SOME DISCRETE INEQUALITIES OF JENSEN TYPE FOR OPERATOR CONVEX FUNCTIONS IN HERMITIAN BANACH $*$-ALGEBRAS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Some inequalities for operator convex functions of selfadjoint elements in Hermitian Banach $*$-Algebras that are related to the Jensen inequality are given.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [10] (see also [1, Theorem 41.5]), then

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [9], Tanahashi and Uchiyama [11] proved the following fundamental properties (see also [7]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [9] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

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In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz,$$

where γ is a close rectifiable curve such that $\sigma(a) \subset \operatorname{ins}(\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [11, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;

(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let G be an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$, then by SMT the element $(1-t)a + tb \in A$ has the spectrum $\sigma((1-t)a + tb) \subset I$ for all $t \in [0,1]$. We say that an analytic function $f(z)$ in G is *operator convex* on I in the Hermitian Banach $*$-algebra A if

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b) \text{ in the order of } A \tag{1.1}$$

for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ and all $t \in [0,1]$.

In the recent paper [6] we obtained the following results:

Theorem 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have*

$$f(b) - f(a) \geq Df(a)(b-a) \tag{1.2}$$

in the order of A, where Df is the Fréchet derivative of f as a function of elements in the Hermitian Banach $*$-algebra A.

Let $f(z)$ be analytic in G, an open convex subset of $\mathbb{C}$ and $a, b \in A$ with $\sigma(a)$, $\sigma(b) \subset G$. Consider the auxiliary function $F_{(a,b)} : [0,1] \to A$ defined by

$$F_{(a,b)}(t) := f((1-t)a + tb).$$

The following characterization result also holds:

Theorem 2. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have for all t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ that*

$$\begin{aligned} F'_{(a,b)}(t_2) &= Df((1-t_2)a + t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a + t_1 b)(b-a) = F'_{(a,b)}(t_1). \end{aligned} \tag{1.3}$$

We also have

$$Df(b)(b-a) \geq F'_{(a,b)}(t) = Df((1-t)a + tb)(b-a) \geq Df(a)(b-a) \tag{1.4}$$

for all $t \in (0,1)$.

Let a_j be selfadjoint elements in A with $\mathrm{Sp}(a_j) \subseteq I$, $j \in \{1,...,n\}$. If $p_j \geq 0$, $j \in \{1,...,n\}$ with $P_n > 0$ and $f(z)$ is *operator convex* on I in the Hermitian Banach $*$-algebra A, then we have the discrete Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i), \tag{1.5}$$

in the order of A.

This is a well known result for selfadjoint operators and operator convex functions in Hilbert spaces and can be proved easily in the more general Hermitian Banach $*$-algebra A by mathematical induction over $n \geq 2$. The details are left to the reader.

A simple proof can also be done by using (1.2) from which we get

$$f(b) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \geq Df\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(b - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right),$$

which implies that

$$f(a_k) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \geq Df\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(a_k - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right),$$

for $k \in \{1,...,n\}$.

If we multiply this inequality by $p_k \geq 0$, $k \in \{1,...,n\}$ with $P_n > 0$ and sum over k then we get

$$\begin{aligned}
&\sum_{k=1}^{n} p_k f(a_k) - \sum_{k=1}^{n} p_k f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \\
&\geq \sum_{k=1}^{n} p_k Df\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(a_k - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \\
&= Df\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{k=1}^{n} p_k a_k - \frac{1}{P_n}\sum_{k=1}^{n} p_k \sum_{i=1}^{n} p_i a_i\right) = 0.
\end{aligned}$$

Motivated by the above results, in this paper we establish some inequalities for operator convex functions of selfadjoint elements in Hermitian Banach $*$-algebras that are related to the Jensen inequality.

2. A Functional of Weights

We consider the functional

$$J_n(\mathbf{p};\mathbf{a},f,I) := \sum_{j=1}^{n} p_j f(a_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j a_j\right) \tag{2.1}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1,...,n\}$ and $P_n > 0$, $a= (a_1, ..., a_n)$ is an n-tuple of selfadjoint elements with $\operatorname{Sp}(a_j) \subseteq I$ for $j \in \{1,...,n\}$ and f is *operator convex* on I in the Hermitian Banach $*$-algebra A.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1,...,n\}$ and $P_n > 0$. For $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1,...,n\}$.

Theorem 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $a= (a_1, ..., a_n)$ an n-tuple of selfadjoint elements with $\operatorname{Sp}(a_j) \subseteq I$, $j \in \{1,...,n\}$, then for any $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q};\mathbf{a},f,I) \geq J_n(\mathbf{p};\mathbf{a},f,I) + J_n(\mathbf{q};\mathbf{a},f,I) \geq 0, \tag{2.2}$$

i.e., $J_n(\cdot;\mathbf{a},f,I)$ is a super-additive functional in the order of A.

Moreover, if $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p};\mathbf{a},f,I) \geq J_n(\mathbf{q};\mathbf{a},f,I) \geq 0, \tag{2.3}$$

i.e., $J_n(\cdot;\mathbf{a},f,I)$ is a monotonic nondecreasing functional in the order of A.

Proof. We have

$$\begin{aligned}
&J_n(\mathbf{p}+\mathbf{q};\mathbf{a},f,I) \\
&= \sum_{j=1}^{n}(p_j+q_j) f(a_j) - (P_n+Q_n) f\left(\frac{1}{P_n+Q_n}\sum_{j=1}^{n}(p_j+q_j) a_j\right)
\end{aligned} \tag{2.4}$$

$$= \sum_{j=1}^{n} (p_j + q_j) f(a_j)$$
$$- (P_n + Q_n) f \left(\frac{P_n \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) + Q_n \left(\frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j \right)}{P_n + Q_n} \right).$$

Now, consider the operators

$$a := \frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \text{ and } b := \frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j.$$

Then $\operatorname{Sp}(a), \operatorname{Sp}(b) \subseteq I$.

Applying the inequality (1.1) for a and b given above and $\lambda = \frac{Q_n}{P_n+Q_n}$ we have

$$f \left(\frac{P_n \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) + Q_n \left(\frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j \right)}{P_n + Q_n} \right) \tag{2.5}$$
$$\leq \frac{P_n}{P_n + Q_n} f \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) + \frac{Q_n}{P_n + Q_n} f \left(\frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j \right)$$

in the order of A.

Making use of (2.4) and (2.5) we have

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{a}, f, I) \tag{2.6}$$
$$\geq \sum_{j=1}^{n} (p_j + q_j) f(a_j) - (P_n + Q_n)$$
$$\times \left[\frac{P_n}{P_n + Q_n} f \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) + \frac{Q_n}{P_n + Q_n} f \left(\frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j \right) \right]$$
$$= \sum_{j=1}^{n} p_j f(a_j) - P_n f \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right)$$
$$+ \sum_{j=1}^{n} q_j f(a_j) - Q_n f \left(\frac{1}{Q_n} \sum_{j=1}^{n} q_j a_j \right)$$
$$= J_n(\mathbf{p}; \mathbf{a}, f, I) + J_n(\mathbf{q}; \mathbf{a}, f, I)$$

in the order of A, and the inequality (2.2) is proved.

Now, let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$. Then by the super-additivity property (2.2) we have

$$J_n(\mathbf{p}; \mathbf{a}, f, I) = J_n((\mathbf{p} - \mathbf{q}) + \mathbf{q}; \mathbf{a}, f, I) \tag{2.7}$$
$$\geq J_n((\mathbf{p} - \mathbf{q}); \mathbf{a}, f, I) + J_n(\mathbf{q}; \mathbf{a}, f, I) \geq J_n(\mathbf{q}; \mathbf{a}, f, I)$$

in the order of A, and the monotonicity property (2.3) is proved. □

Corollary 1. *Assume that the function f is operator convex on I in A and the n-tuple of selfadjoint elements$(a_1, ..., a_n)$ satisfies the condition $\operatorname{Sp}(a_j) \subseteq I$ for any*

$j\in\{1,...,n\}$. If $\mathbf{p}$, $\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m, M such that

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q} \tag{2.8}$$

then

$$mJ_n(\mathbf{q};\mathbf{a},f,I)\leq J_n(\mathbf{p};\mathbf{a},f,I)\leq MJ_n(\mathbf{q};\mathbf{a},f,I) \tag{2.9}$$

in the order of A.

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{p};\mathbf{a},f,I)=\alpha J_n(\mathbf{p};\mathbf{a},f,I)$.

Utilising the monotonicity property (2.3) we have

$$J_n(m\mathbf{q};\mathbf{a},f,I)\leq J_n(\mathbf{p};\mathbf{a},f,I)\leq J_n(M\mathbf{q};\mathbf{a},f,I),$$

which imply the desired result (2.9). □

Remark 1. *We observe that if all $q_j>0$, $j\in\{1,...,n\}$, then we have the inequality*

$$\min_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{a},f,I)\leq J_n(\mathbf{p};\mathbf{a},f,I)\leq\max_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{a},f,I) \tag{2.10}$$

in the order of A.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}$, $j\in\{1,...,n\}$, then we have the inequalities

$$n\min_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{a},f,I)\leq J_n(\mathbf{p};\mathbf{a},f,I)\leq n\max_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{a},f,I) \tag{2.11}$$

where

$$J_n(\mathbf{a},f,I):=\frac{1}{n}\sum_{j=1}^{n}f(a_j)-f\left(\frac{1}{n}\sum_{j=1}^{n}a_j\right). \tag{2.12}$$

For $n=2$ and by choosing $p_1=\alpha,p_2=1-\alpha$ with $\alpha\in[0,1]$, we get from (2.11) the inequalities

$$\begin{aligned}&2\min\{\alpha,1-\alpha\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\\&\leq(1-\alpha)f(a)+\alpha f(b)-f((1-\alpha)a+\alpha b)\\&\leq 2\max\{\alpha,1-\alpha\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right],\end{aligned} \tag{2.13}$$

in the order of A, where f is operator convex on I in A and a and b are two bounded selfadjoint elements in A with $\mathrm{Sp}(a)$, $\mathrm{Sp}(b)\subseteq I$.

We have some refinements of the power inequality as follows.

Remark 2. *Assume that* $(a_1,...,a_n)$ *is an* n*-tuple of positive elements in* A*. If* $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ *and* $q_j > 0$ *for* $j \in \{1,...,n\}$ *, then we have*

$$\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left(\sum_{j=1}^{n} q_j a_j^{-1} - Q_n^2\left(\sum_{j=1}^{n} q_j a_j\right)^{-1}\right) \tag{2.14}$$
$$\le \sum_{j=1}^{n} p_j a_j^{-1} - P_n^2\left(\sum_{j=1}^{n} p_j a_j\right)^{-1}$$
$$\le \max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left(\sum_{j=1}^{n} q_j a_j^{-1} - Q_n^2\left(\sum_{j=1}^{n} q_j a_j\right)^{-1}\right)$$

in the order of A*.*

When $q_j = \frac{1}{n}$, $j \in \{1,...,n\}$ we get from (2.14) the inequality

$$n\min_{j\in\{1,\dots,n\}}\{p_j\}\left(\frac{1}{n}\sum_{j=1}^{n} a_j^{-1} - n\left(\sum_{j=1}^{n} a_j\right)^{-1}\right) \tag{2.15}$$
$$\le \sum_{j=1}^{n} p_j a_j^{-1} - P_n^2\left(\sum_{j=1}^{n} p_j a_j\right)^{-1}$$
$$\le n\max_{j\in\{1,\dots,n\}}\{p_j\}\left(\frac{1}{n}\sum_{j=1}^{n} a_j^{-1} - n\left(\sum_{j=1}^{n} a_j\right)^{-1}\right).$$

The case for two elements is as follows:

$$2\min\{\alpha,1-\alpha\}\left[\frac{a^{-1}+b^{-1}}{2} - \left(\frac{a+b}{2}\right)^{-1}\right] \tag{2.16}$$
$$\le (1-\alpha)\,a^{-1} + \alpha b^{-1} - ((1-\alpha)\,a + \alpha b)^{-1}$$
$$\le 2\max\{\alpha,1-\alpha\}\left[\frac{a^{-1}+b^{-1}}{2} - \left(\frac{a+b}{2}\right)^{-1}\right],$$

where a and b are positive.

3. A Functional of Indices

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathrm{S}(A)$ the linear space of all sequences of selfadjoint elements defined on the complex Hilbert space, i.e.,

$$\mathcal{S}(A) = \left\{\mathbf{a} = (a_k)_{k\in\mathbb{N}} \mid a_k \text{ are selfadjoint elements on } A \text{ for all } k \in \mathbb{N}\right\}$$

and $\mathcal{S}_+(\mathbb{R})$ the family of nonnegative real sequences.

We consider the functional

$$J(K,\mathbf{p};\mathbf{a},f,I):=\sum_{k\in K}p_kf(a_k)-P_Kf\left(\frac{1}{P_K}\sum_{k\in K}p_ka_k\right) \tag{3.1}$$

where $K\in\mathcal{P}_f(\mathbb{N})$, $\mathbf{p}\in\mathcal{S}_+(\mathbb{R})$,$a\in\mathcal{S}(A)$) with $P_K:=\sum_{k\in K}p_k>0$ and $f:I\to\mathbb{R}$ is a operator convex function on the interval I in A.

Theorem 4. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I in A and $\mathbf{p}\in\mathcal{S}_+(\mathbb{R})$,$a\in\mathcal{S}(A)$. Assume that $\operatorname{Sp}(a_k)\subseteq I$ for any $k\in\mathbb{N}$.*

If $K,\ L\in\mathcal{P}_f(\mathbb{N})\diagdown\{\emptyset\}$ with $K\cap L=\emptyset$ and $P_K,\ P_L>0$, then we have the inequality

$$J(K\cup L,\mathbf{p};\mathbf{a},f,I)\geq J(K,\mathbf{p};\mathbf{a},f,I)+J(L,\mathbf{p};\mathbf{a},f,I)\geq 0, \tag{3.2}$$

i.e., $J(\cdot,\mathbf{p};\mathbf{a},f,I)$ is super-additive as an index set functional in the order of A.

If $\emptyset\neq K\subset L$ then we have

$$J(L,\mathbf{p};\mathbf{a},f,I)\geq J(K,\mathbf{p};\mathbf{a},f,I)\geq 0, \tag{3.3}$$

i.e., $J(\cdot,\mathbf{p};\mathbf{a},f,I)$ is monotonic as an index set functional in the order of A.

Proof. If $K,\ L\in\mathcal{P}_f(\mathbb{N})\diagdown\{\emptyset\}$ with $K\cap L=\emptyset$ and $P_K,\ P_L>0$, then we have the equality

$$\begin{aligned}&J(K\cup L,\mathbf{p};\mathbf{a},f,I)\\&=\sum_{k\in K\cup L}p_kf(a_k)-P_{K\cup L}f\left(\frac{1}{P_{K\cup L}}\sum_{k\in K\cup L}p_ka_k\right)\\&=\sum_{k\in K}p_kf(a_k)+\sum_{k\in L}p_kf(a_k)\\&-(P_K+P_L)f\left(\frac{P_K\cdot\frac{1}{P_K}\sum_{k\in K}p_ka_k+P_L\cdot\frac{1}{P_L}\sum_{k\in L}p_ka_k}{P_K+P_L}\right).\end{aligned} \tag{3.4}$$

Consider the elements

$$a=\frac{1}{P_K}\sum_{k\in K}p_ka_k \text{ and } b=\frac{1}{P_L}\sum_{k\in L}p_ka_k.$$

We have that $\operatorname{Sp}(a),\operatorname{Sp}(b)\subseteq I$.

Utilising the inequality (**??**) for the elements a and b as above and $\lambda=\frac{P_L}{P_K+P_Lw}$ we have

$$\begin{aligned}&\frac{P_K}{P_K+P_L}f\left(\frac{1}{P_K}\sum_{k\in K}p_ka_k\right)+\frac{P_L}{P_K+P_L}f\left(\frac{1}{P_L}\sum_{k\in L}p_ka_k\right)\\&\geq f\left(\frac{P_K\cdot\frac{1}{P_K}\sum_{k\in K}p_ka_k+P_L\cdot\frac{1}{P_L}\sum_{k\in L}p_ka_k}{P_K+P_L}\right).\end{aligned} \tag{3.5}$$

On making use of (3.4) and (3.5) we have

$$
\begin{aligned}
(3.6)\qquad & J(K\cup L,\mathbf{p};\mathbf{a},f,I)\\
&=\sum_{k\in K\cup L}p_kf(a_k)-P_{K\cup L}f\left(\frac{1}{P_{K\cup L}}\sum_{k\in K\cup L}p_ka_k\right)\\
&\geq\sum_{k\in K}p_kf(a_k)+\sum_{k\in L}p_kf(a_k)-(P_K+P_L)\\
&\times\left[\frac{P_K}{P_K+P_L}f\left(\frac{1}{P_K}\sum_{k\in K}p_ka_k\right)+\frac{P_L}{P_K+P_L}f\left(\frac{1}{P_L}\sum_{k\in L}p_ka_k\right)\right]\\
&=\sum_{k\in K}p_kf(a_k)-P_Kf\left(\frac{1}{P_K}\sum_{k\in K}p_ka_k\right)\\
&+\sum_{k\in L}p_kf(a_k)-P_Lf\left(\frac{1}{P_L}\sum_{k\in L}p_ka_k\right)\\
&=J(K,\mathbf{p};\mathbf{a},f,I)+J(L,\mathbf{p};\mathbf{a},f,I)
\end{aligned}
$$

and the inequality (3.2) is proved.

If $\emptyset\neq K\subset L$ with $L\diagdown K\neq\emptyset$ then we have by (3.2)

$$
\begin{aligned}
J(L,\mathbf{p};\mathbf{a},f,I)&=J(K\cup(L\diagdown K),\mathbf{p};\mathbf{a},f,I)\\
&\geq J(K,\mathbf{p};\mathbf{a},f,I)+J(L\diagdown K,\mathbf{p};\mathbf{a},f,I)\geq J(K,\mathbf{p};\mathbf{a},f,I)
\end{aligned}
$$

and the inequality (3.3) is proved. □

We consider the functionals:

$$
O_n(\mathbf{p};\mathbf{a},f,I):=\sum_{j=1}^{n}p_{2j-1}f(a_{2j-1})-\sum_{j=1}^{n}p_{2j-1}f\left(\frac{1}{\sum_{j=1}^{n}p_{2j-1}}\sum_{j=1}^{n}p_{2j-1}a_{2j-1}\right)
$$

and

$$
E_n(\mathbf{p};\mathbf{a},f,I):=\sum_{j=1}^{n}p_{2j}f(a_{2j})-\sum_{j=1}^{n}p_{2j}f\left(\frac{1}{\sum_{j=1}^{n}p_{2j}}\sum_{j=1}^{n}p_{2j}a_{2j}\right).
$$

We can state the following corollary.

Corollary 2. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I in A and $\mathbf{p}=(p_1,...,p_{2n})$, a$=(a_1,...,a_{2n})$ with $p_k>0$, a_k selfadjoint elements and such that $\operatorname{Sp}(a_k)\subseteq I$ for any $k\in\{1,...,2n\}$, $n\geq1$. The we have the inequality*

$$
J_{2n}(\mathbf{p};\mathbf{a},f,I)\geq O_n(\mathbf{p};\mathbf{a},f,I)+E_n(\mathbf{p};\mathbf{a},f,I)\geq0 \tag{3.7}
$$

in the order of A, where, as in (2.1)

$$
J_{2n}(\mathbf{p};\mathbf{a},f,I)=\sum_{j=1}^{2n}p_jf(a_j)-P_{2n}f\left(\frac{1}{P_{2n}}\sum_{j=1}^{2n}p_ja_j\right).
$$

The proof follows by (3.2) on choosing $K=\{2,...,2n\}$ and $L=\{1,...,2n-1\}$.

Corollary 3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I in A and $\mathbf{p} = (p_1, ..., p_n)$, $a= (a_1, ..., a_n)$ with $p_k > 0$, a_k selfadjoint elements and such that $\operatorname{Sp}(a_k) \subseteq I$ for any $k \in \{1, ..., n\}$, $n \geq 2$. Then we have the inequality*

$$J_k (\mathbf{p}; \mathbf{a}, f, I) \geq J_{k-1} (\mathbf{p}; \mathbf{a}, f, I) \geq 0 \tag{3.8}$$

for any $k \in \{1, ..., n\}$ with $n \geq k \geq 2$.

We also have that

$$J_n (\mathbf{p}; \mathbf{a}, f, I) \geq p_j f (a_j) + p_k f (a_k) - (p_j + p_k) f \left(\frac{p_j a_j + p_k a_k}{p_j + p_k} \right) \geq 0 \tag{3.9}$$

for any k, $j \in \{1, ..., n\}$ in the order of A.

The proof follows by the monotonicity property (3.3).

Remark 3. *Utilising the inequality for the operator convex function $f(t) = t^{-1}$, we have for any k, $j \in \{1, ..., n\}$ the inequality*

$$\begin{aligned} &\sum_{j=1}^{n} p_j a_j^{-1} - P_n^2 \left(\sum_{j=1}^{n} p_j a_j \right)^{-1} \\ &\geq p_j a_j^{-1} + p_k a_k^{-1} - (p_j + p_k) \left(\frac{p_j a_j + p_k a_k}{p_j + p_k} \right)^{-1} \geq 0, \end{aligned} \tag{3.10}$$

for the positive elements $(a_1, ..., a_n)$.

4. A Reverse Inequality

The following result also holds:

Theorem 5. *If the function f is operator convex on $[m, M]$ in A and if the n-tuple of selfadjoint elements $(a_1, ..., a_n)$ has the property that $\operatorname{Sp}(a_j) \subseteq [m, M]$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^{n} p_j > 0$ we have*

$$\begin{aligned} 0 &\leq \frac{1}{P_n} \sum_{j=1}^{n} p_j f (a_j) - f \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) \\ &\leq \frac{2}{M - m} \left[\frac{f (m) + f (M)}{2} - f \left(\frac{m + M}{2} \right) \right] \end{aligned} \tag{4.1}$$

in the order of A.

Proof. Since the function $f : [m, M] \to \mathbb{R}$ is operator convex, then it is convex and we have the inequality

$$f (t) = f \left(\frac{(M - t) m + (t - m) M}{M - m} \right) \leq \frac{(M - t) f (m) + (t - m) f (M)}{M - m}$$

for any $t \in [m, M]$.

Utilising Lemma 1 for a selfadjoint element a with spectrum $\operatorname{Sp}(a) \subseteq [m, M]$, we have in the order of A

$$f (a_j) \leq \frac{f (m) (M 1_H - a_j) + f (M) (a_j - m 1_H)}{M - m} \tag{4.2}$$

for any $j \in \{1, ..., n\}$.

If we multiply the inequality (4.2) by p_j and sum over j from 1 to n we get

$$\frac{1}{P_n}\sum_{j=1}^{n}p_jf(a_j) \leq \frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1_H\right)}{M-m} \tag{4.3}$$

in the order of A.

Therefore we have

$$0\leq \frac{1}{P_n}\sum_{j=1}^{n}p_jf(a_j)-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right) \leq \frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1_H\right)}{M-m} -f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right) \tag{4.4}$$

in the order of A, which is a reverse of Jensen's inequality that is of interest in itself.

Now, from the scalar version of (2.13) we have

$$\begin{aligned} 0&\leq (1-t)f(m)+tf(M)-f((1-t)m+tM)\\ &\leq 2\max\{t,1-t\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\end{aligned} \tag{4.5}$$

for any $t\in[m,M]$, where $f:[m,M]\to\mathbb{R}$ is a continuous convex function on $[m,M]$.

Utilising Lemma 1 for a selfadjoint element c with $0\leq c\leq 1$ we have from (4.5) that

$$\begin{aligned} 0&\leq f(m)(1-c)+f(M)c-f((1-c)m+cM)\\ &\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\end{aligned} \tag{4.6}$$

in the order of A.

Writing the inequality (4.6) for the operator

$$0\leq c=\frac{\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1}{M-m}\leq 1_H$$

we have

$$\begin{aligned}
(4.7)\qquad & \frac{f(m)\left(M1-\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1\right)}{M-m} \\
& -f\left[\frac{m\left(M1-\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)+M\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1\right)}{M-m}\right] \\
& =\frac{f(m)\left(M1-\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j-m1\right)}{M-m} \\
& -f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right) \\
& \le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]
\end{aligned}$$

in the order of A.

By making use of the inequality (4.4) we deduce (4.1). □

Remark 4. *Assume that $(a_1,...,a_n)$ are positive and with $\operatorname{Sp}(a_j)\subseteq[m,M]$ for any $j\in\{1,...,n\}$. Then we have the inequality*

$$\begin{aligned}
(4.8)\qquad 0 & \le \frac{1}{P_n}\sum_{j=1}^{n}p_ja_j^{-1}-\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)^{-1} \\
& \le \frac{2}{M-m}\left[\frac{m^{-1}+M^{-1}}{2}-\left(\frac{m+M}{2}\right)^{-1}\right].
\end{aligned}$$

5. A Refinement of Jensen Inequality

The following result provides an additive refinement of Jensen inequality (1.5).

Theorem 6. *If the function $f:I\to\mathbb{R}$ is operator convex and if the n-tuple of selfadjoint elements $(a_1,...,a_n)$ is such that $\operatorname{Sp}(a_j)\subseteq I$ for any $j\in\{1,...,n\}$, then for any $p_j\ge 0$ with $j\in\{1,...,n\}$ and $P_k:=\sum_{j=1}^{k}p_j>0$, $\overline{P}_k:=P_n-P_k>0$, with $k\in\{1,...,n-1\}$ we have*

$$\begin{aligned}
(5.1)\qquad 0 & \le \left(1-\frac{\left|P_k-\overline{P}_k\right|}{P_n}\right)\left[\frac{f\left(\frac{1}{P_k}\sum_{j=1}^{k}p_ja_j\right)+f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_ja_j\right)}{2}\right. \\
& \left. -f\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k}p_ja_j+\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_ja_j}{2}\right)\right] \\
& \le \frac{1}{P_n}\sum_{j=1}^{n}p_jf(a_j)-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j\right)
\end{aligned}$$

for any $k\in\{1,...,n-1\}$ in the order of A.

Proof. Consider the n-tuple of selfadjoint elements$(a_1, ..., a_n)$ and define the elements

$$a = \frac{1}{P_k} \sum_{j=1}^{k} p_j a_j \text{ and } b = \frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j.$$

Then $\mathrm{Sp}(a), \mathrm{Sp}(b) \subseteq I$ for $k \in \{1, ..., n-1\}$.

Applying the first inequality in (2.13) for a and b as above and $\alpha = \frac{\overline{P}_k}{P_n}$, for $k \in \{1, ..., n-1\}$, we have

$$\begin{aligned} &2 \min \left\{ \frac{\overline{P}_k}{P_n}, \frac{P_k}{P_n} \right\} \\ &\times \left[\frac{f\left(\frac{1}{P_k} \sum_{j=1}^{k} p_j a_j\right) + f\left(\frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j\right)}{2} \right. \\ &\left. - f\left(\frac{\frac{1}{P_k} \sum_{j=1}^{k} p_j a_j + \frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j}{2} \right) \right] \\ &\leq \frac{P_k}{P_n} f\left(\frac{1}{P_k} \sum_{j=1}^{k} p_j a_j \right) + \frac{\overline{P}_k}{P_n} f\left(\frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j \right) \\ &- f\left(\frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right), \end{aligned} \tag{5.2}$$

in the order of A.

By Jensen's inequality (1.5) we have

$$f\left(\frac{1}{P_k} \sum_{j=1}^{k} p_j a \right) \leq \frac{1}{P_k} \sum_{j=1}^{k} p_j f(a_j)$$

and

$$f\left(\frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j \right) \leq \frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j f(a_j),$$

which imply that

$$\frac{P_k}{P_n} f\left(\frac{1}{P_k} \sum_{j=1}^{k} p_j a_j \right) + \frac{\overline{P}_k}{P_n} f\left(\frac{1}{\overline{P}_k} \sum_{j=k+1}^{n} p_j a_j \right) \leq \frac{1}{P_n} \sum_{j=1}^{n} p_j f(a_j) \tag{5.3}$$

in the order of A.

Since

$$\min \left\{ \frac{\overline{P}_k}{P_n}, \frac{P_k}{P_n} \right\} = \frac{1}{2} - \frac{1}{2P_n} \left| P_k - \overline{P}_k \right|$$

then we get from (5.2) and (5.3) the desired result (5.1). □

Remark 5. *If the elements $(a_1, ..., a_n)$ are positive, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_k := \sum_{j=1}^{k} p_j > 0$, $\overline{P}_k := P_n - P_k > 0$, with $k \in \{1, ..., n-1\}$,* *we have*

$$0\leq\left(1-\frac{\left|P_k-\overline{P}_k\right|}{P_n}\right)\left[\frac{P_k\left(\sum_{j=1}^{k}p_ja_j\right)^{-1}+\overline{P}_k\left(\sum_{j=k+1}^{n}p_ja_j\right)^{-1}}{2}\right.$$
$$\left.-\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k}p_ja_j+\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_ja_j}{2}\right)^{-1}\right]$$
$$\leq\frac{1}{P_n}\sum_{j=1}^{n}p_ja_j^{-1}-P_n\left(\sum_{j=1}^{n}p_ja_j\right)^{-1}. \tag{5.4}$$

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.

[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[3] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach $*$-algebras, *Oper. Matrices* **12** (2018), no. 4, 1009–1026. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 162. [http://rgmia.org/papers/v19/v19a162.pdf].

[4] S. S. Dragomir, Multiplicative inequalities for weighted geometric mean in Hermitian unital Banach *-algebras. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **112** (2018), no. 4, 1349–1365.

[5] S. S. Dragomir, Inequalities of Jensen's type for positive linear functionals on Hermitian unital Banach $*$-algebras, *Bull. Aust. Math. Soc.* (First published online 2020), page 1 of 11 doi:10.1017/S000497271900131X. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 142. 10 pp. [https://rgmia.org/papers/v19/v19a172.pdf].

[6] S. S. Dragomir, Inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach $*$-algebras, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art.

[7] B. Q. Feng, The geometric means in Banach $*$-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.

[8] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990.

[9] T. Okayasu, The Löwner-Heinz inequality in Banach $*$-algebra, *Glasgow Math. J.* **42** (2000), 243-246.

[10] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.

[11] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach $*$-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa