GRÜSS TYPE INTEGRAL INEQUALITIES FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. In this paper we obtain in this paper some upper bounds for the quantity

$$\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\|,$$

where f is a Fréchet differentiable function on the open and convex set C and x, $y \in C$, with $x \neq y$ while $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function on $[0,1]$ satisfying the condition

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1].$$

Some applications for functions defined on Banach algebras are also given.

1. Introduction

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 1. *Let F be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\,dx = 1$. If $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left| \alpha(x) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.1}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - \alpha(x)) \left(\overline{\alpha(x)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.2}$$

for a.e. $x \in \Omega$, and $g : \Omega \to F$ is a Bochner measurable function such that $\rho\alpha g$ and ρg are Bochner integrable on Ω, then,

$$\begin{aligned} &\left\| \int_\Omega \rho(x)\alpha(x) g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx \int_\Omega \rho(x) g(x)\,dx \right\| \\ &\leq \frac{1}{2} |\Gamma - \gamma| \int_\Omega \rho(x) \left\| g(x) - \int_\Omega \rho(y) g(y)\,dy \right\| dx. \end{aligned} \tag{1.3}$$

The constant $\frac{1}{2}$ in (1.3) is the best possible.

The following dual result also holds:

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Theorem 2. *Let F and Ω, ρ be as above. If $g:\Omega\to X$ is Bochner measurable on Ω and there exist vector $v\in X$ and $r>0$ such that*

$$\|g(x)-v\|\le r \text{ for a.e. } x\in\Omega$$

and $\alpha:\Omega\to\mathbb{K}$ is a Lebesgue integrable function with $\rho\alpha g$, ρg Bochner integrable functions on Ω, then we have the sharp inequalities

$$\left\|\int_\Omega \rho(x)\alpha(x)g(x)\,dx-\int_\Omega\rho(x)\alpha(x)\,dx\int_\Omega\rho(x)g(x)\,dx\right\| \tag{1.4}$$
$$\le r\int_\Omega\rho(x)\left|\alpha(x)-\int_\Omega\rho(y)\alpha(y)\,dy\right|dx$$
$$\le r\left[\int_\Omega\rho(x)|\alpha(x)|^2\,dx-\left|\int_\Omega\rho(x)\alpha(x)\,dx\right|^2\right]^{\frac12}.$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega=[0,1]$. Also let $\rho(t)=1$ and $g(t)=f((1-t)x+ty)$ for $t\in[0,1]$ and x, $y\in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f:C\subset E\to F$ is continuous on C and x, $y\in C$, $x\ne y$. If $p:[0,1]\to\mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma\in\mathbb{K}$ with*

$$\left|p(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac12|\Gamma-\gamma| \tag{1.5}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-p(t))\left(\overline{p(t)}-\overline{\gamma}\right)\right]\ge0 \tag{1.6}$$

for a.e. $t\in[0,1]$, then,

$$\left\|\int_0^1 p(t)f((1-t)x+ty)\,dt-\int_0^1p(t)\,dt\int_0^1f((1-t)x+ty)\,dt\right\| \tag{1.7}$$
$$\le\frac12|\Gamma-\gamma|\int_0^1\left\|f((1-t)x+ty)-\int_0^1f((1-s)x+sy)\,ds\right\|dt.$$

The constant $\frac12$ in (1.7) is the best possible.

If there exists a vector v and $r>0$ such that

$$\|f((1-t)x+ty)-v\|\le r \text{ for a.e. } t\in[0,1], \tag{1.8}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\left\|\int_0^1 q(t)f((1-t)x+ty)\,dt-\int_0^1q(t)\,dt\int_0^1f((1-t)x+ty)\,dt\right\| \tag{1.9}$$
$$\le r\int_0^1\left|q(t)-\int_0^1q(s)\,ds\right|dt\le r\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac12}.$$

We observe that, if there exists two vectors z, $w\in F$ such that

$$\left\|f((1-t)x+ty)-\frac{z+w}{2}\right\|\le\frac12\|w-z\| \text{ for a.e. } t\in[0,1], \tag{1.10}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\begin{aligned}(1.11)\quad &\left\|\int_0^1 q(t)\,f((1-t)x+ty)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\|\\ &\le\frac{1}{2}\|w-z\|\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt\\ &\le\frac{1}{2}\|w-z\|\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

For some recent results on Grüss' type inequalities, see [1]-[5] and [7]-[16].

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset if a normed vector space, f a real-valued function defined on O, $a\in O$ and u a nonzero element of E. The function f_u given by $t\mapsto f(a+tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0):=\lim_{t\to 0}\frac{f(a+tu)-f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative (directional derivative)* of f at a in the direction u. If $\nabla_a f(u)$ is defined and $\lambda\in\mathbb{R}\setminus\{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u)=\lambda\nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u.

Let E and F be normed vector spaces, and O be an open subset of F. A function $f:O\to F$ is called *Fréchet differentiable* at $x\in O$ if there exists a bounded linear operator $A:E\to F$ such that

$$\lim_{\|h\|\to 0}\frac{\|f(x+h)-f(x)-Ah\|}{\|h\|}=0.$$

If there exists such an operator A, it is unique, so we write $Df(x)=A$ and call it the *Fréchet derivative* of f at x.

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O\ni x\mapsto Df(x)\in\mathcal{B}(E,F)$ is continuous. A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a linear operation. If f is Fréchet differentiable at x, it is also Gâteaux differentiable there, and $\nabla_x f(u)=Df(x)(u)$ for all $u\in E$.

Motivated by the above results, we obtain in this paper some upper bounds for the quantity

$$\left\|\int_0^1 p(t)\,f((1-t)x+ty)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\|,$$

where f is a Fréchet differentiable function on the open and convex set C and x, $y\in C$, with $x\neq y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function on $[0,1]$ satisfying the condition

$$(1.12)\qquad 0\le\int_0^\tau p(s)\,ds\le\int_0^1 p(s)\,ds\text{ for all }\tau\in[0,1].$$

Some applications for functions defined on Banach algebras are also given.

2. Main Results

We need the following preliminary result:

Lemma 1. *Let f be a Fréchet differentiable function on the open and convex set C and $x, y \in C$, with $x \neq y$. Then the auxiliary function $\varphi_{(x,y)}(t) := f((1-t)x + ty)$, $t \in [0,1]$, is differentiable on $(0,1)$ and*

$$\varphi'_{(x,y)}(t) = Df((1-t)x + ty)(y-x). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(x,y)}(0+) = Df(x)(y-x) \tag{2.2}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y-x). \tag{2.3}$$

Proof. For the sake of completeness, we give here a short proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t + h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \frac{f((1-t-h)x + (t+h)y) - f((1-t)x+ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x+ty)}{h}. \end{aligned} \tag{2.4}$$

Since f is Fréchet differentiable hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) &= \lim_{h\to 0} \frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x+ty)}{h} \\ &= Df((1-t)x+ty)(y-x), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(x,y)}(h) - \varphi_{(x,y)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)x + hy) - f(x)}{h} \\ &= \lim_{h\to 0+} \frac{f(x + h(y-x)) - f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x. This proves (2.2).

The equality (2.3) follows in a similar way. □

Our main result is incorporated in the following Grüss' type inequality:

Theorem 3. *Let f be a Fréchet differentiable function on the open and convex set C and $x, y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function satisfying the condition*

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{2.5}$$

then

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.6}$$
$$\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau \left\|Df((1-\tau)x+\tau y)(y-x)\right\| d\tau$$
$$+\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\left\|Df((1-\tau)x+\tau y)(y-x)\right\| d\tau$$
$$=: M(f,p;x,y).$$

Proof. We need the following identity that is of interest in itself.

Let f be a Fréchet differentiable function on the open and convex set C and x, $y \in C$, with $x \neq y$. If $p:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality

$$\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \tag{2.7}$$
$$= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau Df((1-\tau)x+\tau y)(y-x)\,d\tau$$
$$+\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1) Df((1-\tau)x+\tau y)(y-x)\,d\tau.$$

Integrating by parts in the Bochner's integral, [13], we have

$$\int_0^\tau t\varphi'_{(x,y)}(t)\,dt + \int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt$$
$$= \tau\varphi_{(x,y)}(\tau) - \int_0^\tau \varphi_{(x,y)}(t)\,dt - (\tau-1)\varphi_{(x,y)}(\tau) - \int_\tau^1 \varphi_{(x,y)}(t)\,dt$$
$$= \varphi_{(x,y)}(\tau) - \int_0^1 \varphi_{(x,y)}(t)\,dt$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $p(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(t)\,dt \tag{2.8}$$
$$= \int_0^1 p(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right) d\tau + \int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right) d\tau.$$

Using integration by parts, we get

$$\int_0^1 p(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right) d\tau \tag{2.9}$$
$$= \int_0^1 \left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right) d\left(\int_0^\tau p(s)\,ds\right)$$
$$= \left(\int_0^\tau p(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)\Bigg|_0^1 - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$

$$= \left(\int_0^1 p(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_0^1 p(s)\,ds - \int_0^\tau p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$

and

$$\begin{aligned}(2.10)\qquad & \int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\tau \\ &= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right) d\left(\int_0^\tau p(s)\,ds\right) \\ &= \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau p(s)\,ds\right)\Bigg|_0^1 \\ &+ \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau,\end{aligned}$$

which, together with (2.8), prove the identity in (2.7).

By taking the norm in (2.7), we get

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\|$$
$$\le \left\|\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau Df((1-\tau)x+\tau y)(y-x)\,d\tau\right\|$$
$$+ \left\|\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,Df((1-\tau)x+\tau y)(y-x)\,d\tau\right\|$$
$$\le \int_0^1 \left|\int_\tau^1 p(s)\,ds\right|\tau\,\|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau$$
$$+ \int_0^1 \left|\int_0^\tau p(s)\,ds\right|(1-\tau)\,\|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau \text{ by condition (2.5)}$$
$$= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\,\|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau$$
$$+ \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau,$$

which proves (2.6). □

Corollary 2. *With the assumptions of Theorem 3 and if*

$$\sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)(y-x)\| < \infty,$$

then

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)x+\tau y)\, d\tau \right\| \tag{2.11}$$
$$\leq \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)(y-x)\| \int_0^1 \tau^2 \breve{p}(\tau)\, d\tau,$$

where

$$\breve{p}(\tau) := \frac{1}{2}[p(\tau)+p(1-\tau)],\ \ \tau\in[0,1].$$

Moreover, if p *is symmetric on* $[0,1]$, *namely* $p(1-t)=p(t)$ *for* $t\in[0,1]$, *then*

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)x+\tau y)\, d\tau \right\| \tag{2.12}$$
$$\leq \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)(y-x)\| \int_0^1 \tau^2 p(\tau)\, d\tau.$$

Proof. We have for $M(f,p;x,y)$ defined by (2.6) that

$$\begin{aligned} M(f,p;x,y) &\leq \int_0^1 \left(\int_\tau^1 p(s)\, ds\right) \tau \|Df((1-\tau)x+\tau y)(y-x)\|\, d\tau \\ &+ \int_0^1 \left(\int_0^\tau p(s)\, ds\right)(1-\tau)\|Df((1-\tau)x+\tau y)(y-x)\|\, d\tau \\ &\leq \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)(y-x)\| \\ &\times \left[\int_0^1 \left(\int_\tau^1 p(s)\, ds\right)\tau d\tau + \int_0^1 \left(\int_0^\tau p(s)\, ds\right)(1-\tau)\, d\tau\right]. \end{aligned} \tag{2.13}$$

Using integration by parts, we obtain

$$\begin{aligned} 0 &\leq \int_0^1 \left(\int_\tau^1 p(s)\, ds\right)\tau d\tau = \frac{1}{2}\int_0^1 \left(\int_\tau^1 p(s)\, ds\right) d\left(\tau^2\right) \\ &= \frac{1}{2}\left[\left(\int_\tau^1 p(s)\, ds\right)\tau^2\Big|_0^1 + \int_0^1 \tau^2 p(\tau)\, d\tau\right] = \frac{1}{2}\int_0^1 \tau^2 p(\tau)\, d\tau \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \int_0^1 \left(\int_0^\tau p(s)\, ds\right)(1-\tau)\, d\tau = -\frac{1}{2}\left[\int_0^1 \left(\int_0^\tau p(s)\, ds\right) d\left((1-\tau)^2\right)\right] \\ &= -\frac{1}{2}\left[\left(\int_0^\tau p(s)\, ds\right)(1-\tau)^2\Big|_0^1 - \int_0^1 (1-\tau)^2 p(\tau)\, d\tau\right] \\ &= \frac{1}{2}\int_0^1 (1-\tau)^2 p(\tau)\, d\tau = \frac{1}{2}\int_0^1 \tau^2 p(1-\tau)\, d\tau \end{aligned}$$

and by (2.13) we get (2.11). □

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \tag{2.14}$$
$$\le \left(\sup_{\tau\in[0,1]} \left[\left(\int_\tau^1 p(s)\,ds \right) \tau \right] + \sup_{\tau\in[0,1]} \left[\left(\int_0^\tau p(s)\,ds \right)(1-\tau) \right] \right)$$
$$\times \int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|\,d\tau.$$

We also have:

Corollary 4. *With the assumptions of Theorem 3 we have, for* $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$, *that*

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \tag{2.15}$$
$$\le \left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds \right)^r \tau^r d\tau \right)^{1/r} + \int_0^1 \left(\left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r\,d\tau \right)^{1/r} \right]$$
$$\times \left(\int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|^q\,d\tau \right)^{1/q}$$
$$\le 2^{1/q} \left[\int_0^1 \left(\left(\int_\tau^1 p(s)\,ds \right)^r \tau^r + \left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r \right) d\tau \right]$$
$$\times \left(\int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|^q\,d\tau \right)^{1/q}.$$

Proof. By Hölder's integral inequality, we have for $r,\ q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, that

$$M(f,p;x,y) \tag{2.16}$$
$$\le \left(\int_0^1 \left(\int_\tau^1 p(s)\,ds \right)^r \tau^r d\tau \right)^{1/r} \left(\int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|^q\,d\tau \right)^{1/q}$$
$$+ \left(\int_0^1 \left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r\,d\tau \right)^{1/r} \left(\int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|^q\,d\tau \right)^{1/q}$$
$$= \left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds \right)^r \tau^r d\tau \right)^{1/r} + \left(\int_0^1 \left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r\,d\tau \right)^{1/r} \right]$$
$$\times \left(\int_0^1 \| Df((1-\tau)x+\tau y)(y-x) \|^q\,d\tau \right)^{1/q},$$

which proves the first inequality in (2.15).

By the convexity of power function, we have

$$\left(\frac{a+b}{2} \right)^r \le \frac{a^r+b^r}{2},\ r>1,\ a,\ b>0$$

namely

$$a+b \le 2^{1-1/r} (a^r+b^r)^{1/r},\ r>1,\ a,\ b>0.$$

Therefore

$$\left(\int_0^1\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r}+\left(\int_0^1\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\,d\tau\right)^{1/r}$$
$$\leq 2^{1-1/r}\left[\int_0^1\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau+\int_0^1\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\,d\tau\right]^{1/r}$$
$$=2^{1-1/r}\left[\int_0^1\left(\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r+\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\right)d\tau\right]$$

and the last part of (2.15) is proved. □

We also have:

Corollary 5. *With the assumptions of Theorem 3,*

$$\left\|\int_0^1 p(\tau)\,f((1-\tau)\,x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)\,x+\tau y)\,d\tau\right\| \tag{2.17}$$
$$\leq\int_0^1 p(s)\,ds\int_0^1\|Df((1-\tau)\,x+\tau y)\,(y-x)\|\,d\tau.$$

Proof. By the condition (2.5) we have

$$0\leq\int_\tau^1 p(s)\,ds\leq\int_0^1 p(s)\,ds,\ 0\leq\int_0^\tau p(s)\,ds\leq\int_0^1 p(s)\,ds$$

for $\tau\in[0,1]$.

Therefore

$$M(f,p;x,y)\leq\int_0^1 p(s)\,ds\int_0^1\tau\,\|Df((1-\tau)\,x+\tau y)\,(y-x)\|\,d\tau$$
$$+\int_0^1 p(s)\,ds\int_0^1(1-\tau)\,\|Df((1-\tau)\,x+\tau y)\,(y-x)\|\,d\tau$$
$$=\int_0^1 p(s)\,ds\int_0^1\|Df((1-\tau)\,x+\tau y)\,(y-x)\|\,d\tau,$$

which proves (2.17). □

Remark 1. *Since the Fréchet derivative satisfies the condition*

$$\|Df(a)\,(b)\|\leq\|Df(a)\|\,\|b\|$$

for $a\in C$ and $b\in E$, then we also have the chain of inequalities

$$\left\|\int_0^1 p(\tau)\,f((1-\tau)\,x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)\,x+\tau y)\,d\tau\right\| \tag{2.18}$$
$$\leq\|y-x\|\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau\,\|Df((1-\tau)\,x+\tau y)\|\,d\tau$$
$$+\|y-x\|\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\|Df((1-\tau)\,x+\tau y)\|\,d\tau$$
$$=:M_1(f,p;x,y),$$

provided that p satisfies the condition (2.5).

Also,

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.19}$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)\| \int_0^1 \tau^2 \breve{p}(\tau)\,d\tau$$

and if p *is symmetric, then*

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.20}$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)\| \int_0^1 \tau^2 p(\tau)\,d\tau.$$

Moreover

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.21}$$
$$\leq \|y-x\| \left(\sup_{\tau\in[0,1]}\left[\left(\int_\tau^1 p(s)\,ds\right)\tau\right] + \sup_{\tau\in[0,1]}\left[\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\right]\right)$$
$$\times \int_0^1 \|Df((1-\tau)x+\tau y)\|\,d\tau$$

and for $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$*, that*

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.22}$$
$$\leq \|y-x\| \left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r} + \int_0^1 \left(\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\,d\tau\right)^{1/r}\right]$$
$$\times \left(\int_0^1 \|Df((1-\tau)x+\tau y)\|^q\,d\tau\right)^{1/q}$$
$$\leq 2^{1/q}\|y-x\| \left[\int_0^1 \left(\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r + \left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\right) d\tau\right]$$
$$\times \left(\int_0^1 \|Df((1-\tau)x+\tau y)\|^q\,d\tau\right)^{1/q}.$$

In addition, we have the simpler inequality

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.23}$$
$$\leq \|y-x\| \int_0^1 p(s)\,ds \int_0^1 \|Df((1-\tau)x+\tau y)\|\,d\tau.$$

3. Some Examples for Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\|\,\|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [8].

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-s)x + sy\|)\, ds. \tag{3.1}$$

We also have:

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|Df((1-\tau)x + \tau y)(y - x)\| \leq \|y - x\| f_a'(\|(1-\tau)x + \tau y\|) \tag{3.2}$$

for all $\tau \in [0, 1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (3.1) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds$$

namely

$$\begin{aligned} \left\|\frac{f(u + \varepsilon v) - f(u)}{\varepsilon}\right\| &\leq \|v\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds, \ \varepsilon \neq 0 \\ &= \|v\| \int_0^1 f_a'(\|u + s\varepsilon v\|)\, ds \end{aligned}$$

and by taking $\varepsilon \to 0$ we get, by the property of integral, that

$$\|Df(u)(v)\| \leq \|v\| f_a'(\|u\|). \tag{3.3}$$

Now, if we take in (3.3) $u = (1-\tau)x + \tau y$ and $v = y - x$, then we get (3.2). □

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If*

$p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function satisfying the condition (2.5), then

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.4}$$
$$\le \|y-x\|\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau f_a'(\|(1-\tau)x+\tau y\|)\,d\tau$$
$$+\|y-x\|\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau) f_a'(\|(1-\tau)x+\tau y\|)\,d\tau$$

for any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$.

The proof follows by Theorem 3 and Lemma 3.
From Corollary 5 we also have:

Corollary 6. *Withe the assumptions of Theorem 4 we have*

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.5}$$
$$\le \|y-x\|\int_0^1 p(s)\,ds\int_0^1 f_a'(\|(1-\tau)x+\tau y\|)\,d\tau.$$

Moreover, we have the upper bounds

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.6}$$
$$\le \|y-x\|\int_0^1 p(s)\,ds\times\begin{cases}\frac{f_a(\|x\|)-f_a(\|y\|)}{\|x\|-\|y\|} & \text{if } \|x\|\ne\|y\|,\\[2mm] f_a'(\|x\|) & \text{if } \|x\|\ne\|y\|\end{cases}$$

and

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.7}$$
$$\le \|y-x\|\frac{1}{2}\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right]\int_0^1 p(s)\,ds$$
$$\le \|y-x\|\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\int_0^1 p(s)\,ds$$

for any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$.

Proof. Since f_a' is increasing, hence

$$\begin{aligned}\int_0^1 f_a'(\|(1-\tau)x+\tau y\|)\,d\tau &\le \int_0^1 f_a'((1-\tau)\|x\|+\tau\|y\|)\,d\tau\\ &=\begin{cases}\frac{f_a(\|x\|)-f_a(\|y\|)}{\|x\|-\|y\|} & \text{if } \|x\|\ne\|y\|,\\[2mm] f_a'(\|x\|) & \text{if } \|x\|\ne\|y\|,\end{cases}\end{aligned}$$

which proves (3.6).

Also, because f_a' is convex, then by Hermite-Hadamard inequality, we derive that

$$\begin{aligned}\int_0^1 f_a'\left(\|(1-\tau)x+\tau y\|\right)d\tau &\leq \frac{1}{2}\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right]\\ &\leq \frac{f_a'(\|x\|)+f_a'(\|y\|)}{2},\end{aligned}$$

which proves (3.7). □

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}f(\lambda) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n = \ln\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);\\ g(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n} = \cos\lambda,\ \ \lambda\in\mathbb{C};\\ h(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1} = \sin\lambda,\ \ \lambda\in\mathbb{C};\\ l(\lambda) &= \sum_{n=0}^{\infty}(-1)^n\lambda^n = \frac{1}{1+\lambda},\ \ \lambda\in D(0,1);\end{aligned}\tag{3.8}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned}f_a(\lambda) &= \sum_{n=1}^{\infty}\frac{1}{n}\lambda^n = \ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1);\\ g_a(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n} = \cosh\lambda,\ \ \lambda\in\mathbb{C};\\ h_a(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh\lambda,\ \ \lambda\in\mathbb{C};\\ l_a(\lambda) &= \sum_{n=0}^{\infty}\lambda^n = \frac{1}{1-\lambda},\ \ \lambda\in D(0,1).\end{aligned}\tag{3.9}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.10}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1).$$

Further, we assume that $p:[0,1] \to \mathbb{R}$ is a Lebesgue integrable function satisfying the condition (2.5).

If we write the inequalities (3.6) and (3.7) for the exponential function $f(x) = \exp x$, then we get

$$\begin{aligned} &\left\| \int_0^1 p(\tau)\exp((1-\tau)x+\tau y)\,d\tau \right. \\ &\qquad \left. - \int_0^1 p(\tau)\,d\tau \int_0^1 \exp((1-\tau)x+\tau y)\,d\tau \right\| \\ &\qquad \le \|y-x\| \int_0^1 p(s)\,ds \times \begin{cases} \frac{\exp(\|x\|)-\exp(\|y\|)}{\|x\|-\|y\|} & \text{if } \|x\| \ne \|y\|, \\ \exp(\|x\|) & \text{if } \|x\| \ne \|y\| \end{cases} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\left\| \int_0^1 p(\tau)\exp((1-\tau)x+\tau y)\,d\tau \right. \\ &\qquad \left. - \int_0^1 p(\tau)\,d\tau \int_0^1 \exp((1-\tau)x+\tau y)\,d\tau \right\| \\ &\le \frac{1}{2}\|y-x\| \left[\exp\left(\left\|\frac{x+y}{2}\right\|\right) + \frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right] \int_0^1 p(s)\,ds \\ &\qquad \le \|y-x\| \frac{\exp(\|x\|)+\exp(\|y\|)}{2} \int_0^1 p(s)\,ds \end{aligned} \tag{3.12}$$

for all x, $y \in \mathcal{B}$.

If we write the inequalities (3.6) and (3.7) for the trigonometric function $f(x) = \sin x$, then we get

$$\begin{aligned}(3.13)\quad & \left\| \int_0^1 p(\tau) \sin((1-\tau)x + \tau y)\, d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \sin((1-\tau)x + \tau y)\, d\tau \right\| \\ & \leq \|y - x\| \int_0^1 p(s)\, ds \times \begin{cases} \frac{\sinh(\|x\|) - \sinh(\|y\|)}{\|x\| - \|y\|} & \text{if } \|x\| \neq \|y\|, \\ \cosh(\|x\|) & \text{if } \|x\| \neq \|y\| \end{cases}\end{aligned}$$

and

$$\begin{aligned}(3.14)\quad & \left\| \int_0^1 p(\tau) \sin((1-\tau)x + \tau y)\, d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \sin((1-\tau)x + \tau y)\, d\tau \right\| \\ & \leq \frac{1}{2} \|y - x\| \left[\cosh\left(\left\| \frac{x+y}{2} \right\| \right) + \frac{\cosh(\|x\|) + \cosh(\|y\|)}{2} \right] \int_0^1 p(s)\, ds \\ & \leq \|y - x\| \frac{\cosh(\|x\|) + \cosh(\|y\|)}{2} \int_0^1 p(s)\, ds\end{aligned}$$

for all $x, y \in \mathcal{B}$.

Also, if we write the inequalities (3.6) and (3.7) for the functions $f(x) = (1 \pm x)^{-1}$, then we get

$$\begin{aligned}(3.15)\quad & \left\| \int_0^1 p(\tau) [1 \pm ((1-\tau)x + \tau y)]^{-1}\, d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 [1 \pm ((1-\tau)x + \tau y)]^{-1}\, d\tau \right\| \\ & \leq \|y - x\| \int_0^1 p(s)\, ds \times \begin{cases} \frac{1}{(1-\|x\|)(1-\|y\|)} & \text{if } \|x\| \neq \|y\|, \\ \frac{1}{(1-\|x\|)^2} & \text{if } \|x\| \neq \|y\| \end{cases}\end{aligned}$$

and

$$\begin{aligned}(3.16)\quad & \left\| \int_0^1 p(\tau) [1 \pm ((1-\tau)x + \tau y)]^{-1}\, d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 [1 \pm ((1-\tau)x + \tau y)]^{-1}\, d\tau \right\| \\ & \leq \frac{1}{2} \|y - x\| \left[\left(1 - \left\| \frac{x+y}{2} \right\| \right)^{-2} + \frac{(1 - \|x\|)^{-2} + (1 - \|y\|)^{-2}}{2} \right] \int_0^1 p(s)\, ds \\ & \leq \|y - x\| \frac{(1 - \|x\|)^{-2} + (1 - \|y\|)^{-2}}{2} \int_0^1 p(s)\, ds\end{aligned}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$.

Finally, by using the logarithmic functions $f(x) = \ln(1 \pm x)^{-1}$ and the inequalities (3.6) and (3.7), we obtain

$$\begin{aligned} (3.17)\quad & \left\| \int_0^1 p(\tau) \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \\ & \leq \|y - x\| \int_0^1 p(s)\, ds \times \begin{cases} \frac{\ln(1-\|x\|)^{-1} - \ln(1-\|y\|)^{-1}}{\|x\| - \|y\|} & \text{if } \|x\| \neq \|y\|, \\ \frac{1}{(1-\|x\|)} \text{ if } \|x\| \neq \|y\| \end{cases} \end{aligned}$$

and

$$\begin{aligned} (3.18)\quad & \left\| \int_0^1 p(\tau) \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right. \\ & \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \\ & \leq \frac{1}{2}\|y - x\| \left[\left(1 - \left\| \frac{x+y}{2} \right\| \right)^{-1} + \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2} \right] \int_0^1 p(s)\, ds \\ & \leq \|y - x\| \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2} \int_0^1 p(s)\, ds \end{aligned}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$.

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