{aligned}$$

provided f'' is monotonic nondecreasing on $[m,M]$ and

$$\begin{aligned}(3.29)\qquad 0&\le \frac{1}{M-\psi(c)}\left\{\frac{f(M)-f(\psi(c))}{M-\psi(c)}-f'(\psi(c))\right\}\left[\psi\left(c^2\right)-(\psi(c))^2\right]\\ &\le f(\psi(c))-f(c)-(\psi(c)-c)f'(c)\\ &\le \frac{1}{\psi(c)-m}\left\{\frac{f(\psi(c))-f(m)}{\psi(c)-m}-f'(m)\right\}\left[\psi\left(c^2\right)-(\psi(c))^2\right],\end{aligned}$$

provided f'' is monotonic nonincreasing on $[m,M]$.

Corollary 7. *With the assumptions of Theorem 5 and if* $\frac{\psi(cf'(c))}{\psi(f'(c))} \in (m, M)$ *with* $\psi(f'(c)) \neq 0$ *then*

$$
\begin{aligned}
(3.30)\qquad & \frac{1}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} \left\{ \frac{f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - f(m)}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} - f'(m) \right\} \\
& \times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \\
& \leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c)) \\
& \leq \frac{1}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} \left\{ \frac{f(M) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} - f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) \right\} \\
& \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right],
\end{aligned}
$$

provided f'' *is monotonic nondecreasing on* $[m, M]$ *and*

$$
\begin{aligned}
(3.31)\qquad 0 & \leq \frac{1}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} \left\{ \frac{f(M) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} - f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) \right\} \\
& \times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right] \\
& \leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c)) \\
& \leq \frac{1}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} \left\{ \frac{f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - f(m)}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} - f'(m) \right\} \\
& \times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right],
\end{aligned}
$$

provided f'' *is monotonic nonincreasing on* $[m, M]$.

4. Some Examples

In this section we provide some simple inequalities that can be derived from the above results by taking particular examples of convex functions such as: power function, exponential and logarithm. They generalize several known results obtained for selfadjoint operators in Hilbert spaces.

For $p \geq 1$, consider the power function $f_p : (0, \infty) \to (0, \infty)$, $f_p(x) = x^p$ which is analytic and convex on $(0, \infty)$ and $0 < c \in A$.

We define the functional $\mathcal{S} : \mathfrak{P}_{f_p,c,1,+}(A) \to \mathbb{R}$ by

$$\mathcal{S}(f_p, c, \varphi) := [\varphi(cf'(c))]^p [\varphi(f'(c))]^{1-p}. \tag{4.1}$$

By Theorem 2 we the have that $\mathcal{S}(f_p, c, \cdot)$ is *subadditive and positive homogeneous* on $\mathfrak{P}_{f_p,c,1,+}(A)$.

Since $f_p''(t) = p(p-1)t^{p-2}$, $t > 0$ then

$$k_p := p(p-1)\begin{cases} M^{p-2} \text{ for } p \in (1,2) \\ \\ m^{p-2} \text{ for } p \in [2,\infty) \end{cases} \tag{4.2}$$

$$\leq f_p''(t) \leq K_p := p(p-1)\begin{cases} m^{p-2} \text{ for } p \in (1,2) \\ \\ M^{p-2} \text{ for } p \in [2,\infty) \end{cases}$$

for any $t \in [m, M]$.

From Remark 2 we get

$$\begin{aligned} 0 &\leq \frac{1}{2}k_p\psi\left[(c-z)^2\right] \leq z^p - pz\psi\left(c^{p-1}\right) + (p-1)\psi(c^p) \\ &\leq \frac{1}{2}K_p\psi\left[(c-z)^2\right], \end{aligned} \tag{4.3}$$

for all $z \in \mathring{I}$.

In particular, we have

$$\begin{aligned} 0 &\leq \frac{1}{2}k_p\left[\psi\left(c^2\right) - (\psi(c))^2\right] \leq (\psi(c))^p - p\psi(c)\psi\left(c^{p-1}\right) + (p-1)\psi(c^p) \\ &\leq \frac{1}{2}K_p\left[\psi\left(c^2\right) - (\psi(c))^2\right]. \end{aligned} \tag{4.4}$$

Also we have

$$\begin{aligned} 0 &\leq \frac{1}{2}k_p\psi\left[\left(c - \frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right] \leq \left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^p - \psi(c^p) \\ &\leq \frac{1}{2}K_p\psi\left[\left(c - \frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right] \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}k_p\psi\left[\left(c - \frac{m+M}{2}\right)^2\right] \\ &\leq \left(\frac{m+M}{2}\right)^p - p\frac{m+M}{2}\psi\left(c^{p-1}\right) + (p-1)\psi(c^p) \\ &\leq \frac{1}{2}K_p\psi\left[\left(c - \frac{m+M}{2}\right)^2\right] \leq \frac{1}{8}K_p(M-m)^2. \end{aligned} \tag{4.6}$$

Moreover, we get

$$0 \le \frac{1}{2} k_p \left\{ \frac{1}{12} (M - m)^2 + \psi \left[\left(c - \frac{m + M}{2} \right)^2 \right] \right\} \le \frac{M^{p+1} - m^{p+1}}{(p + 1)(M - m)} - p \frac{m + M}{2} \psi \left(c^{p-1} \right) + (p - 1) \psi (c^p) \le \frac{1}{2} K_p \left\{ \frac{1}{12} (M - m)^2 + \psi \left[\left(c - \frac{m + M}{2} \right)^2 \right] \right\} \tag{4.7}$$

and

$$0 \le \frac{1}{24} k_p (M - m)^2 \le \frac{M^{p+1} - m^{p+1}}{(p + 1)(M - m)} - p \frac{m + M}{2} \psi \left(c^{p-1} \right) + (p - 1) \psi (c^p) \le \frac{1}{6} K_p (M - m)^2 . \tag{4.8}$$

The case of logarithmic function is also of interest. We take the function $f(t) = -\ln t$ and $0 < c \in A$. Define

$$\mathcal{S}(-\ln, c, \varphi) := \ln \left(\frac{\varphi(1)}{\varphi(c^{-1})} \right) \varphi \left(c^{-1} \right).$$

By Theorem 2 we observe that $\mathcal{S}(-\ln, c, \cdot)$ is *subadditive and positive homogeneous* on $\mathfrak{P}_{-\ln, c, 1, +}(A)$.

Moreover, we have $f''(t) = \frac{1}{t^2}$. Therefore for $t \in [m, M] \subset (0, \infty)$ we get

$$\frac{1}{M^2} \le f''(t) \le \frac{1}{m^2}.$$

By using Remark 2 we obtain

$$0 \le \frac{1}{2M^2} \psi \left[(c - z)^2 \right] \le \psi (\ln c) + z \psi \left(c^{-1} \right) - \ln (z) - 1 \le \frac{1}{2m^2} \psi \left[(c - z)^2 \right], \tag{4.9}$$

for all $z \in (m, M)$.

In particular, we have

$$0 \le \frac{1}{2M^2} \left[\psi \left(c^2 \right) - (\psi (c))^2 \right] \le \psi (\ln c) + \psi (c) \psi \left(c^{-1} \right) - \ln (\psi (c)) - 1 \le \frac{1}{2m^2} \left[\psi \left(c^2 \right) - (\psi (c))^2 \right]. \tag{4.10}$$

We also have

$$0 \le \frac{1}{2M^2} \psi \left[\left(c - \frac{1}{\psi (c^{-1})} \right)^2 \right] \le \ln \left(\psi \left(c^{-1} \right) \right) + \psi (\ln c) \le \frac{1}{2m^2} \psi \left[\left(c - \frac{1}{\psi (c^{-1})} \right)^2 \right] \tag{4.11}$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{2M^2}\psi\left[\left(c-\frac{m+M}{2}\right)^2\right] \\
&\leq \psi(\ln c)+\frac{m+M}{2}\psi\left(c^{-1}\right)-\ln\left(\frac{m+M}{2}\right)-1 \\
&\leq \frac{1}{2m^2}\psi\left[\left(c-\frac{m+M}{2}\right)^2\right].
\end{aligned} \tag{4.12}
$$

Finally, we have

$$
\begin{aligned}
0 &\leq \frac{1}{2M^2}\left\{\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right\} \\
&\leq \psi(\ln c)+\frac{m+M}{2}\psi\left(c^{-1}\right) \\
&-\frac{M\ln M-m\ln m-M+m}{M-m}-1 \\
&\leq \frac{1}{2m^2}\left\{\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right\}
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{24}\left(1-\frac{m}{M}\right)^2 \leq \psi(\ln c)+\frac{m+M}{2}\psi\left(c^{-1}\right) \\
&-\frac{M\ln M-m\ln m-M+m}{M-m}-1 \\
&\leq \frac{1}{6}\left(\frac{M}{m}-1\right)^2.
\end{aligned} \tag{4.14}
$$

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