WEIGHTED NORM INEQUALITIES OF MIDPOINT TYPE FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. In this paper we establish some upper bounds for the quantity

$$\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \left(\int_0^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\|$$

in the case that $f : C \subset E \to F$ is a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F, x, $y \in C$ and $p : [0,1] \to [0,\infty)$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0,1]$. Applications for Banach algebras are also provided.

1. Introduction

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset if a normed vector space, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function f_u given by $t \mapsto f(a+tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0) := \lim_{t\to 0} \frac{f(a+tu) - f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative (directional derivative)* of f at a in the direction u. If $\nabla_a f(u)$ is defined and $\lambda \in \mathbb{R}\setminus\{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u) = \lambda \nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u.

Let E and F be normed vector spaces, and O be an open subset of F. A function $f : O \to F$ is called *Fréchet differentiable* at $x \in O$ if there exists a bounded linear operator $A : E \to F$ such that

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

If there exists such an operator A, it is unique, so we write $Df(x) = A$ and call it the *Fréchet derivative* of f at x.

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O \ni x \mapsto Df(x) \in \mathcal{B}(E,F)$ is continuous. A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a

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linear operation. If f is Fréchet differentiable at x, it is also Gâteaux differentiable there, and $\nabla_x f(u) = Df(x)(u)$ for all $u \in E$.

We say that the function $f : O \subset E \to F$ is L-*Lipschitzian* on O with the constant $L > 0$ if

$$\|f(x) - f(y)\| \le L\|x - y\| \text{ for all } x,\ y \in O.$$

In [13] we established among others the following midpoint and trapezoid type inequalities for L-Lipschitzian functions f on an open and convex subset C in E

$$\left\| \int_0^1 f((1-t)t + ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \le \frac{1}{4} L \|x - y\| \tag{1.1}$$

and

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)t + ty)\,dt \right\| \le \frac{1}{4} L \|x - y\| \tag{1.2}$$

for all x, $y \in C$. The constant $\frac{1}{4}$ is best possible in both inequalities (1.1) and (1.2).

For Hermite-Hadamard's type inequalities, namely

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(u)\,du \le \frac{f(a) + f(b)}{2}, \tag{1.3}$$

where $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$, see for instance [6], [7], [8], [19], [21], [22], [23], [25], [26], [27], [28], [29], [30], [31] and the references therein.

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left\| \int_0^1 p(t) f((1-t)x + ty)\,dt - \left(\int_0^1 p(s)\,ds \right) f\left(\frac{x+y}{2}\right) \right\|$$

in the case that $f : C \subset E \to F$ is a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F, x, $y \in C$ and $p : [0, 1] \to [0, \infty)$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$. Applications for Banach algebras are also provided.

2. General Results for Banach Spaces

Consider a function $f : C \subset E \to F$ that is defined on the open and convex set C. We have the following properties for the *auxiliary function*

$$\varphi_{(x,y)}(t) := f((1-t)x + ty),\ t \in [0, 1],$$

where x, $y \in C$.

Lemma 1. *Assume that the function $f : C \subset E \to F$ is Fréchet differentiable on the open and convex set C. Then for all x, $y \in C$ the auxiliary function $\varphi_{(x,y)}$ is differentiable on $(0, 1)$ and*

$$\varphi'_{(x,y)}(t) = Df((1-t)x + ty)(y - x). \tag{2.1}$$

Also we have for the lateral derivatives

$$\varphi'_{(x,y)}(0+) = Df(x)(y - x) \tag{2.2}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y - x). \tag{2.3}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} & \frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h} \\ & = \frac{f((1-t-h)x+(t+h)y)-f((1-t)x+ty)}{h} \\ & = \frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h}. \end{aligned} \tag{2.4}$$

Since f is Fréchet differentiable, hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) & = \lim_{h\to 0} \frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h} \\ & = \lim_{h\to 0} \frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h} \\ & = Df((1-t)x+ty)(y-x), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) & = \lim_{h\to 0+} \frac{\varphi_{(x,y)}(h)-\varphi_{(x,y)}(0)}{h} \\ & = \lim_{h\to 0+} \frac{f((1-h)x+hy)-f(x)}{h} \\ & = \lim_{h\to 0+} \frac{f(x+h(y-x))-f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x. This proves (2.2).

The equality (2.3) follows in a similar way. □

The following result holds:

Theorem 1. *Let $f : C \subset E \to F$ be a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$. Then for all $x, y \in C$*

$$\begin{aligned} & \left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \left(\int_0^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\| \\ & \leq \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) \|Df((1-t)x+ty)(y-x)\|\,dt \\ & + \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) \|Df((1-t)x+ty)(y-x)\|\,dt \\ & =: B(f,p,x,y). \end{aligned} \tag{2.5}$$

Moreover, we have the upper bounds

$$\begin{aligned} B(f,p,x,y) & \leq \frac{1}{2}\int_0^1 p(s)\,ds \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ & \leq \frac{1}{2}\int_0^1 p(s)\,ds \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|, \end{aligned} \tag{2.6}$$

$$B(f,p,x,y) \leq \left[\frac{1}{2}\int_0^1 p(s)\,ds - \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right)dt\right] \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \leq \frac{1}{2}\int_0^1 p(s)\,ds \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \tag{2.7}$$

and

$$B(f,p,x,y) \leq \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q} \leq \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \tag{2.8}$$

for $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$.

Proof. Let $x,\ y\in C$, with $x\neq y$. Using the integration by parts formula for Bochner's integral, [24] we have

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \left(\int_t^1 p(s)\,ds\right)\varphi_{(x,y)}(t)\Big]_{1/2}^1 + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= -\left(\int_{1/2}^1 p(s)\,ds\right)\varphi_{(x,y)}(1/2) + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= -\left(\int_{1/2}^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right) + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1/2} \left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \left(\int_0^t p(s)\,ds\right)\varphi_{(x,y)}(t)\Big]_0^{1/2} - \int_0^{1/2} p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= \left(\int_0^{1/2} p(s)\,ds\right)\varphi_{(x,y)}(1/2) - \int_0^{1/2} p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= \left(\int_0^{1/2} p(s)\,ds\right) f\left(\frac{x+y}{2}\right) - \int_0^{1/2} p(t)\,\varphi_{(x,y)}(t)\,dt.
\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}
&\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)\varphi_{(x,y)}'(t)\,dt-\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)\varphi_{(x,y)}'(t)\,dt\\
&=\int_{1/2}^{1}p(t)\,\varphi_{(x,y)}(t)\,dt+\int_{0}^{1/2}p(t)\,\varphi_{(x,y)}(t)\,dt\\
&-\left(\int_{1/2}^{1}p(s)\,ds\right)f\left(\frac{x+y}{2}\right)-\left(\int_{0}^{1/2}p(s)\,ds\right)f\left(\frac{x+y}{2}\right).
\end{aligned}$$

By the symmetry of p we obtain

$$\int_{1/2}^{1}p(s)\,ds=\int_{0}^{1/2}p(s)\,ds=\frac{1}{2}\int_{0}^{1}p(s)\,ds$$

and then we get the following identity of interest in itself in terms of the Fréchet derivative

$$\begin{aligned}
&\int_{0}^{1}p(t)\,f((1-t)\,x+ty)\,dt-\left(\int_{0}^{1}p(s)\,ds\right)f\left(\frac{x+y}{2}\right)\\
&=\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\,dt\\
&-\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\,dt
\end{aligned}\tag{2.9}$$

for all $x,\,y\in C$.

If we take the norm in (2.5), then we get

$$\begin{aligned}
&\left\|\int_{0}^{1}p(t)\,f((1-t)\,x+ty)\,dt-\left(\int_{0}^{1}p(s)\,ds\right)f\left(\frac{x+y}{2}\right)\right\|\\
&\leq\left\|\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\,dt\right\|\\
&+\left\|\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\,dt\right\|\\
&\leq\int_{1/2}^{1}\left\|\left(\int_{t}^{1}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\right\|dt\\
&+\int_{0}^{1/2}\left\|\left(\int_{0}^{t}p(s)\,ds\right)Df((1-t)\,x+ty)\,(y-x)\right\|dt\\
&=\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)\|Df((1-t)\,x+ty)\,(y-x)\|\,dt\\
&+\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)\|Df((1-t)\,x+ty)\,(y-x)\|\,dt\\
&=B(f,p,x,y)\,,
\end{aligned}$$

which proves the inequality in (2.5).

Now, observe, by the properties of p, that

$$\int_t^1 p(s)\,ds \leq \int_{1/2}^1 p(s)\,ds \text{ for } t \in [1/2, 1]$$

and

$$\int_0^t p(s)\,ds \leq \int_0^{1/2} p(s)\,ds \text{ for } t \in [0, 1/2].$$

Therefore

$$\begin{aligned}
B(f,p,x,y) &\leq \left(\int_{1/2}^1 p(s)\,ds\right) \int_{1/2}^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\
&+ \left(\int_0^{1/2} p(s)\,ds\right) \int_0^{1/2} \|Df((1-t)x+ty)(y-x)\|\,dt \\
&= \frac{1}{2}\int_0^1 p(s)\,ds \int_{1/2}^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\
&+ \frac{1}{2}\int_0^1 p(s)\,ds \int_0^{1/2} \|Df((1-t)x+ty)(y-x)\|\,dt \\
&= \frac{1}{2}\int_0^1 p(s)\,ds \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt,
\end{aligned}$$

which proves the first part of (2.6). The second part is obvious.

We also have

$$\begin{aligned}
B(f,p,x,y) &\leq \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) dt \\
&+ \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) dt \\
&= \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \\
&\times \left(\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) dt + \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) dt\right).
\end{aligned} \tag{2.10}$$

Observe that

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) dt + \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) dt \\
&= \int_{1/2}^1 \left(\int_0^1 p(s)\,ds - \int_0^t p(s)\,ds\right) dt + \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) dt \\
&= \frac{1}{2}\int_0^1 p(s)\,ds + \left(\int_0^{1/2} \left(\int_0^t p(s)\,ds\right) dt - \int_{1/2}^1 \left(\int_0^t p(s)\,ds\right) dt\right) \\
&= \frac{1}{2}\int_0^1 p(s)\,ds - \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right) dt
\end{aligned}$$

and by (2.10) we get the first part of (2.7).

If we use the Čebyšev's inequality for two nondecreasing functions g and h on the interval $[a,b]$, namely

$$(b-a)\int_a^b g(t)\,h(t)\,dt \geq \int_a^b g(t)\,dt\int_a^b h(t)\,dt,$$

then we have

$$\int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right)dt \geq \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)dt\int_0^1\left(\int_0^t p(s)\,ds\right)dt = 0,$$

which proves the last part of (2.7).

If we use Hölder's integral inequality, then we also have for $r,\,q>1,\ \frac{1}{r}+\frac{1}{q}=1$ that

$$\begin{aligned}
B(f,p,x,y) &\leq \left(\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)^r dt\right)^{1/r} \\
&\times\left(\int_{1/2}^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q} \\
&+\left(\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right)^{1/r} \\
&\times\left(\int_0^{1/2} \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q} \\
&\leq \left[\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)^r dt+\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \\
&\times\left[\left(\int_{1/2}^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)\right. \\
&\left.+\int_0^{1/2} \|Df((1-t)x+ty)(y-x)\|^q\,dt\right]^{1/q} \\
&= \left[\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)^r dt+\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \\
&\times\left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},
\end{aligned}$$

which proves (2.8). □

Remark 1. *Since the Fréchet derivative satisfies the condition*

$$\|Df(a)(b)\| \leq \|Df(a)\|\,\|b\|$$

for $a \in C$ and $b \in E$, then we also have the chain of inequalities

$$\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \left(\int_0^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\| \tag{2.11}$$
$$\leq \|y-x\| \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right) \|Df((1-t)x+ty)\|\,dt$$
$$+ \|y-x\| \int_0^{1/2} \left(\int_0^t p(s)\,ds\right) \|Df((1-t)x+ty)\|\,dt$$
$$=: D(f,p,x,y)$$

for all x, $y \in C$.

Moreover, we have

$$D(f,p,x,y) \leq \frac{1}{2}\|y-x\| \int_0^1 p(s)\,ds \int_0^1 \|Df((1-t)x+ty)\|\,dt \tag{2.12}$$
$$\leq \frac{1}{2}\|y-x\| \int_0^1 p(s)\,ds \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|,$$

$$D(f,p,x,y) \tag{2.13}$$
$$\leq \|y-x\| \left[\frac{1}{2}\int_0^1 p(s)\,ds - \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right)dt\right]$$
$$\times \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|$$
$$\leq \frac{1}{2}\|y-x\| \int_0^1 p(s)\,ds \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|$$

and

$$D(f,p,x,y) \tag{2.14}$$
$$\leq \|y-x\| \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r}$$
$$\times \left(\int_0^1 \|Df((1-t)x+ty)\|^q\,dt\right)^{1/q}$$
$$\leq \|y-x\| \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r}$$
$$\times \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|$$

for all x, $y \in C$, where r, $q > 1$ with $\frac{1}{r}+\frac{1}{q}=1$.

A function $h : C \subset E \to F$ is *norm convex* on C if

$$\|h((1-t)x+ty)\| \leq (1-t)\|h(x)\| + t\|h(y)\|$$

for all x, $y \in C$ and $t \in [0,1]$.

Theorem 2. *With the assumptions of Theorem 1 and if the Fréchet derivative $Df : C \to \mathcal{B}(E, F)$ is norm convex on C, then for all x, $y \in C$,*

$$\left\| \int_0^1 p(t) f((1-t)x + ty)\, dt - \left(\int_0^1 p(s)\, ds \right) f\left(\frac{x+y}{2} \right) \right\| \leq \frac{1}{2} (\|Df(x)\| + \|Df(y)\|) \|y - x\| \int_0^1 p(t) \left| t^2 - \frac{1}{4} \right| dt. \tag{2.15}$$

Proof. By the norm convexity of the derivative, we have

$$\int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) \|Df((1-t)x + ty)\|\, dt \leq \int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) [(1-t)\|Df(x)\| + t\|Df(y)\|]\, dt \tag{2.16}$$

and

$$\int_0^{1/2} \left(\int_0^t p(s)\, ds \right) \|Df((1-t)x + ty)\|\, dt \leq \int_0^{1/2} \left(\int_0^t p(s)\, ds \right) [(1-t)\|Df(x)\| + t\|Df(y)\|]\, dt. \tag{2.17}$$

Therefore

$$\int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) [(1-t)\|Df(x)\| + t\|Df(y)\|]\, dt = \|Df(x)\| \int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) (1-t)\, dt + \|Df(y)\| \int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) t dt.$$

Integrating by parts, we get

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) (1-t)\, dt \\
&= -\frac{1}{2} \int_{1/2}^1 \left(\int_t^1 p(s)\, ds \right) d\left[(1-t)^2 \right] \\
&= -\frac{1}{2} \left[\left(\int_t^1 p(s)\, ds \right) (1-t)^2 \Big|_{1/2}^1 + \int_{1/2}^1 p(t)(1-t)^2\, dt \right] \\
&= -\frac{1}{2} \left[-\frac{1}{4} \int_{1/2}^1 p(s)\, ds + \int_{1/2}^1 p(t)(1-t)^2\, dt \right] \\
&= \frac{1}{2} \int_{1/2}^1 p(t) \left[\frac{1}{4} - (1-t)^2 \right] dt
\end{aligned}$$

and

$$\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)tdt=\frac{1}{2}\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)d\left(t^{2}\right)$$
$$=\frac{1}{2}\left[\left(\int_{t}^{1}p(s)\,ds\right)t^{2}\Big|_{1/2}^{1}+\int_{1/2}^{1}p(t)\,t^{2}dt\right]$$
$$=\frac{1}{2}\left[\int_{1/2}^{1}p(t)\,t^{2}dt-\frac{1}{4}\int_{1/2}^{1}p(s)\,ds\right]$$
$$=\frac{1}{2}\int_{1/2}^{1}p(t)\left(t^{2}-\frac{1}{4}\right)dt,$$

which implies, by (2.16), that

$$\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)\|Df((1-t)\,x+ty)\|\,dt$$
$$=\frac{1}{2}\|Df(x)\|\int_{1/2}^{1}p(t)\left[\frac{1}{4}-(1-t)^{2}\right]dt$$
$$+\frac{1}{2}\|Df(y)\|\int_{1/2}^{1}p(t)\left(t^{2}-\frac{1}{4}\right)dt. \tag{2.18}$$

Similarly,

$$\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)\|Df((1-t)\,x+ty)\|\,dt$$
$$\leq\|Df(x)\|\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)(1-t)\,dt+\|Df(y)\|\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)tdt.$$

Integrating by parts we also have

$$\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)(1-t)\,dt$$
$$=-\frac{1}{2}\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)d\left[(1-t)^{2}\right]$$
$$=-\frac{1}{2}\left[\left(\int_{0}^{t}p(s)\,ds\right)(1-t)^{2}\Big|_{0}^{1/2}-\int_{0}^{1/2}p(t)\,(1-t)^{2}\,dt\right]$$
$$=-\frac{1}{2}\left[\frac{1}{4}\left(\int_{0}^{1/2}p(s)\,ds\right)-\int_{0}^{1/2}p(t)\,(1-t)^{2}\,dt\right]$$
$$=\frac{1}{2}\int_{0}^{1/2}p(t)\left[(1-t)^{2}-\frac{1}{4}\right]dt$$

and

$$\begin{aligned}\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)tdt &= \frac{1}{2}\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)d\left(t^2\right)\\ &= \frac{1}{2}\left[\left(\int_0^t p(s)\,ds\right)t^2\Big|_0^{1/2} - \int_0^{1/2} p(t)\,t^2dt\right]\\ &= \frac{1}{2}\left[\frac{1}{4}\left(\int_0^{1/2} p(s)\,ds\right) - \int_0^{1/2} p(t)\,t^2dt\right]\\ &= \frac{1}{2}\int_0^{1/2} p(t)\left(\frac{1}{4}-t^2\right)dt\end{aligned}$$

and by (2.17) we get

$$\begin{aligned}&\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\|Df((1-t)x+ty)\|\,dt\\ &\leq \frac{1}{2}\|Df(x)\|\int_0^{1/2} p(t)\left[(1-t)^2-\frac{1}{4}\right]dt\\ &+\frac{1}{2}\|Df(y)\|\int_0^{1/2} p(t)\left(\frac{1}{4}-t^2\right)dt.\end{aligned} \tag{2.19}$$

If we add (2.18) with (2.19), then we obtain

$$\begin{aligned}&\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)\|Df((1-t)x+ty)\|\,dt\\ &+\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\|Df((1-t)x+ty)\|\,dt\\ &\leq \frac{1}{2}\|Df(x)\|\int_{1/2}^1 p(t)\left[\frac{1}{4}-(1-t)^2\right]dt+\frac{1}{2}\|Df(y)\|\int_{1/2}^1 p(t)\left(t^2-\frac{1}{4}\right)dt\\ &\frac{1}{2}\|Df(x)\|\int_0^{1/2} p(t)\left[(1-t)^2-\frac{1}{4}\right]dt+\frac{1}{2}\|Df(y)\|\int_0^{1/2} p(t)\left(\frac{1}{4}-t^2\right)dt\\ &=\frac{1}{2}\|Df(x)\|\left[\int_{1/2}^1 p(t)\left[\frac{1}{4}-(1-t)^2\right]dt+\int_0^{1/2} p(t)\left[(1-t)^2-\frac{1}{4}\right]dt\right]\\ &+\frac{1}{2}\|Df(y)\|\left[\int_{1/2}^1 p(t)\left(t^2-\frac{1}{4}\right)dt+\int_0^{1/2} p(t)\left(\frac{1}{4}-t^2\right)dt\right]\\ &=\frac{1}{2}\|Df(x)\|\int_0^1 p(t)\left|(1-t)^2-\frac{1}{4}\right|dt+\frac{1}{2}\|Df(y)\|\int_0^1 p(t)\left|t^2-\frac{1}{4}\right|dt.\end{aligned}$$

If we change the variable $1-t=s$, $s\in[0,1]$ and use the symmetry of p we derive

$$\int_0^1 p(t)\left|(1-t)^2-\frac{1}{4}\right|dt=\int_0^1 p(1-s)\left|s^2-\frac{1}{4}\right|ds=\int_0^1 p(s)\left|s^2-\frac{1}{4}\right|ds,$$

which implies that

$$\begin{aligned}
&\int_{1/2}^{1}\left(\int_{t}^{1}p(s)\,ds\right)\|Df((1-t)x+ty)\|\,dt \\
&+\int_{0}^{1/2}\left(\int_{0}^{t}p(s)\,ds\right)\|Df((1-t)x+ty)\|\,dt \\
&\leq \frac{1}{2}\|Df(x)\|\int_{0}^{1}p(t)\left|t^{2}-\frac{1}{4}\right|dt+\frac{1}{2}\|Df(y)\|\int_{0}^{1}p(t)\left|t^{2}-\frac{1}{4}\right|dt \\
&=\frac{1}{2}(\|Df(x)\|+\|Df(y)\|)\int_{0}^{1}p(t)\left|t^{2}-\frac{1}{4}\right|dt.
\end{aligned}$$

By making use of (2.11) we get the desired result (2.15). □

If we consider the uniform weight $p\equiv 1$ in Theorem 1 then we get for x, $y\in C$ that

$$\begin{aligned}
&\left\|\int_{0}^{1}f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\| \qquad (2.20)\\
&\leq\int_{1/2}^{1}(1-t)\|Df((1-t)x+ty)(y-x)\|\,dt \\
&+\int_{0}^{1/2}t\|Df((1-t)x+ty)(y-x)\|\,dt \\
&=:B(f,x,y).
\end{aligned}$$

Moreover, we have the upper bounds

$$B(f,x,y)\leq\frac{1}{2}\int_{0}^{1}\|Df((1-t)x+ty)(y-x)\|\,dt, \qquad (2.21)$$

$$B(f,p,x,y)\leq\frac{1}{4}\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\| \qquad (2.22)$$

and

$$B(f,p,x,y)\leq\frac{1}{2(r+1)^{1/r}}\left(\int_{0}^{1}\|Df((1-t)x+ty)(y-x)\|^{q}\,dt\right)^{1/q} \qquad (2.23)$$

provided $f:C\subset E\to F$ is a function of class C^{1} on C, for r, $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$.

If $Df:C\to\mathcal{B}(E,F)$ is norm convex on C, then by Theorem 2 we get

$$\left\|\int_{0}^{1}f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\|\leq\frac{1}{8}\|y-x\|(\|Df(x)\|+\|Df(y)\|) \qquad (2.24)$$

for x, $y\in C$.

Consider the symmetric weight $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ in Theorem 1, then for all x, $y \in C$ we get

$$\begin{aligned}
&\left\| \int_0^1 \left|t - \frac{1}{2}\right| f((1-t)x + ty)\,dt - \frac{1}{4} f\left(\frac{x+y}{2}\right) \right\| \\
&\le \frac{1}{2} \int_{1/2}^1 t(1-t) \|Df((1-t)x+ty)(y-x)\|\,dt \\
&+ \frac{1}{2} \int_0^{1/2} t(1-t) \|Df((1-t)x+ty)(y-x)\|\,dt \\
&= \frac{1}{2} \int_0^1 t(1-t) \|Df((1-t)x+ty)(y-x)\|\,dt \\
&=: B_1(f,x,y).
\end{aligned} \tag{2.25}$$

Moreover, we have by (2.25) the upper bounds

$$B_1(f,x,y) \le \frac{1}{8} \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt, \tag{2.26}$$

$$B_1(f,x,y) \le \frac{1}{12} \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \tag{2.27}$$

and

$$\begin{aligned}
B_1(f,x,y) &\le \left[\int_{1/2}^1 \left(\int_t^1 \left(s - \frac{1}{2}\right) ds \right)^r dt + \int_0^{1/2} \left(\int_0^t \left(\frac{1}{2} - s\right) ds \right)^r dt \right]^{1/r} \\
&\times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \\
&= \left[\int_{1/2}^1 \left(\frac{1}{2}(1-t)t \right)^r dt + \int_0^{1/2} \left(\frac{1}{2}(1-t)t \right)^r dt \right]^{1/r} \\
&\times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \\
&= \left(\int_0^1 \left(\frac{1}{2}(1-t)t \right)^r dt \right)^{1/r} \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \\
&= \frac{1}{2} [\beta(r+1,r+1)]^{1/r} \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q},
\end{aligned}$$

namely

$$\begin{aligned}
&B_1(f,x,y) \\
&\le \frac{1}{2} [\beta(r+1,r+1)]^{1/r} \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q},
\end{aligned} \tag{2.28}$$

where r, $q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$ and $\beta(\cdot,\cdot)$ is *Beta function.*

If the Fréchet derivative $Df : C \to \mathcal{B}(E, F)$ is norm convex on C, then for all x, $y \in C$, we have from (2.15) that

$$\left\| \int_0^1 \left| t - \frac{1}{2} \right| f((1-t)x + ty)\, dt - \frac{1}{4} f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{24} \|y - x\| \left(\|Df(x)\| + \|Df(y)\| \right). \tag{2.29}$$

3. Some Examples for Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If a, $b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [12].

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any x, $y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-s)x + sy\|)\, ds. \tag{3.1}$$

We also have:

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any x, $y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|Df((1-t)x + ty)(y - x)\| \leq \|y - x\| f_a'(\|(1-t)x + ty\|) \tag{3.2}$$

for all $t \in [0, 1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (3.1) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds$$

namely

$$\begin{aligned}\left\|\frac{f(u+\varepsilon v)-f(u)}{\varepsilon}\right\| &\le \|v\|\int_0^1 f_a'(\|(1-s)u+s(u+\varepsilon v)\|)\,ds,\ \varepsilon\ne 0\\ &= \|v\|\int_0^1 f_a'(\|u+s\varepsilon v\|)\,ds\end{aligned}$$

and by taking $\varepsilon\to 0$ we get, by the property of integral, that

$$\|Df(u)(v)\|\le\|v\|\,f_a'(\|u\|)\,. \tag{3.3}$$

Now, if we take in (3.3) $u=(1-t)x+ty$ and $v=y-x$, then we obtain (3.2). □

We have the following result:

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Also, let $p:[0,1]\to[0,\infty)$ be a Lebesgue integrable and symmetric function on $[0,1]$. For any x, $y\in\mathcal{B}$ with $\|x\|$, $\|y\|<R$ we have*

$$\begin{aligned}&\left\|\int_0^1 p(t)f((1-t)x+ty)\,dt-\left(\int_0^1 p(s)\,ds\right)f\left(\frac{x+y}{2}\right)\right\|\\ &\le\|y-x\|\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)f_a'(\|(1-t)x+ty\|)\,dt\\ &+\|y-x\|\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)f_a'(\|(1-t)x+ty\|)\,dt\\ &=:L(f,p,x,y)\,.\end{aligned} \tag{3.4}$$

Moreover, we have the upper bounds

$$\begin{aligned}L(f,p,x,y)&\le\frac{1}{2}\|y-x\|\int_0^1 p(s)\,ds\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\ &\le\frac{1}{2}\|y-x\|\int_0^1 p(s)\,ds\sup_{t\in[0,1]}f_a'(\|(1-t)x+ty\|)\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}L(f,p,x,y)&\le\|y-x\|\left[\frac{1}{2}\int_0^1 p(s)\,ds-\int_0^1\operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right)dt\right]\\ &\quad\times\sup_{t\in[0,1]}f_a'(\|(1-t)x+ty\|)\\ &\le\frac{1}{2}\|y-x\|\int_0^1 p(s)\,ds\sup_{t\in[0,1]}f_a'(\|(1-t)x+ty\|)\end{aligned} \tag{3.6}$$

and

$$L(f,p,x,y) \le \|y-x\| \left[\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \left(\int_0^1 \left[f_a'(\|(1-t)x+ty\|)\right]^q dt\right)^{1/q} \le \|y-x\| \left[\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|) \tag{3.7}$$

for $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$.

The proof follows by Theorem 1 and Lemma 3.

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\left\|\int_0^1 p(t)\,f((1-t)x+ty)\,dt - \left(\int_0^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right)\right\| \le \frac{1}{2}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right| dt, \tag{3.8}$$

for any $x,\ y\in\mathcal{B}$ *with* $\|x\|,\ \|y\|<R$.

Proof. Since f_a' is monotonic nondecreasing and convex, then for $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$ we have

$$\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right) f_a'(\|(1-t)x+ty\|)\,dt \le \int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right) f_a'((1-t)\|x\|+t\|y\|)\,dt \le \int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)\left[(1-t)f_a'(\|x\|)+tf_a'(\|y\|)\right] dt$$

and, similarly

$$\int_0^{1/2}\left(\int_0^t p(s)\,ds\right) f_a'(\|(1-t)x+ty\|)\,dt \le \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\left[(1-t)f_a'(\|x\|)+tf_a'(\|y\|)\right] dt.$$

By applying a similar procedure to the one from Theorem 2 we deduce that

$$\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right) f_a'(\|(1-t)x+ty\|)\,dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right) f_a'(\|(1-t)x+ty\|)\,dt \le \frac{1}{2}\left[f_a'(\|x\|)+f_a'(\|y\|)\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right| dt$$

and by (3.4) we get (3.8). □

Remark 2. *If we take* $p \equiv 1$ *in (3.8), then we get*

$$\left\| \int_0^1 f\left((1-t)x + ty\right) dt - f\left(\frac{x+y}{2}\right) \right\| \le \frac{1}{8} \|y-x\| \left[f_a'\left(\|x\|\right) + f_a'\left(\|y\|\right)\right], \tag{3.9}$$

while for $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ *we get*

$$\begin{aligned} &\left\| \int_0^1 \left|t - \frac{1}{2}\right| f\left((1-t)x + ty\right) dt - \frac{1}{4} f\left(\frac{x+y}{2}\right) \right\| \\ &\le \frac{1}{24} \|y-x\| \left[f_a'\left(\|x\|\right) + f_a'\left(\|y\|\right)\right] \end{aligned} \tag{3.10}$$

for any $x, y \in \mathcal{B}$ *with* $\|x\|, \|y\| < R$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos\lambda, \ \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin\lambda, \ \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \end{aligned} \tag{3.11}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_a(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \\ g_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh\lambda, \ \lambda \in \mathbb{C}; \\ h_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh\lambda, \ \lambda \in \mathbb{C}; \\ l_a(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1). \end{aligned} \tag{3.12}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\begin{aligned}(3.13)\qquad \exp(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n} \qquad \lambda\in\mathbb{C},\\ \frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda\in D(0,1);\\ \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda\in D(0,1);\\ \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda\in D(0,1)\\ {}_2F_1(\alpha,\beta,\gamma,\lambda) &= \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^{n}, \alpha,\beta,\gamma>0,\\ &\lambda\in D(0,1).\end{aligned}$$

From the inequality (3.8) we then have

$$\begin{aligned}(3.14)\qquad &\left\|\int_0^1 p(t)\exp((1-t)x+ty)\,dt-\left(\int_0^1 p(s)\,ds\right)\exp\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{2}\|y-x\|\left[\exp(\|x\|)+\exp(\|y\|)\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right|dt,\end{aligned}$$

$$\begin{aligned}(3.15)\qquad &\left\|\int_0^1 p(t)\sin((1-t)x+ty)\,dt-\left(\int_0^1 p(s)\,ds\right)\sin\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{2}\|y-x\|\left[\cosh(\|x\|)+\cosh(\|y\|)\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right|dt,\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad &\left\|\int_0^1 p(t)\cos((1-t)x+ty)\,dt-\left(\int_0^1 p(s)\,ds\right)\cos\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{2}\|y-x\|\left[\sinh(\|x\|)+\sinh(\|y\|)\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right|dt,\end{aligned}$$

for all $x,\,y\in\mathcal{B}$.

From the inequality (3.8) we also have

$$\begin{aligned}(3.17)\qquad &\left\|\int_0^1 p(t)\left(1\pm[(1-t)x+ty]\right)^{-1}dt-\left(\int_0^1 p(s)\,ds\right)\left[1\pm\left(\frac{x+y}{2}\right)\right]^{-1}\right\|\\ &\le\frac{1}{2}\|y-x\|\left[(1-\|x\|)^{-2}+(1-\|y\|)^{-2}\right]\int_0^1 p(t)\left|t^2-\frac{1}{4}\right|dt,\end{aligned}$$

and

$$\left\| \int_0^1 p(t) \ln \left(1 \pm [(1-t)x + ty]\right)^{-1} dt - \left(\int_0^1 p(s)\,ds\right) \ln \left[1 \pm \left(\frac{x+y}{2}\right)\right]^{-1} \right\| \leq \frac{1}{2} \|y-x\| \left[(1-\|x\|)^{-1} + (1-\|y\|)^{-1}\right] \int_0^1 p(t) \left|t^2 - \frac{1}{4}\right| dt, \tag{3.18}$$

for all x, $y \in \mathcal{B}$ with $\|x\|$, $\|y\| < 1$.

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