INEQUALITIES OF HERMITE-HADAMARD TYPE FOR OPERATOR CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH ∗-ALGEBRAS

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Abstract. We establish in this paper some inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach ∗-algebras.

1. Introduction

We need some preliminary concepts and facts about Banach ∗-algebras.

Let A be a unital Banach ∗-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach ∗-algebra [9] (see also [1, Theorem 41.5]), then

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [8], Tanahashi and Uchiyama [10] proved the following fundamental properties (see also [6]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [8] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach ∗-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Hermitian unital Banach ∗-algebra, Hermite-Hadamard type inequalities, Operator convex functions.

RGMIA Res. Rep. Coll. 23 (2020), Art. 17, 12 pp. Received 31/01/20

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz,$$

where γ is a close rectifiable curve such that $\sigma(a) \subset \operatorname{ins}(\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [6], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [10, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let G be an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$, then by SMT the element $(1-t)a + tb \in A$ has the spectrum $\sigma((1-t)a + tb) \subset I$ for all $t \in [0,1]$. We say that an analytic function $f(z)$ in G is *operator convex* on I in the Hermitian Banach $*$-algebra A if

$$f((1-t)a + tb) \leq (1-t) f(a) + t f(b) \text{ in the order of } A \tag{1.1}$$

for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ and all $t \in [0,1]$.

It is well known that, if E is a Banach space and $g:[0,1]\to E$ is a continuous function, then g is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_0^1 g(t)\,dt$.

By taking the integral in (1.1), then we get

$$\int_0^1 f((1-t)a+tb)\,dt \le \int_0^1 [(1-t)f(a)+tf(b)]\,dt = \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Since for c, $d\in A$ with $\sigma(c)$, $\sigma(d)\subset I$, we have

$$f\left(\frac{c+d}{2}\right) \le \frac{f(c)+f(d)}{2} \tag{1.3}$$

hence by taking $c=(1-t)a+tb$ and $d=ta+(1-t)b$, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{f((1-t)a+tb)+f(ta+(1-t)b)}{2} \tag{1.4}$$

for all a, $b\in A$ with $\sigma(a)$, $\sigma(b)\subset I$ and all $t\in[0,1]$.

By integrating over t in (1.4) we derive

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\int_0^1 [f((1-t)a+tb)+f(ta+(1-t)b)]\,dt$$

and since

$$\int_0^1 f((1-t)a+tb)\,dt = \int_0^1 f(ta+(1-t)b)\,dt,$$

hence

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f((1-t)a+tb)\,dt. \tag{1.5}$$

Therefore, by (1.2) and (1.5) we obtain the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f((1-t)a+tb)\,dt \le \frac{f(a)+f(b)}{2} \tag{1.6}$$

for *operator convex* functions f on I in the Hermitian Banach $*$-algebra A and a, $b\in A$ with $\sigma(a)$, $\sigma(b)\subset I$.

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach $*$-algebras.

2. Some Preliminary Results

We have:

Lemma 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and a, $b\in A$ with $\sigma(a)\subset G$. Then the Fréchet derivative $Df(a)(b)$ exists and*

$$Df(a)(b) = \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}b(z-a)^{-1}\,dz, \tag{2.1}$$

where γ is a close rectifiable curve such that $\sigma(a)\subset \operatorname{ins}(\gamma)\subset G$.

Proof. Let $\delta > 0$ such that $\sigma(a+\varepsilon b) \subset G$ for $\varepsilon \in (-\delta, \delta)$. Chose γ a close rectifiable curve such that $\sigma(a), \sigma(a+\varepsilon b) \subset \operatorname{ins}(\gamma) \subset G$ for $\varepsilon \in (-\delta, \delta)$. Using the analytic functional calculus, we have

$$\begin{aligned} f(a+\varepsilon b) - f(a) &= \frac{1}{2\pi i}\int_\gamma f(z)(z-a-\varepsilon b)^{-1}dz - \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz \\ &= \frac{1}{2\pi i}\int_\gamma f(z)\left[(z-a-\varepsilon b)^{-1} - (z-a)^{-1}\right]dz. \end{aligned} \tag{2.2}$$

