INEQUALITIES OF HERMITE-HADAMARD TYPE FOR OPERATOR CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH *-ALGEBRAS

SILVESTRU SEVER DRAGOMIR

Abstract. We establish in this paper some inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach *-algebras.

1. Introduction

We need some preliminary concepts and facts about Banach *-algebras.

Let $A$ be a unital Banach *-algebra with unit 1. An element $a \in A$ is called selfadjoint if $a^* = a$. $A$ is called Hermitian if every selfadjoint element $a$ in $A$ has real spectrum $\sigma(a)$, namely $\sigma(a) \subseteq \mathbb{R}$.

We say that an element $a$ is nonnegative and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subseteq [0, \infty)$. We say that $a$ is positive and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$.

Thus $a > 0$ implies that its inverse $a^{-1}$ exists. Denote the set of all invertible elements of $A$ by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Based on this fact, Okayasu [8], Tanahashi and Uchiyama [10] proved the following fundamental properties (see also [6]):

(i) If $a, b \in A$, then $a \geq 0, b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0, b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < c < a$ if and only if $c bc < cac$, also $0 < b \leq a$ if and only if $c bc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [8] showed that the L"{o}wner-Heinz inequality remains valid in a Hermitian unital Banach *-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ ($a \geq b$) implies that $a^p > b^p$ ($a^p \geq b^p$).

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

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Let $a \in A$ and $a > 0$, then $0 \notin \sigma (a)$ and the fact that $\sigma (a)$ is a compact subset of $\mathbb{C}$ implies that $\inf \{ z : z \in \sigma (a) \} > 0$ and $\sup \{ z : z \in \sigma (a) \} < \infty$. Choose $\gamma$ to be close rectifiable curve in $\{ \Re z > 0 \}$, the right half open plane of the complex plane, such that $\sigma (a) \subseteq \text{ins} (\gamma)$, the inside of $\gamma$. Let $G$ be an open subset of $\mathbb{C}$ with $\sigma (a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f (a)$ in $A$ by

$$f (a) := \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a)^{-1} \, dz,$$

where $\gamma$ is a close rectifiable curve such that $\sigma (a) \subseteq \text{ins} (\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f (a)$ does not depend on the choice of $\gamma$ and the Spectral Mapping Theorem (SMT)

$$\sigma (f (a)) = f (\sigma (a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z - a)^{-1} \, dz,$$

where $z^\alpha$ is the principal $\alpha$-power of $z$. Since $A$ is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since $z^\alpha$ is analytic in $\{ \Re z > 0 \}$, then by (SMT) we have

$$\sigma (a^\alpha) = (\sigma (a))^\alpha = \{ z^\alpha : z \in \sigma (a) \} \subset (0, \infty).$$

Following [6], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [10, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha + \beta};$

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha};$

(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f (\cdot)$ is analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f (z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma (u) \subset I$, then by (SMT) we have

$$\sigma (f (u)) = f (\sigma (u)) \subset f (I) \subset [0, \infty)$$

meaning that $f (u) \geq 0$ in the order of $A$.

Therefore, we can state the following fact that will be used to establish various inequalities in $A$, see also [3].

**Lemma 1.** Let $f (z)$ and $g (z)$ be analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f (z) \geq g (z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma (u) \subset I$ we have $f (u) \geq g (u)$ in the order of $A$.

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let $G$ be an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If $a, b \in A$ with $\sigma (a), \sigma (b) \subset I$, then by SMT the element $(1 - t)a + tb \in A$ has the spectrum $\sigma ((1 - t)a + tb) \subset I$ for all $t \in [0, 1]$. We say that an analytic function $f (z)$ in $G$ is operator convex on $I$ in the Hermitian Banach $*$-algebra $A$ if

$$f ((1 - t)a + tb) \leq (1 - t)f (a) + tf (b) \quad \text{in the order of } A \quad \text{for all } a, b \in A \text{ with } \sigma (a), \sigma (b) \subset I \text{ and all } t \in [0, 1].$$
It is well known that, if \( E \) is a Banach space and \( g : [0, 1] \to E \) is a continuous function, then \( g \) is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by \( \int_0^1 g(t) \, dt \).

