INEQUALITIES OF FÉJER'S TYPE FOR OPERATOR CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. We establish in this paper some inequalities of Féjer's type for operator convex functions on Hermitian unital Banach $*$-algebras.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [10] (see also [1, Theorem 41.5]), then

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [9], Tanahashi and Uchiyama [11] proved the following fundamental properties (see also [7]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [9] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Hermitian unital Banach $*$-algebra, Hermite-Hadamard type inequalities, Féjer inequalities, Operator convex functions.

RGMIA Res. Rep. Coll. 23 (2020), Art. 18, 12 pp. Received 31/01/20

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re}z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz,$$

where γ is a close rectifiable curve such that $\sigma(a)\subset\operatorname{ins}(\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^\alpha:=\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a)^{-1}dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha\in A$. Moreover, since z^α is analytic in $\{\operatorname{Re}z>0\}$, then by (SMT) we have

$$\sigma(a^\alpha)=(\sigma(a))^\alpha=\{z^\alpha:z\in\sigma(a)\}\subset(0,\infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^\alpha\in A$ with $a^\alpha>0$ and $(a^2)^{1/2}=a$, [11, Lemma 6];
(ix) If $0<a\in A$ and $\alpha,\beta\in\mathbb{R}$, then $a^\alpha a^\beta=a^{\alpha+\beta}$;
(x) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $(a^\alpha)^{-1}=(a^{-1})^\alpha=a^{-\alpha}$;
(xi) If $0<a$, $b\in A$, $\alpha,\beta\in\mathbb{R}$ and $ab=ba$, then $a^\alpha b^\beta=b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I\subset G$ assume that $f(z)\geq0$ for any $z\in I$. If $u\in A$ such that $\sigma(u)\subset I$, then by (SMT) we have

$$\sigma(f(u))=f(\sigma(u))\subset f(I)\subset[0,\infty)$$

meaning that $f(u)\geq0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I\subset G$, assume that $f(z)\geq g(z)$ for any $z\in I$. Then for any $u\in A$ with $\sigma(u)\subset I$ we have $f(u)\geq g(u)$ in the order of A.*

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let G be an open subset of $\mathbb{C}$ and $I\subset G$ a real interval. If a, $b\in A$ with $\sigma(a)$, $\sigma(b)\subset I$, then by SMT the element $(1-t)a+tb\in A$ has the spectrum $\sigma((1-t)a+tb)\subset I$ for all $t\in[0,1]$. We say that an analytic function $f(z)$ in G is *operator convex* on I in the Hermitian Banach $*$-algebra A if

$$f((1-t)a+tb)\leq(1-t)f(a)+tf(b)\text{ in the order of }A \tag{1.1}$$

for all a, $b\in A$ with $\sigma(a)$, $\sigma(b)\subset I$ and all $t\in[0,1]$.

In the recent paper [6] we obtained the following results:

Theorem 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have*

$$f(b) - f(a) \geq Df(a)(b-a) \tag{1.2}$$

in the order of A, where Df is the Fréchet derivative of f as a function of elements in the Hermitian Banach $*$-algebra A.

Let $f(z)$ be analytic in G, an open convex subset of $\mathbb{C}$ and a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset G$. Consider the auxiliary function $F_{(a,b)} : [0,1] \to A$ defined by

$$F_{(a,b)}(t) := f((1-t)a + tb).$$

Theorem 2. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have for all t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ that*

$$\begin{aligned} F'_{(a,b)}(t_2) &= Df((1-t_2)a + t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a + t_1 b)(b-a) = F'_{(a,b)}(t_1). \end{aligned} \tag{1.3}$$

We also have

$$Df(b)(b-a) \geq F'_{(a,b)}(t) = Df((1-t)a + tb)(b-a) \geq Df(a)(b-a) \tag{1.4}$$

for all $t \in (0,1)$.

It is well known that, if E is a Banach space and $g : [0,1] \to E$ is a continuous function, then g is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_0^1 g(t)\,dt$.