Using the resolvent identity

$$(z-c)^{-1} - (z-a)^{-1} = (z-c)^{-1}(c-a)(z-a)^{-1}$$

we also have

$$\begin{aligned} &\frac{1}{2\pi i}\int_\gamma f(z)\left[(z-a-\varepsilon b)^{-1} - (z-a)^{-1}\right]dz \\ &= \frac{\varepsilon}{2\pi i}\int_\gamma f(z)\left[(z-a-\varepsilon b)^{-1} b (z-a)^{-1}\right]dz. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3) we get

$$\frac{f(a+\varepsilon b) - f(a)}{\varepsilon} = \frac{1}{2\pi i}\int_\gamma f(z)\left[(z-a-\varepsilon b)^{-1} b (z-a)^{-1}\right]dz \tag{2.4}$$

for $\varepsilon \in (-\delta, \delta)$, $\varepsilon \neq 0$.

By taking the limit over $\varepsilon \to 0$ and using the properties of the complex integral, we obtain (2.1). □

Corollary 1. *Let $f(z)$ be analytic in G, an open convex subset of $\mathbb{C}$ and a, $b \in A$ with $\sigma(a), \sigma(b) \subset G$. The auxiliary function $F_{(a,b)} : [0,1] \to A$ defined by $F_{(a,b)}(t) := f((1-t)a + tb)$ is differentiable on $[0,1]$,*

$$\begin{aligned} F'_{(a,b)}(t) &= Df((1-t)a+tb)(b-a) \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(z-(1-t)a-tb)^{-1}(b-a)(z-(1-t)a-tb)^{-1}dz, \end{aligned} \tag{2.5}$$

$$\begin{aligned} F'_{(a,b)}(0+) &= Df(a)(b-a) \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}(b-a)(z-a)^{-1}dz, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F'_{(a,b)}(1-) &= Df(b)(b-a) \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(z-b)^{-1}(b-a)(z-b)^{-1}dz, \end{aligned} \tag{2.7}$$

where γ is a close rectifiable curve such that $\sigma(a), \sigma(b) \subset \operatorname{ins}(\gamma) \subset G$.

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{F_{(a,b)}(t+h) - F_{(a,b)}(t)}{h} \\ &= \frac{f((1-t-h)a + (t+h)b) - f((1-t)a + tb)}{h} \\ &= \frac{f((1-t)a + tb + h(b-a)) - f((1-t)a + tb)}{h}. \end{aligned} \tag{2.8}$$

Since f is Fréchet differentiable, hence by taking the limit over $h \to 0$ in (2.8) we get

$$\begin{aligned} F'_{(a,b)}(t) &= \lim_{h\to 0} \frac{F_{(a,b)}(t+h) - F_{(a,b)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)a + tb + h(b-a)) - f((1-t)a + tb)}{h} \\ &= Df((1-t)a + tb)(b-a) \text{ (and by Lemma 2)} \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(z-(1-t)a - tb)^{-1}(b-a)(z-(1-t)a-tb)^{-1}\,dz, \end{aligned}$$

which proves (2.5).

Also, we have

$$\begin{aligned} F'_{(a,b)}(0+) &= \lim_{h\to 0+} \frac{F_{(a,b)}(h) - F_{(a,b)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)a + hb) - f(a)}{h} \\ &= \lim_{h\to 0+} \frac{f(a + h(b-a)) - f(a)}{h} = Df(a)(b-a) \\ &= \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}(b-a)(z-a)^{-1}\,dz \text{ (by Lemma 2)}, \end{aligned}$$

which proves (2.6).

The equality (2.7) goes in a similar way. □

Theorem 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have*

$$f(b) - f(a) \geq Df(a)(b-a) \tag{2.9}$$

in the order of A.

Proof. Assume that $f(z)$ is operator convex on I and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ and $t \in (0,1)$. Then by (1.1) we have

$$f(a + t(b-a)) - f(a) \leq t[f(b) - f(a)]$$

for $t \in (0,1)$, which implies that

$$\frac{f(a + t(b-a)) - f(a)}{t} \leq f(b) - f(a) \tag{2.10}$$

for $t \in (0,1)$.

Since, by Lemma 2 the Fréchet derivative $Df(a)((b-a))$ exists, hence by taking the limit over $t \to 0+$ in (2.10) we get (2.9).

Let c, $d\in A$ with $\sigma(c), \sigma(d)\subset I$ and $t\in[0,1]$. If we chose in (2.9) $b=d$ and $a=(1-t)d+tc$, then we get

$$f(d)-f((1-t)d+tc)\geq tDf((1-t)d+tc)(d-c) \tag{2.11}$$

and if we choose $b=c$ and $a=(1-t)d+tc$, then we get

$$f(c)-f((1-t)d+tc)\geq(1-t)Df((1-t)d+tc)(c-d). \tag{2.12}$$

If we multiply (2.11) by $(1-t)$ and (2.12) by t and add the obtained inequalities, we get

$$\begin{aligned}&(1-t)f(d)+tf(c)-(1-t)f((1-t)d+tc)-tf((1-t)d+tc)\\&\geq(1-t)tDf((1-t)d+tc)(d-c)+t(1-t)Df((1-t)d+tc)(c-d)\end{aligned}$$

namely

$$\begin{aligned}&(1-t)f(d)+tf(c)-f((1-t)d+tc)\\&\geq(1-t)tDf((1-t)d+tc)(d-c)-t(1-t)Df((1-t)d+tc)(d-c)=0,\end{aligned}$$

which proves the operator convexity of f. □

The above result can be used to prove the operator convexity of some simple functions.

Proposition 1. *Assume that the element q is selfadjoint in A. The function $f(x):=qx^2q$ satisfies the property (2.9) for any selfadjoint element a, $b\in A$.*

Proof. We have for u selfadjoint in A that

$$\begin{aligned}Df(a)(u)&=\lim_{\varepsilon\to0}\frac{f(a+\varepsilon u)-f(a)}{\varepsilon}=\lim_{\varepsilon\to0}\frac{q(a+\varepsilon u)^2q-qa^2q}{\varepsilon}\\&=\lim_{\varepsilon\to0}\frac{q\left(a^2+\varepsilon au+\varepsilon ua+\varepsilon^2u^2\right)q-qa^2q}{\varepsilon}\\&=\lim_{\varepsilon\to0}\frac{q\left(\varepsilon au+\varepsilon ua+\varepsilon^2u^2\right)q}{\varepsilon}=q(au+ua)q.\end{aligned}$$

Therefore

$$\begin{aligned}&f(b)-f(a)-Df(a)(b-a)\\&=qb^2q-qa^2q-q[a(b-a)+(b-a)a]q\\&=qb^2q-qa^2q-q\left(ab-a^2+ba-a^2\right)q\\&=qb^2q-qabq-qbaq+qa^2q=q(b-a)^2q\geq0\end{aligned}$$

for any selfadjoint element a, $b\in A$ and the proposition is proved. □

Corollary 2. *The function $f(z)=z^2$ is operator convex on $\mathbb{R}$ in the Hermitian Banach $*$-algebra A.*

Proposition 2. *Assume that the element $q>0$ in A. The function $f(x):=xqx$ satisfies the property (2.9) for any selfadjoint element a, $b\in A$.*

Proof. We have for u selfadjoint in A that

$$\begin{aligned} Df(a)(u) &= \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon u)-f(a)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{(a+\varepsilon u)\,q\,(a+\varepsilon u)-aqa}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{(aq+\varepsilon uq)(a+\varepsilon u)-aqa}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{aqa+\varepsilon uqa+\varepsilon aqu+\varepsilon^2 uqu-aqa}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\varepsilon uqa+\varepsilon aqu+\varepsilon^2 uqu}{\varepsilon} = uqa+aqu. \end{aligned}$$

Therefore

$$\begin{aligned} & f(b)-f(a)-Df(a)(b-a) \\ &= bqb-aqa-(b-a)\,qa-aq\,(b-a) \\ &= bqb-aqa-bqa+aqa-aqb+aqa \\ &= bqb-bqa-aqb+aqa=(b-a)\,q\,(b-a)\geq 0 \end{aligned}$$

for any selfadjoint element a, $b\in A$ and the proposition is proved. □

Proposition 3. *Assume that the element q is selfadjoint in A. The function $f(x):=qx^{-1}q$ satisfies the property (2.9) for any positive elements a, $b\in A$.*

Proof. For a, $b\in A$ with a, $b>0$

$$\begin{aligned} & Df(a)(b-a) \\ &= \lim_{\varepsilon\to 0+,}\frac{f(a+\varepsilon(b-a))-f(a)}{\varepsilon} = \lim_{\varepsilon\to 0+}\frac{q\left((1-\varepsilon)a+\varepsilon b\right)^{-1}q-qa^{-1}q}{\varepsilon} \\ &= \lim_{\varepsilon\to 0+}\frac{q\left[\left((1-\varepsilon)a+\varepsilon b\right)^{-1}-a^{-1}\right]q}{\varepsilon} = q\lim_{\varepsilon\to 0+}\left[\frac{\left((1-\varepsilon)a+\varepsilon b\right)^{-1}-a^{-1}}{\varepsilon}\right]q. \end{aligned}$$

We have for c, $d>0$ that

$$d^{-1}-c^{-1}=d^{-1}(c-d)\,c^{-1}.$$

Therefore

$$\left((1-\varepsilon)a+\varepsilon b\right)^{-1}-a^{-1}=\varepsilon\left((1-\varepsilon)a+\varepsilon b\right)^{-1}(a-b)\,a^{-1}$$

for $\varepsilon\in(0,1)$, and then

$$Df(a)(b-a)=q\lim_{\varepsilon\to 0+}\left[\left((1-\varepsilon)a+\varepsilon b\right)^{-1}(a-b)\,a^{-1}\right]q=qa^{-1}(a-b)\,a^{-1}q$$

for a, $b\in A$ with a, $b>0$.

Now, we have

$$\begin{aligned} & f(b)-f(a)-Df(a)(b-a) \\ &= qb^{-1}q-qa^{-1}q-qa^{-1}(a-b)\,a^{-1}q \\ &= q\left(b^{-1}-a^{-1}\right)q-qa^{-1}(a-b)\,a^{-1}q \\ &= qb^{-1}(a-b)\,a^{-1}q-qa^{-1}(a-b)\,a^{-1}q=\left(qb^{-1}-qa^{-1}\right)(a-b)\,a^{-1}q \\ &= q\left(b^{-1}-a^{-1}\right)bb^{-1}(a-b)\,a^{-1}q=q\left(b^{-1}-a^{-1}\right)b\left(b^{-1}-a^{-1}\right)q\geq 0 \end{aligned}$$

for a, $b\in A$ with a, $b>0$.

This proves the statement. □

Corollary 3. *The function $f(z) = z^{-1}$ is operator convex on $(0, \infty)$ in the Hermitian Banach $*$-algebra A.*

We also have:

Theorem 2. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have for all t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ that*

$$\begin{aligned} F'_{(a,b)}(t_2) &= Df((1-t_2)a + t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a + t_1 b)(b-a) = F'_{(a,b)}(t_1). \end{aligned} \tag{2.13}$$

We also have

$$Df(b)(b-a) \geq F'_{(a,b)}(t) = Df((1-t)a + tb)(b-a) \geq Df(a)(b-a) \tag{2.14}$$

for all $t \in (0,1)$.

Proof. Assume that the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A.

Let c, $d \in A$ with $\sigma(c)$, $\sigma(d) \subset I$, then by taking $b = d$ and $a = c$, we have

$$f(d) - f(c) \geq Df(c)(d-c)$$

and by taking $b = c$ and $a = d$, we have

$$f(c) - f(d) \geq Df(d)(c-d)$$

which imply the double inequality

$$Df(d)(d-c) \geq f(d) - f(c) \geq Df(c)(d-c) \tag{2.15}$$

for all c, $d \in A$ with $\sigma(c)$, $\sigma(d) \subset I$.

Let t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ and $a, b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$. Then $\sigma((1-t_1)a + t_1 b)$, $\sigma((1-t_2)a + t_2 b) \subset I$ and by (2.15) for $d = (1-t_2)a + t_2 b$ and $c = (1-t_1)a + t_1 b$ we get

$$\begin{aligned} &Df((1-t_2)a + t_2 b)((1-t_2)a + t_2 b - (1-t_1)a - t_1 b) \\ &\geq f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b) \\ &\geq Df((1-t_1)a + t_1 b)((1-t_2)a + t_2 b - (1-t_1)a - t_1 b), \end{aligned}$$

namely

$$\begin{aligned} &(t_2 - t_1) Df((1-t_2)a + t_2 b)(b-a) \\ &\geq f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b) \\ &\geq (t_2 - t_1) Df((1-t_1)a + t_1 b)(b-a), \end{aligned}$$

which implies that

$$\begin{aligned} &Df((1-t_2)a + t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a + t_1 b)(b-a), \end{aligned}$$

for all $t_2,\ t_1 \in [0,1]$ with $t_1 < t_2$ and $a,\ b \in A$ with $\sigma(a),\ \sigma(b) \subset I$, and the inequality (2.13) is proved.

Now, if the condition (2.13) is valid, then by taking $t_1 = 0$ and $t_1 = 1$, then we get

$$Df(b)(b-a) \geq f(b) - f(a) \geq Df(a)(b-a)$$

for all $a,\ b \in A$ with $\sigma(a),\ \sigma(b) \subset I$, and by Theorem 1 it follows that $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A.

If we take $t_1 = 0$ and $t_2 = t \in (0,1]$ in (2.13), then we get

$$Df((1-t)a + tb)(b-a) \geq Df(a)(b-a).$$

Also, if we take $t_1 = t \in [0,1)$ and $t_2 = 1$ in (2.13), then we get

$$Df(b)(b-a) \geq Df((1-t)a + tb)(b-a).$$

□

3. Reverse Hermite-Hadamard Inequalities

We have the following reverse of the first operator Hermite-Hadamard inequality:

Theorem 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A then for all $a,\ b \in A$ with $\sigma(a),\ \sigma(b) \subset I$ we have*

$$0 \leq \int_0^1 f((1-t)a + tb)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{3.1}$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\int_0^{1/2} tF'_{(a,b)}(t)\,dt = \frac{1}{2}F_{(a,b)}\left(\frac{1}{2}\right) - \int_0^{1/2} F_{(a,b)}(t)\,dt = \frac{1}{2}f\left(\frac{a+b}{2}\right) - \int_0^{1/2} f((1-t)a + tb)\,dt \tag{3.2}$$

and

$$\int_{1/2}^1 (t-1)F'_{(a,b)}(t)\,dt = \frac{1}{2}F_{(a,b)}\left(\frac{1}{2}\right) - \int_{1/2}^1 f((1-t)a + tb)\,dt = \frac{1}{2}f\left(\frac{a+b}{2}\right) - \int_{1/2}^1 f((1-t)a + tb)\,dt. \tag{3.3}$$

If we add these two equalities, we get the following identity of interest

$$\int_0^1 f((1-t)a + tb)\,dt - f\left(\frac{a+b}{2}\right) = \int_{1/2}^1 (1-t)F'_{(a,b)}(t)\,dt - \int_0^{1/2} tF'_{(a,b)}(t)\,dt. \tag{3.4}$$

From Theorem 2 we have

$$F'_{(a,b)}(1/2) \leq F'_{(a,b)}(t) \leq F'_{(a,b)}(1-) = Df(b)(b-a),\ \ t \in [1/2,1) \tag{3.5}$$

and

$$Df(a)(b-a) = F'_{(a,b)}(0+) \le F'_{(a,b)}(t) \le F'_{(a,b)}(1/2), \; t \in (0, 1/2]. \tag{3.6}$$

This implies that

$$(1-t) F'_{(a,b)}(1/2) \le (1-t) F'_{(a,b)}(t) \le (1-t) Df(b)(b-a)$$

for $t \in [1/2, 1)$ and

$$-tF'_{(a,b)}(1/2) \le -tF'_{(a,b)}(t) \le -tDf(a)(b-a)$$

for $t \in (0, 1/2]$.

By integrating these inequalities on the corresponding intervals, we get

$$\frac{1}{8} F'_{(a,b)}(1/2) \le \int_{1/2}^{1} (1-t) F'_{(a,b)}(t)\, dt \le \frac{1}{8} Df(b)(b-a)$$

and

$$-\frac{1}{8} F'_{(a,b)}(1/2) \le -\int_{0}^{1/2} tF'_{(a,b)}(t)\, dt \le -\frac{1}{8} Df(a)(b-a).$$

By addition, we deduce that

$$0 \le \int_{1/2}^{1} (1-t) F'_{(a,b)}(t) - \int_{0}^{1/2} tF'_{(a,b)}(t)\, dt \le \frac{1}{8} [Df(b)(b-a) - Df(a)(b-a)]$$

and by the identity (3.4) we get (3.1). □

We have the following reverse of the second operator Hermite-Hadamard inequality:

Theorem 4. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have*

$$\begin{aligned} 0 &\le \frac{f(a) + f(b)}{2} - \int_0^1 f((1-t)a + tb)\, dt \\ &\le \frac{1}{8} [Df(b)(b-a) - Df(a)(b-a)]. \end{aligned} \tag{3.7}$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\begin{aligned} \int_0^1 \left(t - \frac{1}{2}\right) F'_{(a,b)}(t)\, dt &= \frac{F_{(a,b)}(1) + F_{(a,b)}(0)}{2} - \int_0^1 F_{(a,b)}(t) \\ &= \frac{f(b) + f(a)}{2} - \int_0^1 f((1-t)a + tb)\, dt. \end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned} &\int_0^1 \left(t - \frac{1}{2}\right) F'_{(a,b)}(t)\, dt \\ &= \int_{1/2}^1 \left(t - \frac{1}{2}\right) F'_{(a,b)}(t)\, dt - \int_0^{1/2} \left(\frac{1}{2} - t\right) F'_{(a,b)}(t)\, dt. \end{aligned} \tag{3.9}$$

Therefore, we have the following identity of interest

$$\frac{f(b)+f(a)}{2}-\int_0^1 f((1-t)a+tb)\,dt$$
$$=\int_{1/2}^1\left(t-\frac{1}{2}\right)F'_{(a,b)}(t)\,dt-\int_0^{1/2}\left(\frac{1}{2}-t\right)F'_{(a,b)}(t)\,dt.$$

From the inequality (3.5) we obtain

$$\left(t-\frac{1}{2}\right)F'_{(a,b)}(1/2)\le\left(t-\frac{1}{2}\right)F'_{(a,b)}(t)$$
$$\le\left(t-\frac{1}{2}\right)Df(b)(b-a),\ \ t\in[1/2,1)$$

and from (3.6)

$$\left(\frac{1}{2}-t\right)Df(a)(b-a)\le\left(\frac{1}{2}-t\right)F'_{(a,b)}(t)$$
$$\le\left(\frac{1}{2}-t\right)F'_{(a,b)}(1/2),\ \ t\in(0,1/2],$$

namely

$$-\left(\frac{1}{2}-t\right)F'_{(a,b)}(1/2)\le-\left(\frac{1}{2}-t\right)F'_{(a,b)}(t)$$
$$\le-\left(\frac{1}{2}-t\right)Df(a)(b-a),\ \ t\in(0,1/2].$$

Integrating these inequalities on the corresponding intervals, we get

$$\frac{1}{8}F'_{(a,b)}(1/2)\le\int_{1/2}^1\left(t-\frac{1}{2}\right)F'_{(a,b)}(t)\,dt\le\frac{1}{8}Df(b)(b-a),$$

and

$$-\frac{1}{8}F'_{(a,b)}(1/2)\le-\int_0^{1/2}\left(\frac{1}{2}-t\right)F'_{(a,b)}(t)\,dt\le-\frac{1}{8}Df(a)(b-a).$$

If we add these two inequalities, we obtain

$$0\le\int_{1/2}^1\left(t-\frac{1}{2}\right)F'_{(a,b)}(t)\,dt-\int_0^{1/2}\left(\frac{1}{2}-t\right)F'_{(a,b)}(t)\,dt$$
$$\le\frac{1}{8}\left[Df(b)(b-a)-Df(a)(b-a)\right],$$

which, by the use of identity (3.9) produces the desired result (3.7). □

For the function $f(z)=z^{-1}$ we have for $a,b>0$ that

$$Df(a)(b-a)=-a^{-1}(b-a)a^{-1}\text{ and }Df(b)(b-a)=-b^{-1}(b-a)b^{-1}.$$

Therefore, by the inequality (3.1) we have

$$0\le\int_0^1((1-t)a+tb)^{-1}\,dt-\left(\frac{a+b}{2}\right)^{-1}\tag{3.10}$$
$$\le\frac{1}{8}\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right]$$

while from (3.7) we have

$$0 \leq \frac{a^{-1}+b^{-1}}{2} - \int_0^1 ((1-t)\,a + tb)^{-1}\,dt \leq \frac{1}{8}\left[a^{-1}(b-a)\,a^{-1} - b^{-1}(b-a)\,b^{-1}\right]. \tag{3.11}$$

for $a,\, b \in A$ with $a,\, b > 0$.

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