By taking the integral in (1.1), then we get
\[
(1.2) \quad \int_0^1 f((1-t)a + tb) \, dt \leq \int_0^1 [(1-t)f(a) + tf(b)] \, dt = \frac{f(a) + f(b)}{2}.
\]

Since for \( c, d \in A \) with \( \sigma(c), \sigma(d) \subset I \), we have
\[
(1.3) \quad f\left(\frac{c + d}{2}\right) \leq \frac{f(c) + f(d)}{2}
\]
hence by taking \( c = (1-t)a + tb \) and \( d = ta + (1-t)b \), we get
\[
(1.4) \quad f\left(\frac{a + b}{2}\right) \leq \frac{f((1-t)a + tb) + f(ta + (1-t)b)}{2}
\]
for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) and all \( t \in [0, 1] \).

By integrating over \( t \) in (1.4) we derive
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2} \int_0^1 [f((1-t)a + tb) + f(ta + (1-t)b)] \, dt
\]
and since
\[
\int_0^1 f((1-t)a + tb) \, dt = \int_0^1 f(ta + (1-t)b) \, dt,
\]
hence
\[
(1.5) \quad f\left(\frac{a + b}{2}\right) \leq \int_0^1 f((1-t)a + tb) \, dt.
\]

Therefore, by (1.2) and (1.5) we obtain the Hermite-Hadamard inequality
\[
(1.6) \quad f\left(\frac{a + b}{2}\right) \leq \int_0^1 f((1-t)a + tb) \, dt \leq \frac{f(a) + f(b)}{2}
\]
for operator convex functions \( f \) on \( I \) in the Hermitian Banach *-algebra \( A \) and \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \).

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach *-algebras.

2. Some Preliminary Results

We have:

**Lemma 2.** Let \( f(z) \) be analytic in \( G \), an open subset of \( \mathbb{C} \) and \( a, b \in A \) with \( \sigma(a) \subset G \). Then the Fréchet derivative \( Df(a)(b) \) exists and
\[
(2.1) \quad Df(a)(b) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z-a)^{-1}b(z-a)^{-1} \, dz,
\]
where \( \gamma \) is a close rectifiable curve such that \( \sigma(a) \subset \text{ins}(\gamma) \subset G \).
Proof. Let $\delta > 0$ such that $\sigma (a + \varepsilon b) \subseteq G$ for $\varepsilon \in (-\delta, \delta)$. Chose $\gamma$ a close rectifiable curve such that $\sigma (a), \sigma (a + \varepsilon b) \subseteq \operatorname{ins} (\gamma) \subseteq G$ for $\varepsilon \in (-\delta, \delta)$. Using the analytic functional calculus, we have

\begin{equation}
 f (a + \varepsilon b) - f (a) = \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a - \varepsilon b)^{-1} \, dz - \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a)^{-1} \, dz
\end{equation}

Using the resolvent identity

\begin{equation}
 (z - c)^{-1} - (z - a)^{-1} = (z - c)^{-1} (c - a) (z - a)^{-1}
\end{equation}

we also have

\begin{equation}
 \frac{1}{2\pi i} \int_{\gamma} f (z) \left[ (z - a - \varepsilon b)^{-1} - (z - a)^{-1} \right] \, dz = \frac{\varepsilon}{2\pi i} \int_{\gamma} f (z) \left[ (z - a - \varepsilon b)^{-1} b (z - a)^{-1} \right] \, dz.
\end{equation}

By (2.2) and (2.3) we get

\begin{equation}
 \frac{f (a + \varepsilon b) - f (a)}{\varepsilon} = \frac{1}{2\pi i} \int_{\gamma} f (z) \left[ (z - a - \varepsilon b)^{-1} b (z - a)^{-1} \right] \, dz
\end{equation}

for $\varepsilon \in (-\delta, \delta), \varepsilon \neq 0$.

By taking the limit over $\varepsilon \to 0$ and using the properties of the complex integral, we obtain (2.1). \hfill \square

**Corollary 1.** Let $f (z)$ be analytic in $G$, an open convex subset of $\mathbb{C}$ and $a, b \in A$ with $\sigma (a), \sigma (b) \subseteq G$. The auxiliary function $F_{(a, b)} : [0, 1] \to A$ defined by $F_{(a, b)} (t) := f ((1 - t) a + t b)$ is differentiable on $[0, 1]$,

\begin{equation}
 F'_{(a, b)} (t) = D f ((1 - t) a + t b) (b - a)
 = \frac{1}{2\pi i} \int_{\gamma} f (z) \left( z - (1 - t) a - t b \right)^{-1} (b - a) \left( z - (1 - t) a - t b \right)^{-1} \, dz,
\end{equation}

\begin{equation}
 F'_{(a, b)} (0^+) = D f (a) (b - a)
 = \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a)^{-1} (b - a) (z - a)^{-1} \, dz,
\end{equation}

and

\begin{equation}
 F'_{(a, b)} (1^-) = D f (b) (b - a)
 = \frac{1}{2\pi i} \int_{\gamma} f (z) (z - b)^{-1} (b - a) (z - b)^{-1} \, dz,
\end{equation}

where $\gamma$ is a close rectifiable curve such that $\sigma (a), \sigma (b) \subseteq \operatorname{ins} (\gamma) \subseteq G$.\\

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Proof. Let \( t \in (0, 1) \) and \( h \neq 0 \) small enough such that \( t + h \in (0, 1) \). Then

\[
(2.8) \quad \frac{F_{(a,b)}(t+h) - F_{(a,b)}(t)}{h} = \frac{f((1-t-h)a + (t+h)b) - f((1-t)a + tb)}{h} = \frac{f((1-t)a + tb + h(b-a)) - f((1-t)a + tb)}{h}.
\]

Since \( f \) is Fréchet differentiable, hence by taking the limit over \( h \to 0 \) in (2.8) we get

\[
F'_{(a,b)}(t) = \lim_{h \to 0} \frac{F_{(a,b)}(t+h) - F_{(a,b)}(t)}{h} = \lim_{h \to 0} \frac{f((1-t)a + tb + h(b-a)) - f((1-t)a + tb)}{h} = \frac{1}{2\pi i} \int_{\gamma} f(z) (z - (1-t)a - tb)^{-1} (b - a) (z - (1-t)a - tb)^{-1} dz,
\]

which proves (2.5).

Also, we have

\[
F'_{(a,b)}(0+) = \lim_{h \to 0^+} \frac{F_{(a,b)}(h) - F_{(a,b)}(0)}{h} = \lim_{h \to 0^+} \frac{f((1-h)a + hb) - f(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} f(z) (z-a)^{-1} (b-a) (z-a)^{-1} dz \quad \text{(by Lemma 2)},
\]

which proves (2.6).

The equality (2.7) goes in a similar way. \( \square \)

**Theorem 1.** Assume that \( f(z) \) is analytic in \( G \text{, an open subset of } \mathbb{C} \text{ and } I \subset G \text{ a real interval.} \) The function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) if and only if for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have

\[
(2.9) \quad f(b) - f(a) \geq Df(a)(b-a)
\]

in the order of \( A \).

**Proof.** Assume that \( f(z) \) is operator convex on \( I \) and \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) and \( t \in (0, 1) \). Then by (1.1) we have

\[
f(a + t(b-a)) - f(a) \leq tf(b) - f(a)
\]

for \( t \in (0, 1) \), which implies that

\[
(2.10) \quad \frac{f(a + t(b-a)) - f(a)}{t} \leq f(b) - f(a)
\]

for \( t \in (0, 1) \).

Since, by Lemma 2 the Fréchet derivative \( Df(a)((b-a)) \) exists, hence by taking the limit over \( t \to 0^+ \) in (2.10) we get (2.9).
Let \( c, d \in A \) with \( \sigma (c) \subseteq I \) and \( t \in [0, 1] \). If we chose in (2.9) \( b = d \) and \( a = (1 - t) d + tc \), then we get

\[
(2.11) \quad f (d) - f ((1 - t) d + tc) \geq t D f ((1 - t) d + tc) (d - c)
\]

and if we choose \( b = c \) and \( a = (1 - t) d + tc \), then we get

\[
(2.12) \quad f (c) - f ((1 - t) d + tc) \geq (1 - t) D f ((1 - t) d + tc) (c - d).
\]

If we multiply (2.11) by \( (1 - t) \) and (2.12) by \( t \) and add the obtained inequalities, we get

\[
(1 - t) f (d) + tf (c) - (1 - t) f ((1 - t) d + tc) - tf ((1 - t) d + tc)
\]

\[
\geq (1 - t) t D f ((1 - t) d + tc) (d - c) + t (1 - t) D f ((1 - t) d + tc) (c - d)
\]

namely

\[
(1 - t) f (d) + tf (c) - f ((1 - t) d + tc)
\]

\[
\geq (1 - t) t D f ((1 - t) d + tc) (d - c) - t (1 - t) D f ((1 - t) d + tc) (d - c) = 0,
\]

which proves the operator convexity of \( f \).

The above result can be used to prove the operator convexity of some simple functions.

**Proposition 1.** Assume that the element \( q \) is selfadjoint in \( A \). The function \( f (x) := qx^2q \) satisfies the property (2.9) for any selfadjoint element \( a, b \in A \).

**Proof.** We have for \( u \) selfadjoint in \( A \) that

\[
D f (a) (u) = \lim_{\varepsilon \to 0} \frac{f (a + \varepsilon u) - f (a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{q (a + \varepsilon u)^2 q - q a^2 q}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{q (a^2 + \varepsilon au + \varepsilon^2 u^2) q - q a^2 q}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{q (\varepsilon au + \varepsilon^2 u^2) q}{\varepsilon} = q (au + ua) q.
\]

Therefore

\[
f (b) - f (a) - D f (a) (b - a)
\]

\[
= qb^2q - qa^2q - q [a (b - a) + (b - a) a] q
\]

\[
= qb^2q - qa^2q - q (ab - a^2 + ba - a^2) q
\]

\[
= qb^2q - qabq - qbaq + qa^2q = q (b - a)^2 q \geq 0
\]

for any selfadjoint element \( a, b \in A \) and the proposition is proved.

**Corollary 2.** The function \( f (z) = z^2 \) is operator convex on \( \mathbb{R} \) in the Hermitian Banach \( * \)-algebra \( A \).

**Proposition 2.** Assume that the element \( q \) is selfadjoint in \( A \). The function \( f (x) := x q x \) satisfies the property (2.9) for any selfadjoint element \( a, b \in A \).
Proof. We have for \( u \) selfadjoint in \( A \) that
\[
Df (a) (u) = \lim_{\varepsilon \to 0} \frac{f (a + \varepsilon u) - f (a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(aq + \varepsilon uq)(a + \varepsilon u) - aq}{\varepsilon}
\]
\[
= \lim_{\varepsilon \to 0} \frac{aq + \varepsilon uq + \varepsilon^2 uqu - aq}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon uqa + \varepsilon^2 uqu}{\varepsilon} = uqa + aq.
\]
Therefore
\[
f (b) - f (a) - Df (a) (b - a) = bqb - aqa - (b - a) qa - aq (b - a)
\]
\[
= bqb - aqa - b qa + aqa - aqb + aq = bqb - aqb + aq = (b - a) q (b - a) \geq 0
\]
for any selfadjoint element \( a, b \in A \) and the proposition is proved. \( \square \)

**Proposition 3.** Assume that the element \( q \) is selfadjoint in \( A \). The function \( f (x) := qx^{-1} q \) satisfies the property (2.9) for any positive elements \( a, b \in A \).

Proof. For \( a, b \in A \) with \( a, b > 0 \)
\[
Df (a) (b - a) = \lim_{\varepsilon \to 0^+} \frac{f (a + \varepsilon (b - a)) - f (a)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{q ((1 - \varepsilon) a + \varepsilon b)^{-1} q - qa^{-1} q}{\varepsilon}
\]
\[
= \lim_{\varepsilon \to 0^+} \frac{q ((1 - \varepsilon) a + \varepsilon b)^{-1} - a^{-1}}{\varepsilon} q = q \lim_{\varepsilon \to 0^+} \left[ \frac{((1 - \varepsilon) a + \varepsilon b)^{-1} - a^{-1}}{\varepsilon} \right] q.
\]
We have for \( c, d > 0 \) that
\[
d^{-1} - c^{-1} = d^{-1} (c - d) c^{-1}.
\]
Therefore
\[
((1 - \varepsilon) a + \varepsilon b)^{-1} - a^{-1} = \varepsilon ((1 - \varepsilon) a + \varepsilon b)^{-1} (a - b) a^{-1}
\]
for \( \varepsilon \in (0, 1) \), and then
\[
Df (a) (b - a) = q \lim_{\varepsilon \to 0^+} \left[ ((1 - \varepsilon) a + \varepsilon b)^{-1} (a - b) a^{-1} \right] q = qa^{-1} (a - b) a^{-1} q
\]
for \( a, b \in A \) with \( a, b > 0 \).

Now, we have
\[
f (b) - f (a) - Df (a) (b - a) = qb^{-1} q - qa^{-1} q - qa^{-1} (a - b) a^{-1} q
\]
\[
= q (b^{-1} - a^{-1}) q - qa^{-1} (a - b) a^{-1} q
\]
\[
= qb^{-1} (a - b) a^{-1} q - qa^{-1} (a - b) a^{-1} q = (qb^{-1} - qa^{-1}) (a - b) a^{-1} q
\]
\[
= q (b^{-1} - a^{-1}) qb^{-1} (a - b) a^{-1} q = q (b^{-1} - a^{-1}) b (b^{-1} - a^{-1}) q \geq 0
\]
for \( a, b \in A \) with \( a, b > 0 \).

This proves the statement. \( \square \)
Corollary 3. The function $f(z) = z^{-1}$ is operator convex on $(0, \infty)$ in the Hermitian Banach $*$-algebra $A$.

We also have:

Theorem 2. Assume that $f(z)$ is analytic in $G$, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on $I$ in the Hermitian Banach $*$-algebra $A$ if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have for all $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$ that

$$F_{(a,b)}^{t_2} = Df((1-t_2)a + t_2b)(b-a)$$

$$\geq f((1-t_2)a + t_2b) - f((1-t_1)a + t_1b)$$

$$\geq Df((1-t_1)a + t_1b)(b-a) = F_{(a,b)}^{t_1}.$$

We also have

$$Df((b-a) \geq F_{(a,b)}^{t_1}(t) = Df((1-t)a + tb)(b-a) \geq Df(a)(b-a)$$

for all $t \in (0,1)$.

Proof. Assume that the function $f(z)$ is operator convex on $I$ in the Hermitian Banach $*$-algebra $A$.

Let $c, d \in A$ with $\sigma(c), \sigma(d) \subset I$, then by taking $b = d$ and $a = c$, we have

$$f(d) - f(c) \geq Df(c)(d-c)$$

and by taking $b = c$ and $a = d$, we have

$$f(c) - f(d) \geq Df(d)(c-d)$$

which imply the double inequality

$$Df(d)(d-c) \geq f(d) - f(c) \geq Df(c)(d-c)$$

for all $c, d \in A$ with $\sigma(c), \sigma(d) \subset I$.

Let $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$ and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$. Then $\sigma((1-t_1)a + t_1b), \sigma((1-t_2)a + t_2b) \subset I$ and by (2.15) for $d = (1-t_2)a + t_2b$ and $c = (1-t_1)a + t_1b$ we get

$$Df((1-t_2)a + t_2b)((1-t_2)a + t_2b - (1-t_1)a - t_1b)$$

$$\geq f((1-t_2)a + t_2b) - f((1-t_1)a + t_1b)$$

$$\geq Df((1-t_1)a + t_1b)((1-t_2)a + t_2b - (1-t_1)a - t_1b),$$

namely

$$(t_2 - t_1)Df((1-t_2)a + t_2b)(b-a)$$

$$\geq f((1-t_2)a + t_2b) - f((1-t_1)a + t_1b)$$

$$\geq (t_2 - t_1)Df((1-t_1)a + t_1b)(b-a),$$

which implies that

$$Df((1-t_2)a + t_2b)(b-a)$$

$$\geq f((1-t_2)a + t_2b) - f((1-t_1)a + t_1b)$$

$$\geq Df((1-t_1)a + t_1b)(b-a),$$
for all $t_2, t_1 \in [0,1]$ with $t_1 < t_2$ and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$, and the inequality (2.13) is proved.

Now, if the condition (2.13) is valid, then by taking $t_1 = 0$ and $t_1 = 1$, then we get

$$Df(b)(b-a) \geq f(b) - f(a) \geq Df(a)(b-a)$$

for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$, and by Theorem 1 it follows that $f(z)$ is operator convex on $I$ in the Hermitian Banach $\ast$-algebra $A$.

If we take $t_1 = 0$ and $t_2 = t \in (0,1]$ in (2.13), then we get

$$Df((1-t)a+tb)(b-a) \geq Df(a)(b-a).$$

Also, if we take $t_1 = t \in [0,1)$ and $t_2 = 1$ in (2.13), then we get

$$Df(b)(b-a) \geq Df((1-t)a+tb)(b-a).$$

□

3. Reverse Hermite-Hadamard Inequalities

We have the following reverse of the first operator Hermite-Hadamard inequality:

**Theorem 3.** Assume that $f(z)$ is analytic in $G$, an open subset of $C$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on $I$ in the Hermitian Banach $\ast$-algebra $A$ then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have

$$0 \leq \int_{0}^{1} f((1-t)a+tb)dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}[Df(b)(b-a) - Df(a)(b-a)].$$

**Proof.** Using integration by parts formula for the Bochner integral, we have

$$\int_{0}^{1/2} tF'_{(a,b)}(t)dt = \frac{1}{2}F_{(a,b)}\left(\frac{1}{2}\right) - \int_{0}^{1/2} F_{(a,b)}(t)dt$$

$$= \frac{1}{2}f\left(\frac{a+b}{2}\right) - \int_{0}^{1/2} f\left((1-t)a+tb\right)dt$$

and

$$\int_{1/2}^{1} (t-1) F'_{(a,b)}(t)dt = \frac{1}{2}F_{(a,b)}\left(\frac{1}{2}\right) - \int_{1/2}^{1} f\left((1-t)a+tb\right)dt$$

$$= \frac{1}{2}f\left(\frac{a+b}{2}\right) - \int_{1/2}^{1} f\left((1-t)a+tb\right)dt.$$

If we add these two equalities, we get the following identity of interest

$$\int_{0}^{1} f\left((1-t)a+tb\right)dt - f\left(\frac{a+b}{2}\right)$$

$$= \int_{1/2}^{1} (1-t) F'_{(a,b)}(t)dt - \int_{0}^{1/2} tF'_{(a,b)}(t)dt.$$

From Theorem 2 we have

$$F'_{(a,b)}(1/2) \leq F'_{(a,b)}(t) \leq F'_{(a,b)}(1-) = Df(b)(b-a), \ t \in [1/2,1)$$
and

\[ Df(a)(b-a) = F'_{(a,b)}(0+) \leq F'_{(a,b)}(t) \leq F'_{(a,b)}(1/2), \quad t \in (0,1/2]. \]

This implies that

\[ (1-t)F'_{(a,b)}(1/2) \leq (1-t)F'_{(a,b)}(t) \leq (1-t)Df(b)(b-a) \]

for \( t \in [1/2,1) \) and

\[ -tF'_{(a,b)}(1/2) \leq -tF'_{(a,b)}(t) \leq -tDf(a)(b-a) \]

for \( t \in (0,1/2]. \)

By integrating these inequalities on the corresponding intervals, we get

\[
\frac{1}{8}F'_{(a,b)}(1/2) \leq \int_{1/2}^{1} (1-t)F'_{(a,b)}(t) dt \leq \frac{1}{8}Df(b)(b-a)
\]

and

\[
-\frac{1}{8}F'_{(a,b)}(1/2) \leq -\int_{0}^{1/2} tF'_{(a,b)}(t) dt \leq -\frac{1}{8}Df(a)(b-a).
\]

By addition, we deduce that

\[
0 \leq \int_{1/2}^{1} (1-t)F'_{(a,b)}(t) dt - \int_{0}^{1/2} tF'_{(a,b)}(t) dt \leq \frac{1}{8} [Df(b)(b-a) - Df(a)(b-a)]
\]

and by the identity (3.4) we get (3.1).

We have the following reverse of the second operator Hermite-Hadamard inequality:

**Theorem 4.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach *-algebra \( A \) then for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have

\[
0 \leq \frac{f(a) + f(b)}{2} - \int_{0}^{1} f((1-t)a + tb) dt \leq \frac{1}{8} [Df(b)(b-a) - Df(a)(b-a)].
\]

**Proof.** Using integration by parts formula for the Bochner integral, we have

\[
\int_{0}^{1} \left( t - \frac{1}{2} \right) F'_{(a,b)}(t) dt = \frac{F_{(a,b)}(1) + F_{(a,b)}(0)}{2} - \int_{0}^{1} F_{(a,b)}(t)
\]

\[ = \frac{f(b) + f(a)}{2} - \int_{0}^{1} f((1-t)a + tb) dt.
\]

Observe that

\[
\int_{0}^{1} \left( t - \frac{1}{2} \right) F'_{(a,b)}(t) dt
\]

\[ = \int_{1/2}^{1} \left( t - \frac{1}{2} \right) F'_{(a,b)}(t) dt - \int_{0}^{1/2} \left( \frac{1}{2} - t \right) F'_{(a,b)}(t) dt.
\]
Therefore, we have the following identity of interest
\[
\frac{f(b) + f(a)}{2} - \int_0^1 f((1-t)a + tb)\,dt
= \int_{1/2}^1 \left( t - \frac{1}{2} \right) F'_{(a,b)}(t)\,dt - \int_0^{1/2} \left( \frac{1}{2} - t \right) F'_{(a,b)}(t)\,dt.
\]

From the inequality (3.5) we obtain
\[
\left( t - \frac{1}{2} \right) F'_{(a,b)}(1/2) \leq \left( t - \frac{1}{2} \right) F_{(a,b)}(t)
\leq \left( t - \frac{1}{2} \right) Df(b)(b-a), \ t \in [1/2, 1)
\]
and from (3.6)
\[
\left( \frac{1}{2} - t \right) Df(a)(b-a) \leq \left( \frac{1}{2} - t \right) F_{(a,b)}(t)
\leq \left( \frac{1}{2} - t \right) F'_{(a,b)}(1/2), \ t \in (0, 1/2],
\]
namely
\[
-\left( \frac{1}{2} - t \right) F'_{(a,b)}(1/2) \leq -\left( \frac{1}{2} - t \right) F_{(a,b)}(t)
\leq -\left( \frac{1}{2} - t \right) Df(a)(b-a), \ t \in (0, 1/2].
\]

Integrating these inequalities on the corresponding intervals, we get
\[
\frac{1}{8} F_{(a,b)}(1/2) \leq \int_{1/2}^1 \left( t - \frac{1}{2} \right) F_{(a,b)}(t)\,dt \leq \frac{1}{8} Df(b)(b-a),
\]
and
\[
-\frac{1}{8} F'_{(a,b)}(1/2) \leq -\int_0^{1/2} \left( \frac{1}{2} - t \right) F_{(a,b)}(t)\,dt \leq -\frac{1}{8} Df(a)(b-a).
\]

If we add these two inequalities, we obtain
\[
0 \leq \int_{1/2}^1 \left( t - \frac{1}{2} \right) F_{(a,b)}(t)\,dt - \int_0^{1/2} \left( \frac{1}{2} - t \right) F_{(a,b)}(t)\,dt
\leq \frac{1}{8} \left[ Df(b)(b-a) - Df(a)(b-a) \right],
\]
which, by the use of identity (3.9) produces the desired result (3.7). \qed

For the function \( f(z) = z^{-1} \) we have for \( a, b > 0 \) that
\[
Df(a)(b-a) = -a^{-1}(b-a) a^{-1} \quad \text{and} \quad Df(b)(b-a) = -b^{-1}(b-a) b^{-1}.
\]

Therefore, by the inequality (3.1) we have
\[
(3.10) \quad 0 \leq \int_0^1 ((1-t)a + tb)^{-1}\,dt - \left( \frac{a + b}{2} \right)^{-1}
\leq \frac{1}{8} \left[ a^{-1}(b-a) a^{-1} - b^{-1}(b-a) b^{-1} \right].
while from (3.7) we have

\[
0 \leq \frac{a^{-1} + b^{-1}}{2} - \int_0^1 ((1 - t) a + tb)^{-1} dt \\
\leq \frac{1}{8} \left[ a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1} \right].
\]

for \( a, b \in A \) with \( a, b > 0 \).

References


1Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

2DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa