LOWER AND UPPER BOUNDS FOR SLATER’S GAP FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH ∗-ALGEBRAS

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Abstract. We establish in this paper some lower and upper bounds for Slater’s gap in the general setting of Hermitian unital Banach ∗-algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

1. Introduction

We need some preliminary concepts and facts about Banach ∗-algebras.

Let A be a unital Banach ∗-algebra with unit 1. An element a ∈ A is called selfadjoint if a∗ = a. A is called Hermitian if every selfadjoint element a in A has real spectrum σ(a), namely σ(a) ⊂ ℜ.

We say that an element a is nonnegative and write this as a ≥ 0 if a∗ = a and σ(a) ⊂ [0, ∞). We say that a is positive and write a > 0 if a ≥ 0 and 0 ∉ σ(a).

Thus a > 0 implies that its inverse a−1 exists. Denote the set of all invertible elements of A by Inv(A). If a, b ∈ Inv(A), then ab ∈ Inv(A) and (ab)−1 = b−1a−1.

Also, saying that a ≥ b means that a − b ≥ 0 and, similarly a > b means that a − b > 0.

The Shirali-Ford theorem asserts that if A is a unital Banach ∗-algebra [10] (see also [1, Theorem 41.5]), then

(SF) a∗a ≥ 0 for every a ∈ A.

Based on this fact, Okayasu [9], Tanahashi and Uchiyama [11] proved the following fundamental properties (see also [7]):

(i) If a, b ∈ A, then a ≥ 0, b ≥ 0 imply a + b ≥ 0 and aα ≥ 0 implies aα ≥ 0;
(ii) If a, b ∈ A, then a > 0, b ≥ 0 imply a + b > 0;
(iii) If a, b ∈ A, then either a ≥ b > 0 or a > b ≥ 0 imply a > 0;
(iv) If a > 0, then a−1 > 0;
(v) If c > 0, then 0 < b < a and only if b < a if and only if b < a = c, also 0 < b ≤ a if and only if b < c ≤ a;
(vi) If 0 < a < 1, then 1 < a−1;
(vii) If 0 < b < a, then 0 < a−1 < b−1, also if 0 < b ≤ a, then 0 < a−1 ≤ b−1.

Okayasu [9] showed that the Löwner-Heinz inequality remains valid in a Hermitian unital Banach ∗-algebra with continuous involution, namely if a, b ∈ A and p ∈ [0, 1] then a > b (a ≥ b) implies that ap > bp (ap ≥ bp).

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In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let \( a \in A \) and \( a > 0 \), then \( 0 \notin \sigma (a) \) and the fact that \( \sigma (a) \) is a compact subset of \( \mathbb{C} \) implies that \( \inf \{ z : z \in \sigma (a) \} > 0 \) and \( \sup \{ z : z \in \sigma (a) \} < \infty \). Choose \( \gamma \) to be close rectifiable curve in \( \{ \text{Re} \ z > 0 \} \), the right half open plane of the complex plane, such that \( \sigma (a) \subset \text{ins} (\gamma) \), the inside of \( \gamma \). Let \( G \) be an open subset of \( \mathbb{C} \) with \( \sigma (a) \subset G \). If \( f : G \to \mathbb{C} \) is analytic, we define an element \( f (a) \) in \( A \) by

\[
f (a) := \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a)^{-1} \, dz.
\]

It is well known (see for instance [2, pp. 201-204]) that \( f (a) \) does not depend on the choice of \( \gamma \) and the Spectral Mapping Theorem (SMT)

\[
\sigma (f (a)) = f (\sigma (a))
\]

holds.

For any \( \alpha \in \mathbb{R} \) we define for \( a \in A \) and \( a > 0 \), the real power

\[
a^{\alpha} := \frac{1}{2\pi i} \int_{\gamma} z^{\alpha} (z - a)^{-1} \, dz,
\]

where \( z^{\alpha} \) is the principal \( \alpha \)-power of \( z \). Since \( A \) is a Banach *-algebra, then \( a^{\alpha} \in A \). Moreover, since \( z^{\alpha} \) is analytic in \( \{ \text{Re} \ z > 0 \} \), then by (SMT) we have

\[
\sigma (a^{\alpha}) = (\sigma (a))^{\alpha} = \{ z^{\alpha} : z \in \sigma (a) \} \subset (0, \infty).
\]

Following [7], we list below some important properties of real powers:

(viii) If \( 0 < a \in A \) and \( \alpha \in \mathbb{R} \), then \( a^{\alpha} \in A \) with \( a^{\alpha} > 0 \) and \( (a^{2})^{1/2} = a \), [11, Lemma 6];

(ix) If \( 0 < a \in A \) and \( \alpha, \beta \in \mathbb{R} \), then \( a^{\alpha}a^{\beta} = a^{\alpha+\beta} \);

(x) If \( 0 < a \in A \) and \( \alpha \in \mathbb{R} \), then \( (a^{\alpha})^{-1} = (a^{-1})^{\alpha} = a^{-\alpha} \);

(xi) If \( 0 < a, b \in A \), \( \alpha, \beta \in \mathbb{R} \) and \( ab = ba \), then \( a^{\alpha}b^{\beta} = b^{\beta}a^{\alpha} \).

Now, assume that \( f (\cdot) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and for the real interval \( I \subset G \) assume that \( f (z) \geq 0 \) for any \( z \in I \). If \( u \in A \) such that \( \sigma (u) \subset I \), then by (SMT) we have

\[
\sigma (f (u)) = f (\sigma (u)) \subset f (I) \subset [0, \infty)
\]

meaning that \( f (u) \geq 0 \) in the order of \( A \).

Therefore, we can state the following fact that will be used to establish various inequalities in \( A \), see also [4].

**Lemma 1.** Let \( f (z) \) and \( g (z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and for the real interval \( I \subset G \), assume that \( f (z) \geq g (z) \) for any \( z \in I \). Then for any \( u \in A \) with \( \sigma (u) \subset I \) we have \( f (u) \geq g (u) \) in the order of \( A \).

**Definition 1.** Assume that \( A \) is a Hermitian unital Banach *-algebra. A linear functional \( \psi : A \to \mathbb{C} \) is positive if for \( a \geq 0 \) we have \( \psi (a) \geq 0 \). We say that it is normalized if \( \psi (1) = 1 \).

We observe that the positive linear functional \( \psi \) preserves the order relation, namely if \( a \geq b \) then \( \psi (a) \geq \psi (b) \) and if \( \beta \geq a \geq \alpha \) with \( \alpha, \beta \) real numbers, then \( \beta \geq \psi (a) \geq \alpha \).

In the recent papers [6] and [5] we obtained between others the following results:
Theorem 1. Let \( f(z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and the real interval \( I \subset G \). Assume that \( f \) is convex (in the usual sense) on the interval \( I \) and \( \psi : A \to \mathbb{C} \) is a positive normalized linear functional on \( A \) and a selfadjoint element \( c \in A \) with \( \sigma(c) \subset I \). If

\[
\psi(f'(c)) \neq 0 \quad \text{and} \quad \frac{\psi(cf'(c))}{\psi(f'(c))} \in I,
\]

then we have the Slater's type inequalities

\[
(1.1) \quad 0 \leq f \left( \frac{\psi(cf'(c))}{\psi(f'(c))} \right) - \psi(f(c)) \leq f' \left( \frac{\psi(cf'(c))}{\psi(f'(c))} \right) \left( \frac{\psi(cf'(c))}{\psi(f'(c))} - \psi(c) \right).
\]

Motivated by the above result, we establish in this paper some new lower and upper bounds for Slater's gap in the general setting of Hermitian unital Banach algebra, analytic convex functions and positive normalized linear functionals. Some examples for power function and logarithm are also provided.

2. Functional Properties

We denote by \( \Phi_1(A) \) the set of all linear, positive functionals defined on \( A \) with the property that, if \( \varphi \in \Phi_1(A) \), then \( \varphi(1) > 0 \). If \( \varphi, \omega \in \Phi_1(A) \) then \( \varphi + \omega \in \Phi_1(A) \) and for all \( \alpha > 0 \) we have \( \alpha \varphi \in \Phi_1(A) \).

For a convex function \( f \) as above and a selfadjoint element \( c \) with \( \sigma(c) \subset I \) we consider the subset of Slater’s functionals

\[
\Phi_{f,c,1,+,}(A) := \{ \varphi \in \Phi_1(A) \mid \varphi(f'(c)) > 0 \text{ and } \frac{\varphi(cf'(c))}{\varphi(f'(c))} \in I \}.
\]

Observe that, if \( \varphi, \omega \in \Phi_{f,c,1,+,}(A) \), then \( \varphi(f'(c)) > 0 \), \( \omega(f'(c)) > 0 \), and

\[
\frac{\varphi(cf'(c))}{\varphi(f'(c))}, \frac{\omega(cf'(c))}{\omega(f'(c))} \in I. \quad \text{Therefore } (\varphi + \omega)(f'(c)) > 0 \text{ and }
\]

\[
\frac{(\varphi + \omega)(cf'(c))}{(\varphi + \omega)(f'(c))} = \frac{\varphi(cf'(c)) + \omega(cf'(c))}{\varphi(f'(c)) + \omega(f'(c))}
\]

\[
= \frac{\varphi(f'(c)) \frac{\varphi(cf'(c))}{\varphi(f'(c))} + \omega(f'(c)) \frac{\omega(cf'(c))}{\omega(f'(c))}}{\varphi(f'(c)) + \omega(f'(c))} \in I
\]

since the interval \( I \) is a convex set.

We conclude that \( \varphi + \omega \in \Phi_{f,c,1,+,}(A) \) and also \( \alpha \varphi \in \Phi_{f,c,1,+,}(A) \) for all \( \alpha > 0 \).

We define the functional \( \mathcal{S} : \Phi_{f,c,1,+,}(A) \to \mathbb{R} \) by

\[
(2.1) \quad \mathcal{S}(f,c,\varphi) := f \left( \frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \varphi(f'(c)).
\]

Theorem 2. Let \( f(z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and the real interval \( I \subset G \). Assume that \( f \) is convex on the interval \( I \) and \( c \in A \) is a selfadjoint element with \( \sigma(c) \subset I \). If \( \varphi, \omega \in \Phi_{f,c,1,+,}(A) \), then

\[
(2.2) \quad \mathcal{S}(f,c,\varphi + \omega) \leq \mathcal{S}(f,c,\varphi) + \mathcal{S}(f,c,\omega),
\]

namely \( \mathcal{S}(f,c,\cdot) \) is subadditive on \( \Phi_{f,c,1,+,}(A) \). It is also positive homogeneous on \( \Phi_{f,c,1,+,}(A) \).
Proof. If $\varphi, \omega \in \mathfrak{P}_{f,c,1,+} (A)$, then $\varphi + \omega \in \mathfrak{P}_{f,c,1,+} (A)$ and

$$
S (f, c, \varphi + \omega) = f \left( \frac{(\varphi + \omega) (cf') (c)}{(\varphi + \omega) (f') (c)} \right) (\varphi + \omega) (f') (c) 
$$

$$
= f \left( \frac{\varphi (f') (c)}{\varphi (f') (c)} \frac{\omega (cf') (c)}{\omega (f') (c)} + \omega (f') (c) \right) (\varphi + \omega) (f') (c) 
$$

$$
\leq \left[ \varphi (f') (c) f \left( \frac{\varphi (cf') (c)}{\varphi (f') (c)} \right) + \omega (f') (c) f \left( \frac{\omega (cf') (c)}{\omega (f') (c)} \right) \right] (\varphi + \omega) (f') (c) 
$$

$$
= \varphi (f') (c) f \left( \frac{\varphi (cf') (c)}{\varphi (f') (c)} \right) + \omega (f') (c) f \left( \frac{\omega (cf') (c)}{\omega (f') (c)} \right) 
$$

$$
= S (f, c, \varphi) + S (f, c, \omega), 
$$

which proves (2.2).

If $\alpha > 0$ and $\varphi \in \mathfrak{P}_{f,c,1,+} (A)$, then $\alpha \varphi \in \mathfrak{P}_{f,c,1,+} (A)$ and

$$
S (f, c, \alpha \varphi) := f \left( \frac{(\alpha \varphi) (cf') (c)}{(\alpha \varphi) (f') (c)} \right) (\alpha \varphi) (f') (c) = \alpha S (f, c, \varphi). 
$$

\[\square\]

Let $X$ be a linear space. A subset $C \subseteq X$ is called a **convex cone** in $X$ provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;

(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is called **subadditive** on $C$ if

(iii) $h (x + y) \leq h (x) + h (y)$ for any $x, y \in C$

and **nonnegative (strictly positive)** on $C$ if, it satisfies

(iv) $h (x) \geq (>) 0$ for each $x \in C$.

The functional $h$ is **s-positive homogeneous** on $C$, for a given $s > 0$, if

(v) $h (\alpha x) = \alpha^s h (x)$ for any $\alpha \geq 0$ and $x \in C$.

In the paper [3] we obtained the following results for superadditive functions:

**Lemma 2.** Let $C$ be a convex cone in the linear space $X$ and $v : C \to (0, \infty)$ an additive functional on $C$. If $h : C \to [0, \infty)$ is a subadditive functional on $C$ and $0 < p < 1$ then the functional

$$(2.3) \quad \Psi_p : C \to [0, \infty), \Psi_p (x) = v^{1 - \frac{1}{p}} (x) h (x)$$

is subadditive on $C$.

**Corollary 1.** Assume that $X, C$ and $v$ are as in Theorem 2. If $h : C \to [0, \infty)$ is a subadditive functional on $C$ and $0 < p, q < 1$ then the two parameters functional

$$(2.4) \quad \Psi_{p,q} : C \to [0, \infty), \Psi_{p,q} (x) = v^{q (1 - \frac{1}{p})} (x) h^q (x)$$

is subadditive on $C$.

**Remark 1.** If we consider the functional $\psi_p (x) := v^{p-1} (x) h^p (x)$ then for $0 < p < 1$ and $h : C \to [0, \infty)$ a subadditive functional on $C$, the functional $\psi_p$ is also subadditive on $C$. 

Corollary 2. Let \( f(z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and the real interval \( I \subset G \). Assume that \( f \) is convex and positive on the interval \( I \) and \( c \in A \) is a selfadjoint element with \( \sigma(c) \subset I \). For \( 0 < p, q < 1 \) we define the functional \( S_{p,q}(f,c,\cdot): \mathfrak{P}_{f,c,1,+}(A) \to [0,\infty) \) by

\[
S_{p,q}(f,c,\varphi) := \left[ f \left( \frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^q [\varphi(f'(c))]^{q(2-\frac{1}{p})}.
\]

Then \( S_{p,q}(f,c,\cdot) \) is subadditive on \( \mathfrak{P}_{f,c,1,+}(A) \).

The proof follows by Corollary 1 on choosing

\[
v(\varphi) = \varphi(f'(c)) \quad \text{and} \quad h(\varphi) = S(f,c,\varphi) := f \left( \frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \varphi(f'(c)).
\]

We also observe that for \( 0 < p < 1 \),

\[
S_{p,p}(f,c,\varphi) := \left[ f \left( \frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^p [\varphi(f'(c))]^{p(2-\frac{1}{p})}
\]

and for \( 0 < q < 1 \),

\[
S_{1/2,q}(f,c,\varphi) := \left[ f \left( \frac{\varphi(cf'(c))}{\varphi(f'(c))} \right) \right]^q
\]

are subadditive on \( \mathfrak{P}_{f,c,1,+}(A) \).

3. Lower and Upper Bounds

We have:

**Theorem 3.** Let \( f(z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and convex on the real interval \( I \subset G \) and \( \psi: A \to \mathbb{C} \) is a positive normalized linear functional on \( A \). For any selfadjoint element \( c \in A \) with \( \sigma(c) \subset I \),

\[
0 \leq \left( \inf_{w \in I} \int_0^1 f'' \left( (1-s)w + sz \right) (1-s) ds \right) \psi \left( c - z \right)^2 \leq f(z) - \psi(f(c)) - z\psi(f'(c)) + \psi(cf'(c)) \\
\leq \left( \sup_{w \in I} \int_0^1 f'' \left( (1-s)w + sz \right) (1-s) ds \right) \psi \left( c - z \right)^2,
\]

for all \( z \in I \).

In particular, we have

\[
0 \leq \left( \inf_{w \in I} \int_0^1 f'' \left( (1-s)w + s\psi(c) \right) (1-s) ds \right) \left[ \psi(c^2) - \psi(c)^2 \right] \leq f(\psi(c)) - \psi(f(c)) - \psi(c) \psi(f'(c)) + \psi(cf'(c)) \\
\leq \left( \sup_{w \in I} \int_0^1 f'' \left( (1-s)w + s\psi(c) \right) (1-s) ds \right) \left[ \psi(c^2) - \psi(c)^2 \right].
\]

**Proof.** Using Taylor’s representation with the integral remainder we can write the following identity

\[
f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t) (z-t)^k + \frac{1}{n!} \int_t^z f^{(n+1)}(s) (z-s)^n ds
\]

for any \( z, t \in I \), the interior of \( I \).
For any integrable function $h$ on an interval and any distinct numbers $c$, $d$ in that interval, we have, by the change of variable $s = (1 - s)c + sd$, $s \in [0, 1]$ that
\[
\int_c^d h(s)\,ds = (d - c) \int_0^1 h((1 - s)c + sd)\,ds.
\]
Therefore,
\[
\int_t^z f^{(n+1)}(s)(z - s)^n\,ds
= (z - t) \int_0^1 f^{(n+1)}((1 - s)t + sz)(z - (1 - s)t - sz)^n\,ds
= (z - t)^{n+1} \int_0^1 f^{(n+1)}((1 - s)t + sz)(1 - s)^n\,ds.
\]
The identity (3.3) can then be written as
\[
f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(z - t)^k
+ \frac{1}{n!} (z - t)^{n+1} \int_0^1 f^{(n+1)}((1 - s)t + sz)(1 - s)^n\,ds.
\]
For $n = 1$ we get
\[
f(z) = f(w) + (z - w)f'(w) + (z - w)^2 \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds
\]
for any $z, w \in \bar{I}$.

Since
\[
0 \leq \inf_{w \in I} \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds \leq \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds
\leq \sup_{w \in I} \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds,
\]
for any $z, w \in \bar{I}$.

Fix $z \in I$. Using Lemma 1 and the inequality (3.7) we obtain for the element $c \in A$ with $\sigma(c) \subset I$ the following inequality in the order of $A$
\[
0 \leq \left(\inf_{w \in I} \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds\right)(z - w)^2
\leq f(z) - f(w) - (z - w)f'(w)
\leq \left(\sup_{w \in I} \int_0^1 f''((1 - s)w + sz)(1 - s)\,ds\right)(z - w)^2,
\]
for any $z, w \in \bar{I}$.
for any $z \in \hat{I}$.

If we take in this inequality the functional $\psi$ we get (3.1).

If we take in (3.1) $z = \psi(c)$, then we get

\[
\inf_{w \in I} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\, ds \psi[(z - \psi(c))^2] 
\leq f(z) - \psi(f(c)) - z\psi(f'(c)) + \psi(cf'(c))
\leq \left( \sup_{w \in I} \int_0^1 f''((1-s)w + s\psi(c))(1-s)\, ds \right) \psi[(z - \psi(c))^2].
\]

Since

\[
\psi[(c - \psi(c))^2] = \psi\left(c^2 - 2\psi(c)c + (\psi(c))^2\right)
= \psi(c^2) - 2(\psi(c))^2 + (\psi(c))^2 = \psi(c^2) - (\psi(c))^2,
\]

hence by (3.7) we get (3.2).

**Corollary 3.** With the assumptions of Theorem 3 and, if, in addition, $\psi(f'(c)) \neq 0$ with $z = \frac{\psi(cf'(c))}{\psi(f'(c))} \in \hat{I}$, then we have the refinement and reverse of Slater’s inequality

\[
0 \leq \left( \inf_{w \in I} \int_0^1 f''\left((1-s)w + s\frac{\psi(cf'(c))}{\psi(f'(c))}\right)(1-s)\, ds \right) 
\times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right]
\leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c))
\leq \left( \sup_{w \in I} \int_0^1 f''\left((1-s)w + s\frac{\psi(cf'(c))}{\psi(f'(c))}\right)(1-s)\, ds \right) 
\times \psi\left[\left(c - \frac{\psi(cf'(c))}{\psi(f'(c))}\right)^2\right].
\]

**Corollary 4.** For any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m, M] \subset I$, we have

\[
0 \leq \left( \inf_{w \in I} \int_0^1 f''\left((1-s)w + s\frac{m+M}{2}\right)(1-s)\, ds \right) \psi\left[\left(c - \frac{m+M}{2}\right)^2\right]
\leq f\left(\frac{m+M}{2}\right) - \psi(f(c)) - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c))
\leq \left( \sup_{w \in I} \int_0^1 f''\left((1-s)w + s\frac{m+M}{2}\right)(1-s)\, ds \right) \psi\left[\left(c - \frac{m+M}{2}\right)^2\right].
\]

We also have:

**Theorem 4.** Let $f(z)$ be analytic in $G$, an open subset of $\mathbb{C}$ and convex on the real interval $I \subset G$ and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on $A$. 
For any selfadjoint element \( c \in A \) with \( \sigma(c) \subseteq [m, M] \subseteq I \), we have

\[
0 \leq \left( \inf_{(w, z) \in I \times I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) \times \left\{ \frac{1}{12} (M - m)^2 + \psi' \left( \frac{c - m + M}{2} \right)^2 \right\}
\]

\[
\leq \frac{1}{M - m} \int_m^M f(z) \, dx - \psi(f(c)) - \frac{m + M}{2} \psi'(f(c)) + \psi(cf'(c))
\]

\[
\leq \left( \sup_{(w, z) \in I \times I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) \times \left\{ \frac{1}{12} (M - m)^2 + \psi' \left( \frac{c - m + M}{2} \right)^2 \right\}.
\]

**Proof.** If we take the integral mean in (3.6) over \( z \in [m, M] \), we get

\[
0 \leq \frac{1}{M - m} \int_m^M \left( \inf_{w \in I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) (z - w)^2 \, dz
\]

\[
\leq \frac{1}{M - m} \int_m^M f(z) \, dx - f(w) - \left( \frac{m + M}{2} - w \right) f'(w)
\]

\[
\leq \frac{1}{M - m} \int_m^M \left( \sup_{w \in I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) (z - w)^2 \, dz,
\]

for all \( w \in I \).

Since

\[
\left( \inf_{(w, z) \in I \times I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) \frac{1}{M - m} \int_m^M (z - w)^2 \, dz
\]

\[
\leq \frac{1}{M - m} \int_m^M \left( \inf_{w \in I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) (z - w)^2 \, dz
\]

and

\[
\frac{1}{M - m} \int_m^M \left( \sup_{w \in I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) (z - w)^2 \, dz
\]

\[
\leq \left( \sup_{(w, z) \in I \times I} \int_0^1 f'' \left( (1 - s) w + sz \right) (1 - s) ds \right) \frac{1}{M - m} \int_m^M (z - w)^2 \, dz
\]

for all \( w \in I \).
Hence

\[
\begin{align*}
(3.12) & \quad \left( \inf_{(w,z) \in \bar{I} \times I} \int_0^1 f'''((1-s)w + sz)(1-s) \, ds \right) \frac{1}{M-m} \int_m^M (z-w)^2 \, dz \\
& \leq \frac{1}{M-m} \int_m^M f(z) \, dx - f(w) - \left( \frac{m+M}{2} - w \right) f'(w) \\
& \leq \left( \sup_{(w,z) \in \bar{I} \times I} \int_0^1 f'''((1-s)w + sz)(1-s) \, ds \right) \frac{1}{M-m} \int_m^M (z-w)^2 \, dz
\end{align*}
\]

for all \( w \in \bar{I} \).

Observe that

\[
\frac{1}{M-m} \int_m^M (z-w)^2 \, dz = \frac{(M-w)^3 + (w-m)^3}{3(M-m)}
\]

\[
= \frac{1}{3} \left[ (w-m)^2 + (M-w)^2 - (w-m)(M-w) \right]
\]

\[
= \frac{1}{3} \left[ \frac{1}{4} (M-m)^2 + 3 \left( w - \frac{m+M}{2} \right)^2 \right]
\]

\[
= \frac{1}{12} (M-m)^2 + \left( w - \frac{m+M}{2} \right)^2
\]

for all \( w \in [m,M] \).

From (3.12) we then get

\[
(3.13) \quad \left( \inf_{(w,z) \in \bar{I} \times I} \int_0^1 f'''((1-s)w + sz)(1-s) \, ds \right)
\]

\[
\times \left[ \frac{1}{12} (M-m)^2 + \left( w - \frac{m+M}{2} \right)^2 \right]
\]

\[
\leq \frac{1}{M-m} \int_m^M f(z) \, dx - f(w) - \left( \frac{m+M}{2} - w \right) f'(w)
\]

\[
\leq \left( \sup_{(w,z) \in \bar{I} \times I} \int_0^1 f'''((1-s)w + sz)(1-s) \, ds \right)
\]

\[
\times \left[ \frac{1}{12} (M-m)^2 + \left( w - \frac{m+M}{2} \right)^2 \right]
\]

for all \( w \in [m,M] \).
Using Lemma 1 and the inequality (3.13) we obtain for the element \( c \in A \) with \( \sigma (c) \subset I \) the following inequality in the order of \( A \)

\[
\begin{align*}
(3.14) \quad & \left( \inf_{(w,z) \in I \times I} \int_0^1 f'' ((1-s) w + sz) (1-s) \, ds \right) \\
& \times \left[ \frac{1}{12} (M - m)^2 + \left( c - \frac{m + M}{2} \right)^2 \right] \\
& \leq \frac{1}{M - m} \int_m^M f (z) \, dx - f (c) - \frac{m + M}{2} f' (c) + cf' (c) \\
& \leq \left( \sup_{(w,z) \in I \times I} \int_0^1 f'' ((1-s) w + sz) (1-s) \, ds \right) \\
& \times \left[ \frac{1}{12} (M - m)^2 + \left( c - \frac{m + M}{2} \right)^2 \right].
\end{align*}
\]

If we take the functional \( \psi \) on (3.14), then we get (3.10).

\[ \square \]

**Corollary 5.** With the assumptions of Theorem 4 we have

\[
(3.15) \quad 0 \leq \frac{1}{12} (M - m)^2 \left( \inf_{(w,z) \in I \times I} \int_0^1 f'' ((1-s) w + sz) (1-s) \, ds \right) \\
\leq \frac{1}{M - m} \int_m^M f (z) \, dx - f (c) - \frac{m + M}{2} \psi (f' (c)) + \psi (cf' (c)) \\
\leq \frac{1}{3} (M - m)^2 \left( \sup_{(w,z) \in I \times I} \int_0^1 f'' ((1-s) w + sz) (1-s) \, ds \right).
\]

The proof follows by (3.10) on observing that

\[
0 \leq \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right] \leq \frac{1}{4} (M - m)^2.
\]

**Remark 2.** If there exists the constants \( 0 < \gamma < \Gamma < \infty \) such that \( \gamma \leq f'' (x) \leq \Gamma \) for almost every \( x \in \bar{I} \), then by (3.1) and (3.2) we get

\[
(3.16) \quad 0 \leq \frac{1}{2} \gamma \psi \left[ (c - z)^2 \right] \leq f (z) - f (c) - z \psi (f' (c)) + \psi (cf' (c)) \\
\leq \frac{1}{2} \Gamma \psi \left[ (c - z)^2 \right],
\]

for all \( z \in \bar{I} \).

In particular, we have

\[
(3.17) \quad 0 \leq \frac{1}{2} \gamma \left[ \psi (c^2) - (\psi (c))^2 \right] \\
\leq f (\psi (c)) - \psi (f (c)) - \psi (c) \psi (f' (c)) + \psi (cf' (c)) \\
\leq \frac{1}{2} \Gamma \left[ \psi (c^2) - (\psi (c))^2 \right].
\]
From (3.8) and (3.9) we have

\[
(3.18) \quad 0 \leq \frac{1}{2} \gamma \psi \left[ \left( \frac{c - \psi (cf' (c))}{\psi (f' (c))} \right)^2 \right] \leq f \left( \frac{\psi (cf' (c))}{\psi (f' (c))} \right) - \psi (f (c))
\]

and

\[
(3.19) \quad 0 \leq \frac{1}{2} \Gamma \psi \left[ \left( \frac{c - cf' (c)}{f' (c)} \right)^2 \right]
\]

\[
\leq f \left( \frac{m + M}{2} \right) - \psi (f (c)) - \frac{m + M}{2} \psi (f' (c)) + \psi (cf' (c))
\]

\[
\leq \frac{1}{2} \Gamma \psi \left[ \left( \frac{c - m + M}{2} \right)^2 \right].
\]

Also, by (3.10) and (3.15) we get

\[
(3.20) \quad 0 \leq \frac{1}{2} \gamma \left\{ \frac{1}{12} (M - m)^2 + \psi \left[ \left( \frac{c - m + M}{2} \right)^2 \right] \right\}
\]

\[
\leq \frac{1}{M - m} \int_m^M f (z) \, dx - \psi (f (c)) - \frac{m + M}{2} \psi (f' (c)) + \psi (cf' (c))
\]

\[
\leq \frac{1}{2} \Gamma \left\{ \frac{1}{12} (M - m)^2 + \psi \left[ \left( \frac{c - m + M}{2} \right)^2 \right] \right\}
\]

and

\[
(3.21) \quad 0 \leq \frac{1}{24} \gamma (M - m)^2
\]

\[
\leq \frac{1}{M - m} \int_m^M f (z) \, dx - \psi (f (c)) - \frac{m + M}{2} \psi (f' (c)) + \psi (cf' (c))
\]

\[
\leq \frac{1}{6} \Gamma (M - m)^2.
\]

We also have:

**Theorem 5.** Let \( f \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and convex on the real
interval \( I \subset G \), \( \psi : A \to \mathbb{C} \) be a positive normalized linear functional on \( A \) and
\( c \in A \) a selfadjoint element with \( \sigma (c) \subset I \). If \( f'' \) is monotonic nondecreasing on
\( [m, M] \subset I \), then

\[
(3.22) \quad \frac{1}{z - m} \left\{ \frac{f (z) - f (m)}{z - m} - f' (m) \right\} \psi \left[ (c - z)^2 \right]
\]

\[
\leq f (z) - f (c) - (z - c) f' (c)
\]

\[
\leq \frac{1}{M - z} \left\{ \frac{f (M) - f (z)}{M - z} - f' (z) \right\} \psi \left[ (c - z)^2 \right]
\]

for \( z \in (m, M) \).
Using this equality, we have

\begin{equation}
0 \leq \frac{1}{M - z} \left\{ \frac{f(M) - f(z)}{M - z} - f'(z) \right\} \psi \left[ (c - z)^2 \right]
\end{equation}

and

\begin{equation}
0 \leq f(z) - f(c) - (z - c) f'(c)
\end{equation}

for \( z \in (m, M) \).

**Proof.** If \( f'' \) is monotonic nonincreasing on \([m, M] \subset \hat{I} \), then

\begin{equation}
f(z) = f(w) + (z - w) f'(w) + (z - w)^2 \int_0^1 f''((1 - s) w + sz) (1 - s) \, ds
\end{equation}

and

\begin{equation}
f(z) = f(w) + (z - w) f'(w) + (z - w)^2 \int_0^1 f''((1 - s) w + sz) (1 - s) \, ds
\end{equation}

First, observe that for \( u, v \in [m, M] \) with \( u \neq v \) we have

\[
\int_0^1 f''((1 - s) v + su) (1 - s) \, ds
\]

\[
= \frac{1}{u - v} \int_0^1 (1 - s) d \left( f'((1 - s) v + su) \right)
\]

\[
= \frac{1}{u - v} \left[ (1 - s) f'((1 - s) v + su) \right]_0^1 + \int_0^1 f'((1 - s) v + su) \, ds
\]

\[
= \frac{1}{u - v} \left\{ -f'(v) + \int_0^1 f'((1 - s) v + su) \, ds \right\}
\]

\[
= \frac{1}{v - u} \left\{ f'(v) - \int_0^1 f'((1 - s) v + su) \, ds \right\}
\]

\[
= \frac{1}{v - u} \left\{ f'(v) - \frac{f(v) - f(u)}{v - u} \right\}.
\]

Using this equality, we have

\[
\int_0^1 f''((1 - s) m + sz) (1 - s) \, ds = \frac{1}{m - z} \left\{ f'(m) - \frac{f(m) - f(z)}{w - z} \right\}
\]

\[
= \frac{1}{z - m} \left\{ \frac{f(z) - f(m)}{z - m} - f'(m) \right\}
\]

and

\[
\int_0^1 f''((1 - s) M + sz) (1 - s) \, ds = \frac{1}{M - z} \left\{ \frac{f(M) - f(z)}{M - z} - f'(z) \right\}.
\]
Then by (3.24) and (3.25) we get

\[
\frac{1}{z-m} \left \{ f(z) - f(m) \right \} \frac{z-m}{(z-w)^2} \]
\[
\leq f(z) - f(w) - (z-w)f'(w) \]
\[
\leq \frac{1}{M-z} \left \{ f(M) - f(z) \right \} \frac{M-z}{(z-w)^2} \]

for all \( w \in (m,M) \) and \( z \in I \).

Fix \( z \in (m,M) \). Using Lemma 1 and the inequality (3.26) we obtain for the element \( c \in A \) with \( \sigma(c) \subset (m,M) \) the following inequality in the order of \( A \)

\[
\frac{1}{z-m} \left \{ f(z) - f(m) \right \} \frac{z-m}{(z-c)^2} \]
\[
\leq f(z) - f(c) - (z-c)f'(c) \]
\[
\leq \frac{1}{M-z} \left \{ f(M) - f(z) \right \} \frac{M-z}{(z-c)^2} \]

for all \( z \in (m,M) \).

Taking the functional \( \psi \) in the inequality (3.27) we get (3.22).

\[\square\]

**Corollary 6.** With the assumptions of Theorem 5 and if \( \psi(c) \in (m,M) \) then

\[
\frac{1}{\psi(c)-m} \left \{ f(\psi(c)) - f(m) \right \} \frac{\psi(c)-m}{(\psi(c))^2} \]
\[
\leq f(\psi(c)) - f(c) - (\psi(c)-c)f'(c) \]
\[
\leq \frac{1}{M-\psi(c)} \left \{ f(M) - f(\psi(c)) \right \} \frac{M-\psi(c)}{(\psi(c))^2} \]

provided \( f'' \) is monotonic nondecreasing on \([m,M]\) and

\[
0 \leq \frac{1}{\psi(c)-m} \left \{ f(M) - f(\psi(c)) \right \} \frac{M-\psi(c)}{\psi(c)-m} \]
\[
\leq f(\psi(c)) - f(c) - (\psi(c)-c)f'(c) \]
\[
\leq \frac{1}{\psi(c)-m} \left \{ f(\psi(c)) - f(m) \right \} \frac{\psi(c)-m}{(\psi(c))^2} \]

provided \( f'' \) is monotonic nonincreasing on \([m,M]\).
Corollary 7. With the assumptions of Theorem 5 and if \( \frac{\psi(cf'(c))}{\psi(f'(c))} \in (m, M) \) with \( \psi(f'(c)) \neq 0 \) then

\[
(3.30) \quad \frac{1}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} \left\{ \frac{f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} - f'(m) \right\}
\]

\[
\times \psi \left[ \left( c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right]
\]

\[
\leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c))
\]

\[
\leq \frac{1}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} \left\{ \frac{f(M) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} - f'(\frac{\psi(cf'(c))}{\psi(f'(c))}) \right\}
\]

\[
\times \psi \left[ \left( c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right],
\]

provided \( f'' \) is monotonic nondecreasing on \([m, M]\) and

\[
(3.31) \quad 0 \leq \frac{1}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} \left\{ \frac{f(M) - f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{M - \frac{\psi(cf'(c))}{\psi(f'(c))}} - f'(\frac{\psi(cf'(c))}{\psi(f'(c))}) \right\}
\]

\[
\times \psi \left[ \left( c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right]
\]

\[
\leq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) - \psi(f(c))
\]

\[
\leq \frac{1}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} \left\{ \frac{f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)}{\frac{\psi(cf'(c))}{\psi(f'(c))} - m} - f'(m) \right\}
\]

\[
\times \psi \left[ \left( c - \frac{\psi(cf'(c))}{\psi(f'(c))} \right)^2 \right],
\]

provided \( f'' \) is monotonic nonincreasing on \([m, M]\).

4. Some Examples

In this section we provide some simple inequalities that can be derived from the above results by taking particular examples of convex functions such as: power function, exponential and logarithm. They generalize several known results obtained for selfadjoint operators in Hilbert spaces.

For \( p \geq 1 \), consider the power function \( f_p : (0, \infty) \to (0, \infty) \), \( f_p(x) = x^p \) which is analytic and convex on \((0, \infty)\) and \( 0 < c \in A \).
We define the functional $S : \mathfrak{F}_{p,c,1,+}(A) \to \mathbb{R}$ by
\begin{equation}
S(f_p, c, \varphi) := [\varphi(c f'(c))]^p [\varphi(f'(c))]^{1-p}.
\end{equation}

By Theorem 2 we have that $S(f_p, c, \cdot)$ is subadditive and positive homogeneous on $\mathfrak{F}_{p,c,1,+}(A)$.

Since $f''_p(t) = p(p-1)t^{p-2}$, $t > 0$ then
\begin{equation}
k_p := p(p-1) \begin{cases} 
M^{p-2} & \text{for } p \in (1, 2) \\
m^{p-2} & \text{for } p \in [2, \infty) 
\end{cases}
\end{equation}

\begin{equation}
\leq f''_p(t) \leq \begin{cases} 
M^{p-2} & \text{for } p \in (1, 2) \\
m^{p-2} & \text{for } p \in [2, \infty) 
\end{cases}
\end{equation}

for any $t \in [m, M]$.

From Remark 2 we get
\begin{equation}
0 \leq \frac{1}{2} k_p \psi \left[ (c - z)^2 \right] \leq z^p - p z \psi(c^{p-1}) + (p - 1) \psi(c^p)
\end{equation}
\begin{equation}
\leq \frac{1}{2} K_p \psi \left[ (c - z)^2 \right],
\end{equation}

for all $z \in \mathfrak{I}$.

In particular, we have
\begin{equation}
0 \leq \frac{1}{2} k_p \left[ \psi(c^2) - (\psi(c))^2 \right] \leq (\psi(c))^p - p \psi(c) \psi(c^{p-1}) + (p - 1) \psi(c^p)
\end{equation}
\begin{equation}
\leq \frac{1}{2} K_p \left[ \psi(c^2) - (\psi(c))^2 \right].
\end{equation}

Also we have
\begin{equation}
0 \leq \frac{1}{2} k_p \psi \left[ \left( c - \frac{\psi(c^p)}{\psi(c^{p-1})} \right)^2 \right] \leq \left( \frac{\psi(c^p)}{\psi(c^{p-1})} \right)^p - \psi(c^p)
\end{equation}
\begin{equation}
\leq \frac{1}{2} K_p \psi \left[ \left( c - \frac{\psi(c^p)}{\psi(c^{p-1})} \right)^2 \right]
\end{equation}

and
\begin{equation}
0 \leq \frac{1}{2} k_p \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right]
\end{equation}
\begin{equation}
\leq \left( \frac{m + M}{2} \right)^p - p \frac{m + M}{2} \psi(c^{p-1}) + (p - 1) \psi(c^p)
\end{equation}
\begin{equation}
\leq \frac{1}{8} K_p (M - m)^2.
\end{equation}
Moreover, we get

(4.7) \[ 0 \leq \frac{1}{2} k_p \left\{ \frac{1}{12} (M-m)^2 + \psi \left[ \left( c - \frac{m+M}{2} \right)^2 \right] \right\} \]

\[ \leq \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} - \frac{m+M}{2} \psi \left( e^{p-1} \right) + (p-1) \psi \left( e^p \right) \]

\[ \leq \frac{1}{2} K_p \left\{ \frac{1}{12} (M-m)^2 + \psi \left[ \left( c - \frac{m+M}{2} \right)^2 \right] \right\} \]

and

(4.8) \[ 0 \leq \frac{1}{24} k_p (M-m)^2 \]

\[ \leq \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} - \frac{m+M}{2} \psi \left( e^{p-1} \right) + (p-1) \psi \left( e^p \right) \]

\[ \leq \frac{1}{6} K_p (M-m)^2. \]

The case of logarithmic function is also of interest. We take the function \( f(t) = -\ln t \) and \( 0 < c \in A \). Define

\[ S(-\ln c, \varphi) := \ln \left( \frac{\varphi(1)}{\varphi(c^{-1})} \right) \varphi \left( c^{-1} \right). \]

By Theorem 2 we observe that \( S(-\ln, \cdot) \) is subadditive and positive homogeneous on \( \mathcal{P}_{-\ln, c, 1, +}(A) \).

Moreover, we have \( f''(t) = \frac{1}{t^2} \). Therefore for \( t \in [m, M] \subset (0, \infty) \) we get

\[ \frac{1}{M^2} \leq f''(t) \leq \frac{1}{m^2}. \]

By using Remark 2 we obtain

(4.9) \[ 0 \leq \frac{1}{2M^2} \psi \left[ (c-z)^2 \right] \leq \psi (\ln c) + z \psi \left( c^{-1} \right) - \ln (z) - 1 \]

\[ \leq \frac{1}{2m^2} \psi \left[ (c-z)^2 \right], \]

for all \( z \in (m, M) \).

In particular, we have

(4.10) \[ 0 \leq \frac{1}{2M^2} \left[ \psi \left( c^2 \right) - (\psi(c))^2 \right] \]

\[ \leq \psi (\ln c) + \psi (c) \psi \left( c^{-1} \right) - \ln (\psi (c)) - 1 \]

\[ \leq \frac{1}{2m^2} \left[ \psi \left( c^2 \right) - (\psi(c))^2 \right]. \]

We also have

(4.11) \[ 0 \leq \frac{1}{2M^2} \psi \left[ \left( c - \frac{1}{\psi(c^{-1})} \right)^2 \right] \leq \ln \left( \psi \left( c^{-1} \right) \right) + \psi (\ln c) \]

\[ \leq \frac{1}{2m^2} \psi \left[ \left( c - \frac{1}{\psi(c^{-1})} \right)^2 \right]. \]
and

\begin{equation}
0 \leq \frac{1}{2M^2} \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right] \\
\leq \psi \left( \ln c \right) + \frac{m + M}{2} \psi \left( c^{-1} \right) - \ln \left( \frac{m + M}{2} \right) - 1 \\
\leq \frac{1}{2m^2} \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right].
\end{equation}

Finally, we have

\begin{equation}
0 \leq \frac{1}{2M^2} \left\{ \frac{1}{12} (M - m)^2 + \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right] \right\} \\
\leq \psi \left( \ln c \right) + \frac{m + M}{2} \psi \left( c^{-1} \right) \\
- \frac{M \ln M - m \ln m - M + m}{M - m} - 1 \\
\leq \frac{1}{2m^2} \left\{ \frac{1}{12} (M - m)^2 + \psi \left[ \left( c - \frac{m + M}{2} \right)^2 \right] \right\}
\end{equation}

and

\begin{equation}
0 \leq \frac{1}{2M} \left( \frac{1 - m}{M} \right)^2 \leq \psi \left( \ln c \right) + \frac{m + M}{2} \psi \left( c^{-1} \right) \\
- \frac{M \ln M - m \ln m - M + m}{M - m} - 1 \\
\leq \frac{1}{6} \left( \frac{M}{m} - 1 \right)^2.
\end{equation}

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