From the operator convexity of the function f on I in the Hermitian Banach $*$-algebra A we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f((1-s)a + sb) + f(sa + (1-s)b)\right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \tag{1.5}$$

for all $s \in [0,1]$ and a, b selfadjoint elements with spectra included in I.

If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s) = p(s)$ for all $s \in [0,1]$, then by multiplying (1.5) with $p(s)$, integrating on $[0,1]$ and taking into account that

$$\int_0^1 p(s) f((1-s)a + sb)\,ds = \int_0^1 p(s) f(sa + (1-s)b)\,ds,$$

we get for a, b selfadjoint elements with spectra included in I

$$\begin{aligned} \left(\int_0^1 p(s)\,ds\right) f\left(\frac{a+b}{2}\right) &\leq \int_0^1 p(s) f(sa + (1-s)b)\,ds \\ &\leq \left(\int_0^1 p(s)\,ds\right) \frac{f(a) + f(b)}{2}, \end{aligned} \tag{1.6}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

Motivated by the above results, we establish in this paper a reverse for each inequality in (1.6) in the case of operator convexity of the function f on I in the Hermitian Banach $*$-algebra A. A particular example of interest for $f(z) = z^{-1}$ is also given.

2. Reverse Operator Féjer Inequalities

We have:

Theorem 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p:[0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have*

$$0 \le \int_0^1 p(t) f((1-t)a + tb)\,dt - \left(\int_0^1 p(t)\,dt\right) f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left(\int_0^1 \left|t - \frac{1}{2}\right| p(t)\,dt\right)[Df(b)(b-a) - Df(a)(b-a)]. \tag{2.1}$$

In particular, for $p \equiv 1$ we get

$$0 \le \int_0^1 f((1-t)a + tb)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}[Df(b)(b-a) - Df(a)(b-a)]. \tag{2.2}$$

Proof. Let $a, b \in A$, with $a \ne b$. Using the integration by parts formula for Bochner's integral, we have

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) F'_{(a,b)}(t)\,dt \\
&= \left(\int_t^1 p(s)\,ds\right) F_{(a,b)}(t)\Big]_{1/2}^1 + \int_{1/2}^1 p(t) F_{(a,b)}(t)\,dt \\
&= -\left(\int_{1/2}^1 p(s)\,ds\right) F_{(a,b)}(1/2) + \int_{1/2}^1 p(t) F_{(a,b)}(t)\,dt \\
&= -\left(\int_{1/2}^1 p(s)\,ds\right) f\left(\frac{a+b}{2}\right) + \int_{1/2}^1 p(t) F_{(a,b)}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1/2} \left(\int_0^t p(s)\,ds\right) F'_{(a,b)}(t)\,dt \\
&= \left(\int_0^t p(s)\,ds\right) F_{(a,b)}(t)\Big]_0^{1/2} - \int_0^{1/2} p(t) F_{(a,b)}(t)\,dt
\end{aligned}$$

$$= \left(\int_0^{1/2} p(s)\,ds \right) F_{(a,b)}(1/2) - \int_0^{1/2} p(t)\,F_{(a,b)}(t)\,dt$$
$$= \left(\int_0^{1/2} p(s)\,ds \right) f\left(\frac{a+b}{2} \right) - \int_0^{1/2} p(t)\,F_{(a,b)}(t)\,dt.$$

By subtracting the second identity from the first, we get

$$\int_{1/2}^1 \left(\int_t^1 p(s)\,ds \right) F'_{(a,b)}(t)\,dt - \int_0^{1/2} \left(\int_0^t p(s)\,ds \right) F'_{(a,b)}(t)\,dt$$
$$= \int_{1/2}^1 p(t)\,F_{(a,b)}(t)\,dt + \int_0^{1/2} p(t)\,F_{(a,b)}(t)\,dt$$
$$- \left(\int_{1/2}^1 p(s)\,ds \right) f\left(\frac{a+b}{2} \right) - \left(\int_0^{1/2} p(s)\,ds \right) f\left(\frac{a+b}{2} \right).$$

and then we get the following identity of interest in itself

$$\int_0^1 p(t)\,F_{(a,b)}(t)\,dt - \int_0^1 p(s)\,ds f\left(\frac{a+b}{2} \right)$$
$$= \int_{1/2}^1 \left(\int_t^1 p(s)\,ds \right) F'_{(a,b)}(t)\,dt - \int_0^{1/2} \left(\int_0^t p(s)\,ds \right) F'_{(a,b)}(t)\,dt. \tag{2.3}$$

By Theorem 2 we have

$$Df(a)(b-a) \le F'_{(a,b)}(t) \le F'_{(a,b)}\left(\frac{1}{2} \right),\ t \in [0,1/2]$$

and

$$F'_{(a,b)}\left(\frac{1}{2} \right) \le F'_{(a,b)}(t) \le Df(b)(b-a),\ t \in [1/2,1]$$

in the order of A.

These imply that

$$\left(\int_0^t p(s)\,ds \right) Df(a)(b-a) \le \left(\int_0^t p(s)\,ds \right) F'_{(a,b)}(t)$$
$$\le \left(\int_0^t p(s)\,ds \right) F'_{(a,b)}\left(\frac{1}{2} \right),\ t \in [0,1/2]$$

and

$$\left(\int_t^1 p(s)\,ds \right) F'_{(a,b)}\left(\frac{1}{2} \right) \le \left(\int_t^1 p(s)\,ds \right) F'_{(a,b)}(t)$$
$$\le \left(\int_t^1 p(s)\,ds \right) Df(b)(b-a),\ t \in [1/2,1],$$

and by integration

$$\int_{1/2}^1 \left(\int_t^1 p(s)\,ds \right) dt F'_{(a,b)}\left(\frac{1}{2} \right) \le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds \right) F'_{(a,b)}(t)\,dt$$
$$\le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds \right) dt Df(b)(b-a)$$

and

$$-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt F'_{(a,b)}\left(\frac{1}{2}\right) \le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)F'_{(a,b)}(t)\,dt$$
$$\le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt Df(a)(b-a).$$

If we add these inequalities, then we get

$$\left[\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\right]F'_{(a,b)}\left(\frac{1}{2}\right) \tag{2.4}$$
$$\le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)F'_{(a,b)}(t)\,dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)F'_{(a,b)}(t)\,dt$$
$$\le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt Df(b)(b-a)$$
$$-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt Df(a)(b-a)$$

in the order of A.

Integrating by parts in the Lebesgue integral, we have

$$\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt = \left(\int_t^1 p(s)\,ds\right)t\Big]_{1/2}^1 + \int_{1/2}^1 tp(t)\,dt$$
$$= \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_{1/2}^1 p(s)\,ds$$
$$= \int_{1/2}^1\left(t-\frac{1}{2}\right)p(t)\,dt$$

and

$$\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt = \left(\int_0^t p(s)\,ds\right)t\Big]_0^{1/2} - \int_0^{1/2} p(t)\,tdt$$
$$= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \int_0^{1/2} p(t)\,tdt$$
$$= \int_0^{1/2}\left(\frac{1}{2}-t\right)p(t)\,dt.$$

Then

$$\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt$$
$$= \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_{1/2}^1 p(s)\,ds - \frac{1}{2}\int_0^{1/2} p(s)\,ds + \int_0^{1/2} p(t)\,tdt$$
$$= \int_0^1 tp(t)\,dt - \int_{1/2}^1 p(s)\,ds = \int_0^1 tp(t)\,dt - \frac{1}{2}\int_0^1 p(t)\,dt$$
$$= \int_0^1\left(t-\frac{1}{2}\right)p(t)\,dt = 0,$$

since the function $q(t) = \left(t - \frac{1}{2}\right) p(t)$ is asymmetric on $[0,1]$.

Also by changing the variable $s = 1 - t$, we have

$$\int_0^{1/2} \left(\frac{1}{2} - t\right) p(t)\, dt = \int_{1/2}^1 \left(s - \frac{1}{2}\right) p(1-s)\, ds = \int_{1/2}^1 \left(s - \frac{1}{2}\right) p(s)\, ds$$
$$= \frac{1}{2} \int_0^1 \left|s - \frac{1}{2}\right| p(s)\, ds.$$

Finally, by utilising (2.4) we obtain the desired result (2.1). □

Remark 1. *If we take* $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ *in (2.1), then we get*

$$0 \le \int_0^1 \left|t - \frac{1}{2}\right| f((1-t)a + tb)\, dt - \frac{1}{4} f\left(\frac{a+b}{2}\right) \le \frac{1}{24} \left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{2.5}$$

If we take $p(t) = t(1-t)$, $t \in [0,1]$ *in (2.1), then we get*

$$0 \le \int_0^1 t(1-t) f((1-t)a + tb)\, dt - \frac{1}{6} f\left(\frac{a+b}{2}\right) \le \frac{1}{64} \left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{2.6}$$

We also have:

Theorem 4. *Assume that* $f(z)$ *is analytic in* G*, an open subset of* $\mathbb{C}$ *and* $I \subset G$ *a real interval. If the function* $f(z)$ *is operator convex on* I *in the Hermitian Banach* $*$*-algebra* A *and* $p : [0,1] \to [0,\infty)$ *is Lebesgue integrable and symmetric, namely* $p(1-t) = p(t)$ *for all* $t \in [0,1]$*, then for all* $a, b \in A$ *with* $\sigma(a), \sigma(b) \subset I$ *we have*

$$0 \le \left(\int_0^1 p(t)\, dt\right) \frac{f(a) + f(b)}{2} - \int_0^1 p(t) f((1-t)a + tb)\, dt \le \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right) p(t)\, dt \left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{2.7}$$

In particular, for $p \equiv 1$ *we get*

$$0 \le \frac{f(a) + f(b)}{2} - \int_0^1 f((1-t)a + tb)\, dt \le \frac{1}{8} \left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{2.8}$$

Proof. Using the integration by parts for Bochner's integral, we have

$$\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$= \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right) F_{(a,b)}(t)\Big]_0^1 - \int_0^1 p(t)\,F_{(a,b)}(t)\,dt$$
$$= \left(\int_0^1 p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right) F_{(a,b)}(1) + \left(\frac{1}{2}\int_0^1 p(s)\,ds\right) F_{(a,b)}(0)$$
$$- \int_0^1 p(t)\,F_{(a,b)}(t)\,dt$$
$$= \left(\int_0^1 p(t)\,dt\right) \frac{f(a)+f(b)}{2} - \int_0^1 p(t)\,F_{(a,b)}(t)\,dt.$$

We also have, by the symmetry of p, that

$$\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$= \int_0^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$= \int_0^{1/2} \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$+ \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$= \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$- \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds\right) F'_{(a,b)}(t)\,dt.$$

Observe that

$$\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \geq 0 \text{ for } t \in [1/2, 1]$$

and

$$\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \geq 0 \text{ for } t \in [0, 1/2]\,.$$

By Theorem 2 we have in the operatorial order the following inequalities

$$\int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) dt F'_{(a,b)}\left(\frac{1}{2}\right)$$
$$\leq \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) F'_{(a,b)}(t)\,dt$$
$$\leq \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right) dt Df(b)(b-a)$$

and

$$
\begin{aligned}
&-\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt F'_{(a,b)}\left(\frac{1}{2}\right)\\
&\le -\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)F'_{(a,b)}(t)\,dt\\
&\le -\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt Df(a)(b-a).
\end{aligned}
$$

If we add these inequalities, then we get

$$
\begin{aligned}
&\left[\int_{1/2}^1\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)dt\right.\\
&\left.-\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt\right]\times F'_{(a,b)}\left(\frac{1}{2}\right)\\
&\le \int_{1/2}^1\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)F'_{(a,b)}(t)\,dt\\
&-\int_{1/2}^1\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)F'_{(a,b)}(t)\,dt\\
&\le \int_{1/2}^1\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)dt Df(b)(b-a)\\
&-\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt Df(a)(b-a).
\end{aligned}
\tag{2.9}
$$

Observe that

$$
\begin{aligned}
&\int_{1/2}^1\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)dt\\
&-\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt\\
&=\int_{1/2}^1\left(\int_0^t p(s)\,ds\right)dt-\frac{1}{2}\int_0^{1/2} p(s)\,ds\\
&-\frac{1}{2}\int_0^{1/2} p(s)\,ds+\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\\
&=\int_0^1\left(\int_0^t p(s)\,ds\right)dt-\int_0^{1/2} p(s)\,ds\\
&=\int_0^1\left(\int_0^t p(s)\,ds\right)dt-\frac{1}{2}\int_0^1 p(t)\,dt
\end{aligned}
$$

$$= \left(\int_0^t p(s)\,ds\right)t\Big]_0^1 - \int_0^1 p(t)\,tdt - \frac{1}{2}\int_0^1 p(t)\,dt$$
$$= \int_0^1 p(t)\,dt - \int_0^1 p(t)\,tdt - \frac{1}{2}\int_0^1 p(t)\,dt = \int_0^1 \left(\frac{1}{2} - t\right)p(t)\,dt = 0$$

since the function $q(t) = \left(t - \frac{1}{2}\right)p(t)$ is asymmetric on $[0,1]$.

Therefore, by the first inequality in (2.9) we get the first inequality in (2.7).

We also have

$$\int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)dt$$
$$= \int_{1/2}^1 \left(\int_0^t p(s)\,ds\right)dt - \frac{1}{2}\int_0^{1/2} p(s)\,ds$$
$$= \left(\int_0^t p(s)\,ds\right)t\Big]_{1/2}^1 - \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_0^{1/2} p(s)\,ds$$
$$= \int_0^1 p(s)\,ds - \frac{1}{2}\int_0^{1/2} p(s)\,ds - \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_0^{1/2} p(s)\,ds$$
$$= \int_0^1 p(s)\,ds - \int_0^{1/2} p(s)\,ds - \int_{1/2}^1 tp(t)\,dt$$
$$= \int_{1/2}^1 p(s)\,ds - \int_{1/2}^1 tp(t)\,dt = \int_{1/2}^1 (1-t)\,p(t)\,dt$$

and

$$\int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds\right)dt$$
$$= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt$$
$$= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \left(\left(\int_0^t p(s)\,ds\right)t\Big]_0^{1/2} - \int_0^{1/2} tp(t)\,dt\right)$$
$$= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \frac{1}{2}\int_0^{1/2} p(s)\,ds + \int_0^{1/2} tp(t)\,dt = \int_0^{1/2} tp(t)\,dt.$$

If we change the variable $s = 1 - t$, then

$$\int_0^{1/2} tp(t)\,dt = -\int_1^{1/2} (1-s)\,p(1-s)\,ds = \int_{1/2}^1 (1-s)\,p(1-s)\,ds \tag{2.10}$$
$$= \int_{1/2}^1 (1-s)\,p(s)\,ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\frac{1}{2}+t\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-t+\frac{1}{2}\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}tp(t)\,dt+\frac{1}{2}\int_{1/2}^1(1-t)\,p(t)\,dt=\int_0^{1/2}tp(t)\,dt\ (\text{by }2.10)
\end{aligned}$$

and by the second inequality in (2.9) we get the second part of (2.7). □

Remark 2. *If we take* $p(t)=\left|t-\frac{1}{2}\right|,\ t\in[0,1]$ *in (2.7), then we get*

$$\begin{aligned}
0&\le\frac{f(a)+f(b)}{8}-\int_0^1\left|t-\frac{1}{2}\right|f((1-t)a+tb)\,dt \\
&\le\frac{1}{48}\left[Df(b)(b-a)-Df(a)(b-a)\right].
\end{aligned}\tag{2.11}$$

If we take $p(t)=t(1-t),\ t\in[0,1]$ *in (2.7), then we get*

$$\begin{aligned}
0&\le\frac{f(a)+f(b)}{12}-\int_0^1 t(1-t)\,f((1-t)a+tb)\,dt \\
&\le\frac{5}{192}\left[Df(b)(b-a)-Df(a)(b-a)\right].
\end{aligned}\tag{2.12}$$

The function $f(z)=z^{-1}$ is operator convex on $(0,\infty)$ in the Hermitian Banach $*$-algebra A and we have for $a,\ b>0$ that

$$Df(a)(b-a)=-a^{-1}(b-a)\,a^{-1}\text{ and }Df(b)(b-a)=-b^{-1}(b-a)\,b^{-1}.$$

If we write the inequalities (2.1) and (2.7) for this function, then we get

$$\begin{aligned}
0&\le\int_0^1 p(t)\left((1-t)a+tb\right)^{-1}dt-\left(\int_0^1 p(t)\,dt\right)\left(\frac{a+b}{2}\right)^{-1}\\
&\le\frac{1}{2}\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\left[a^{-1}(b-a)\,a^{-1}-b^{-1}(b-a)\,b^{-1}\right]
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
0&\le\left(\int_0^1 p(t)\,dt\right)\frac{a^{-1}+b^{-1}}{2}-\int_0^1 p(t)\left((1-t)a+tb\right)^{-1}dt\\
&\le\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\left[a^{-1}(b-a)\,a^{-1}-b^{-1}(b-a)\,b^{-1}\right]
\end{aligned}\tag{2.14}$$

for all $a,\ b>0$.

From (2.5) we have

$$\begin{aligned}
0&\le\int_0^1\left|t-\frac{1}{2}\right|\left((1-t)a+tb\right)^{-1}dt-\frac{1}{4}\left(\frac{a+b}{2}\right)^{-1}\\
&\le\frac{1}{24}\left[a^{-1}(b-a)\,a^{-1}-b^{-1}(b-a)\,b^{-1}\right]
\end{aligned}\tag{2.15}$$

while from (2.11) we get

$$0 \le \frac{a^{-1}+b^{-1}}{8} - \int_0^1 \left| t - \frac{1}{2} \right| ((1-t)\,a + tb)^{-1}\,dt \le \frac{1}{48}\left[a^{-1}(b-a)\,a^{-1} - b^{-1}(b-a)\,b^{-1}\right] \quad (2.16)$$

for all a, $b > 0$.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.

[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[3] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach $*$-algebras, *Oper. Matrices* **12** (2018), no. 4, 1009–1026. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 162. [http://rgmia.org/papers/v19/v19a162.pdf].

[4] S. S. Dragomir, Multiplicative inequalities for weighted geometric mean in Hermitian unital Banach *-algebras. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **112** (2018), no. 4, 1349–1365.

[5] S. S. Dragomir, Inequalities of Jensen's type for positive linear functionals on Hermitian unital Banach $*$-algebras, *Bull. Aust. Math. Soc.* (First published online 2020), page 1 of 11 doi:10.1017/S000497271900131X. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 142. 10 pp. [https://rgmia.org/papers/v19/v19a172.pdf].

[6] S. S. Dragomir, Inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach $*$-algebras, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art.

[7] B. Q. Feng, The geometric means in Banach $*$-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.

[8] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990.

[9] T. Okayasu, The Löwner-Heinz inequality in Banach $*$-algebra, *Glasgow Math. J.* **42** (2000), 243-246.

[10] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.

[11] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach $*$-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa