

WEIGHTED NORM INEQUALITIES OF TRAPEZOID TYPE FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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ABSTRACT. In this paper we establish some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s) ds \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\|$$

in the case that $f : C \subset E \rightarrow F$ is a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F , $x, y \in C$ and $p : [0, 1] \rightarrow [0, \infty)$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$. Applications for Banach algebras are also provided.

1. INTRODUCTION

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset of a normed vector space, f a real-valued function defined on O , $a \in O$ and u a nonzero element of E . The function f_u given by $t \mapsto f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0) := \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative* (*directional derivative*) of f at a in the direction u . If $\nabla_a f(u)$ is defined and $\lambda \in \mathbb{R} \setminus \{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u) = \lambda \nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u .

Let E and F be normed vector spaces, and O be an open subset of E . A function $f : O \rightarrow F$ is called *Fréchet differentiable* at $x \in O$ if there exists a bounded linear operator $A : E \rightarrow F$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

If there exists such an operator A , it is unique, so we write $Df(x) = A$ and call it the *Fréchet derivative* of f at x .

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O \ni x \mapsto Df(x) \in \mathcal{B}(E, F)$ is continuous. A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a

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linear operation. If f is Fréchet differentiable at x , it is also Gâteaux differentiable there, and $\nabla_x f(u) = Df(x)(u)$ for all $u \in E$.

We say that the function $f : O \subset E \rightarrow F$ is L -Lipschitzian on O with the constant $L > 0$ if

$$\|f(x) - f(y)\| \leq L \|x - y\| \text{ for all } x, y \in O.$$

In [13] we established among others the following midpoint and trapezoid type inequalities for L -Lipschitzian functions f on an open and convex subset C in E

$$(1.1) \quad \left\| \int_0^1 f((1-t)t + ty) dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{4} L \|x - y\|$$

and

$$(1.2) \quad \left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)t + ty) dt \right\| \leq \frac{1}{4} L \|x - y\|$$

for all $x, y \in C$. The constant $\frac{1}{4}$ is best possible in both inequalities (1.1) and (1.2).

For Hermite-Hadamard's type inequalities, namely

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) du \leq \frac{f(a) + f(b)}{2},$$

where $f : [a, b] \rightarrow \mathbb{R}$ is convex on $[a, b]$, see for instance [6], [7], [8], [19], [21], [22], [23], [25], [26], [27], [28], [29], [30], [31] and the references therein.

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s) ds \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\|$$

in the case that $f : C \subset E \rightarrow F$ is a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F , $x, y \in C$ and $p : [0, 1] \rightarrow [0, \infty)$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$. Applications for Banach algebras are also provided.

2. GENERAL RESULTS FOR BANACH SPACES

Consider a function $f : C \subset E \rightarrow F$ that is defined on the open and convex set C . We have the following properties for the *auxiliary function*

$$\varphi_{(x,y)}(t) := f((1-t)x + ty), \quad t \in [0, 1],$$

where $x, y \in C$.

Lemma 1. *Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Then for all $x, y \in C$ the auxiliary function $\varphi_{(x,y)}$ is differentiable on $(0, 1)$ and*

$$(2.1) \quad \varphi'_{(x,y)}(t) = Df((1-t)x + ty)(y - x).$$

Also we have for the lateral derivatives

$$(2.2) \quad \varphi'_{(x,y)}(0+) = Df(x)(y - x)$$

and

$$(2.3) \quad \varphi'_{(x,y)}(1-) = Df(y)(y - x).$$

Proof. Let $t \in (0, 1)$ and $h \neq 0$ small enough such that $t + h \in (0, 1)$. Then

$$(2.4) \quad \begin{aligned} & \frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \frac{f((1-t-h)x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}. \end{aligned}$$

Since f is Fréchet differentiable, hence by taking the limit over $h \rightarrow 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) &= \lim_{h \rightarrow 0} \frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \\ &= Df((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) &= \lim_{h \rightarrow 0+} \frac{\varphi_{(x,y)}(h) - \varphi_{(x,y)}(0)}{h} \\ &= \lim_{h \rightarrow 0+} \frac{f((1-h)x + hy) - f(x)}{h} \\ &= \lim_{h \rightarrow 0+} \frac{f(x + h(y-x)) - f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x . This proves (2.2).

The equality (2.3) follows in a similar way. \square

The following result holds:

Theorem 1. *Let $f : C \subset E \rightarrow F$ be a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0, 1] \rightarrow [0, \infty)$ a Lebesgue integrable and symmetric function on $[0, 1]$. Then for all $x, y \in C$*

$$(2.5) \quad \begin{aligned} & \left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ & \quad + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ & = \int_0^1 \left| \int_t^{1/2} p(s) ds \right| \|Df((1-t)x + ty)(y-x)\| dt \\ & =: B(f, p, x, y). \end{aligned}$$

Moreover, we have the upper bounds

$$(2.6) \quad B(f, p, x, y) \leq \frac{1}{2} \int_0^1 p(s) ds \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \leq \frac{1}{2} \int_0^1 p(s) ds \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|,$$

$$(2.7) \quad B(f, p, x, y) \leq \left(\frac{1}{2} \int_0^1 p(s) ds - \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) tp(t) dt \right) \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \\ \leq \frac{1}{2} \int_0^1 p(s) ds \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|$$

and

$$(2.8) \quad B(f, p, x, y) \leq \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ \leq \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|$$

for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.

Proof. Using the integration by parts for Bochner's integral, [24], we have

$$(2.9) \quad \int_0^1 \left(\int_0^t p(s) ds - \frac{1}{2} \int_0^1 p(s) ds \right) \varphi'_{(x,y)}(t) dt \\ = \left(\int_0^t p(s) ds - \frac{1}{2} \int_0^1 p(s) ds \right) \varphi_{(x,y)}(t) \Big|_0^1 - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ = \left(\int_0^1 p(s) ds - \frac{1}{2} \int_0^1 p(s) ds \right) \varphi_{(x,y)}(1) + \left(\frac{1}{2} \int_0^1 p(s) ds \right) \varphi_{(x,y)}(0) \\ - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ = \left(\frac{1}{2} \int_0^1 p(s) ds \right) [\varphi_{(x,y)}(0) + \varphi_{(x,y)}(1)] \\ - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ = \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) \varphi_{(x,y)}(t) dt.$$

Observe that, by the symmetry of p we have

$$\int_0^{1/2} p(s) ds = \int_{1/2}^1 p(s) ds = \frac{1}{2} \int_0^1 p(s) ds$$

and then

$$\begin{aligned} & \int_0^1 \left(\int_0^t p(s) ds - \frac{1}{2} \int_0^1 p(s) ds \right) \varphi'_{(x,y)}(t) dt \\ &= \int_0^1 \left(\int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right) \varphi'_{(x,y)}(t) dt. \end{aligned}$$

Then we get by (2.9) the following identity of interest

$$\begin{aligned} (2.10) \quad & \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \\ &= \int_0^1 \left(\int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right) Df((1-t)x + ty)(y-x) dt. \end{aligned}$$

If we take the norm in (2.10) and use the properties of the integrals, then we obtain

$$\begin{aligned} & \left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \int_0^1 \left\| \left(\int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right) Df((1-t)x + ty)(y-x) \right\| dt \\ &= \int_0^1 \left| \int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right| \|Df((1-t)x + ty)(y-x)\| dt \\ &= \int_0^{1/2} \left| \int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right| \|Df((1-t)x + ty)(y-x)\| dt \\ &+ \int_{1/2}^1 \left| \int_0^t p(s) ds - \int_0^{1/2} p(s) ds \right| \|Df((1-t)x + ty)(y-x)\| dt \\ &= \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ &+ \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ &= B(f, p, x, y), \end{aligned}$$

which proves the inequality (2.5).

Since

$$\int_t^{1/2} p(s) ds \leq \int_0^{1/2} p(s) ds \text{ for } t \in [0, 1/2]$$

and

$$\int_{1/2}^t p(s) ds \leq \int_{1/2}^1 p(s) ds \text{ for } t \in [1/2, 1],$$

hence

$$\begin{aligned}
B(f, p, x, y) &\leq \int_0^{1/2} p(s) ds \int_0^{1/2} \|Df((1-t)x + ty)(y-x)\| dt \\
&\quad + \int_{1/2}^1 p(s) ds \int_{1/2}^1 \|Df((1-t)x + ty)(y-x)\| dt \\
&= \frac{1}{2} \int_0^1 p(s) ds \int_0^{1/2} \|Df((1-t)x + ty)(y-x)\| dt \\
&\quad + \frac{1}{2} \int_0^1 p(s) ds \int_{1/2}^1 \|Df((1-t)x + ty)(y-x)\| dt \\
&= \frac{1}{2} \int_0^1 p(s) ds \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt,
\end{aligned}$$

which proves the first part of (2.6). The second part is obvious.

We also have

$$\begin{aligned}
(2.11) \quad B(f, p, x, y) &\leq \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) dt \\
&\quad + \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) dt \\
&= \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \\
&\quad \times \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) dt \right].
\end{aligned}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) dt \\
&= \left(\int_t^{1/2} p(s) ds \right) t \Big|_0^{1/2} + \int_0^{1/2} tp(t) dt \\
&\quad + \left(\int_{1/2}^t p(s) ds \right) t \Big|_{1/2}^1 - \int_{1/2}^1 tp(t) dt \\
&= \int_0^{1/2} tp(t) dt + \int_{1/2}^1 p(s) ds - \int_{1/2}^1 tp(t) dt \\
&= \frac{1}{2} \int_0^1 p(s) ds - \int_0^1 \operatorname{sgn} \left(t - \frac{1}{2} \right) tp(t) dt
\end{aligned}$$

and by (2.11) we get the first inequality in (2.7).

If we use the weighted Čebyšev's inequality for two nondecreasing functions g and h on the interval $[a, b]$, and p positive on $[a, b]$, namely

$$\int_a^b p(t) dt \int_a^b g(t) h(t) p(t) dt \geq \int_a^b g(t) p(t) dt \int_a^b h(t) p(t) dt,$$

then we have

$$\int_a^b p(t) dt \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) tp(t) dt \geq \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) p(t) dt \int_0^1 tp(t) dt = 0$$

since $z(t) := \operatorname{sgn}\left(t - \frac{1}{2}\right) p(t)$ is asymmetric on $[a, b]$.

Using Hölder's inequality, we have for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$ that

$$\begin{aligned} & B(f, p, x, y) \\ & \leq \left(\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt \right)^{1/r} \left(\int_0^{1/2} \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ & + \left(\int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right)^{1/r} \left(\int_{1/2}^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ & = \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ & \times \left(\int_0^{1/2} \|Df((1-t)x + ty)(y-x)\|^q dt + \int_{1/2}^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ & = \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ & \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}, \end{aligned}$$

which proves the first inequality in (2.8). The last part is obvious. \square

Remark 1. Since the Fréchet derivative satisfies the condition

$$\|Df(a)(b)\| \leq \|Df(a)\| \|b\|$$

for $a \in C$ and $b \in E$, then we also have the chain of inequalities

$$\begin{aligned} (2.12) \quad & \left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & + \|y - x\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & = \|y - x\| \int_0^1 \left| \int_t^{1/2} p(s) ds \right| \|Df((1-t)x + ty)\| dt \\ & =: B(f, p, x, y) \end{aligned}$$

for all $x, y \in C$.

Moreover, we have the upper bounds

$$(2.13) \quad \begin{aligned} B(f, p, x, y) &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \int_0^1 \|Df((1-t)x + ty)\| dt \\ &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \sup_{t \in [0,1]} \|Df((1-t)x + ty)\|, \end{aligned}$$

$$(2.14) \quad \begin{aligned} B(f, p, x, y) &\leq \|y - x\| \left(\frac{1}{2} \int_0^1 p(s) ds - \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) tp(t) dt \right) \\ &\quad \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)\| \\ &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \sup_{t \in [0,1]} \|Df((1-t)x + ty)\| \end{aligned}$$

and

$$(2.15) \quad \begin{aligned} B(f, p, x, y) &\leq \|y - x\| \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ &\quad \times \left(\int_0^1 \|Df((1-t)x + ty)\|^q dt \right)^{1/q} \\ &\leq \|y - x\| \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\ &\quad \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)\| \end{aligned}$$

for all $x, y \in C$, where $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.

A function $h : C \subset E \rightarrow F$ is norm convex on C if

$$\|h((1-t)x + ty)\| \leq (1-t)\|h(x)\| + t\|h(y)\|$$

for all $x, y \in C$ and $t \in [0, 1]$.

Theorem 2. *With the assumptions of Theorem 1 and if the Fréchet derivative $Df : C \rightarrow \mathcal{B}(E, F)$ is norm convex on C , then for all $x, y \in C$,*

$$(2.16) \quad \begin{aligned} &\left\| \left(\int_0^1 p(s) ds \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ &\leq \frac{1}{2} \|y - x\| [\|Df(x)\| + \|Df(y)\|] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt. \end{aligned}$$

Proof. Observe that, if the Fréchet derivative $Df : C \rightarrow \mathcal{B}(E, F)$ is norm convex on C , then for all $x, y \in C$

$$\begin{aligned} & \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & \leq \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) [(1-t)\|Df(x)\| + t\|Df(y)\|] dt \\ & = \|Df(x)\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) (1-t) dt \\ & \quad + \|Df(y)\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) t dt \end{aligned}$$

and

$$\begin{aligned} & \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & \leq \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) [(1-t)\|Df(x)\| + t\|Df(y)\|] dt \\ & = \|Df(x)\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) (1-t) dt + \|Df(y)\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) t dt. \end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned} (2.17) \quad & \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & \leq \|Df(x)\| \\ & \quad \times \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) (1-t) dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) (1-t) dt \right] \\ & \quad + \|Df(y)\| \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) t dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) t dt \right]. \end{aligned}$$

Using the integration by parts, we have

$$\begin{aligned} & \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) (1-t) dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) (1-t) dt \\ & = -\frac{1}{2} \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) d[(1-t)^2] - \frac{1}{2} \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) d[(1-t)^2] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \left[\left(\int_t^{1/2} p(s) ds \right) (1-t)^2 \Big|_0^{1/2} + \int_0^{1/2} (1-t)^2 p(t) dt \right] \\
&\quad - \frac{1}{2} \left[\left(\int_{1/2}^t p(s) ds \right) (1-t)^2 \Big|_{1/2}^1 - \int_{1/2}^1 (1-t)^2 p(t) dt \right] \\
&= -\frac{1}{2} \left[\int_0^{1/2} (1-t)^2 p(t) dt - \int_0^{1/2} p(s) ds \right] - \frac{1}{2} \left[- \int_{1/2}^1 (1-t)^2 p(t) dt \right] \\
&= \frac{1}{2} \left[\int_{1/2}^1 (1-t)^2 p(t) dt - \int_0^{1/2} (1-t)^2 p(t) dt + \int_0^{1/2} p(s) ds \right] =: K.
\end{aligned}$$

If we change the variable $1-t=s$, $t \in [1/2, 1]$, then by the symmetry of p on $[a, b]$,

$$\int_{1/2}^1 (1-t)^2 p(t) dt = \int_0^{1/2} s^2 p(1-s) ds = \int_0^{1/2} s^2 p(s) ds.$$

Therefore

$$\begin{aligned}
K &= \frac{1}{2} \left[\int_0^{1/2} t^2 p(t) dt - \int_0^{1/2} (1-t)^2 p(t) dt + \int_0^{1/2} p(t) dt \right] \\
&= \frac{1}{2} \int_0^{1/2} (t^2 - 1 + 2t - t^2 + 1) p(t) dt = \int_0^{1/2} tp(t) dt.
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) t dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) t dt \\
&= \frac{1}{2} \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) d(t^2) + \frac{1}{2} \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) d(t^2) \\
&= \frac{1}{2} \left[\left(\int_t^{1/2} p(s) ds \right) t^2 \Big|_0^{1/2} + \int_0^{1/2} t^2 p(t) dt \right] \\
&\quad + \frac{1}{2} \left[\left(\int_{1/2}^t p(s) ds \right) t^2 \Big|_{1/2}^1 - \int_{1/2}^1 t^2 p(t) dt \right] \\
&= \frac{1}{2} \left[\int_0^{1/2} t^2 p(t) dt + \int_{1/2}^1 p(s) ds - \int_{1/2}^1 t^2 p(t) dt \right] =: M.
\end{aligned}$$

If we change the variable $1-t=s$, $t \in [1/2, 1]$, then by the symmetry of p

$$\int_{1/2}^1 t^2 p(t) dt = \int_0^{1/2} (1-s)^2 p(1-s) ds = \int_0^{1/2} (1-s)^2 p(s) ds$$

and $\int_{1/2}^1 p(s) ds = \int_0^{1/2} p(t) dt$, then

$$\begin{aligned} M &= \frac{1}{2} \left[\int_0^{1/2} t^2 p(t) dt + \int_0^{1/2} p(t) dt - \int_0^{1/2} (1-t)^2 p(ts) dt \right] \\ &= \frac{1}{2} \int_0^{1/2} (t^2 + 1 - 1 + 2t - t^2) p(t) dt = \int_0^{1/2} tp(t) dt. \end{aligned}$$

Therefore by (2.17) we get

$$\begin{aligned} (2.18) \quad & \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & \leq (\|Df(x)\| + \|Df(y)\|) \int_0^{1/2} tp(t) dt. \end{aligned}$$

Observe that, by the symmetry of p we get

$$\begin{aligned} \int_0^1 \left| \frac{1}{2} - t \right| p(t) dt &= \int_0^{1/2} \left(\frac{1}{2} - t \right) p(t) dt + \int_{1/2}^1 \left(t - \frac{1}{2} \right) p(t) dt \\ &= \int_0^{1/2} \left(\frac{1}{2} - t \right) p(t) dt + \int_0^{1/2} \left(1 - s - \frac{1}{2} \right) p(1-s) ds \\ &= \int_0^{1/2} \left(\frac{1}{2} - t \right) p(t) dt + \int_0^{1/2} \left(\frac{1}{2} - s \right) p(s) dt \\ &= 2 \int_0^{1/2} \left(\frac{1}{2} - t \right) p(t) dt = \int_0^{1/2} p(t) dt - 2 \int_0^{1/2} tp(t) dt \\ &= \frac{1}{2} \int_0^1 p(t) dt - 2 \int_0^{1/2} tp(t) dt, \end{aligned}$$

therefore

$$2 \int_0^{1/2} tp(t) dt = \frac{1}{2} \int_0^1 p(t) dt - \int_0^1 \left| \frac{1}{2} - t \right| p(t) dt = \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt$$

namely

$$\int_0^{1/2} tp(t) dt = \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt.$$

By (2.18) we derive

$$\begin{aligned} & \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)\| dt \\ & \leq \frac{1}{2} (\|Df(x)\| + \|Df(y)\|) \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt \end{aligned}$$

and by (2.12) we get (2.16). \square

Let $f : C \subset E \rightarrow F$ be a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0, 1] \rightarrow [0, \infty)$ a Lebesgue integrable and symmetric function. If we take the uniform weight $p \equiv 1$ in Theorem 1, then for all $x, y \in C$ we get

$$(2.19) \quad \left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) dt \right\| \\ \leq \int_0^1 \left| t - \frac{1}{2} \right| \|Df((1-t)x + ty)(y-x)\| dt =: B_1(f, x, y).$$

Moreover, we have the upper bounds

$$(2.20) \quad B_1(f, x, y) \leq \frac{1}{2} \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt,$$

$$(2.21) \quad B_1(f, x, y) \leq \frac{1}{4} \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|$$

and

$$(2.22) \quad B_1(f, x, y) \leq \frac{1}{2(r+1)^{1/r}} \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}$$

for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.

If we take in Theorem 2 $p(t) = |t - \frac{1}{2}|$, $t \in [0, 1]$, then

$$(2.23) \quad \left\| \frac{f(x) + f(y)}{8} - \int_0^1 \left| t - \frac{1}{2} \right| f((1-t)x + ty) dt \right\| \\ \leq \frac{1}{2} \int_0^1 \left(t - \frac{1}{2} \right)^2 \|Df((1-t)x + ty)(y-x)\| dt \\ =: B_2(f, x, y).$$

Moreover, we have the upper bounds

$$(2.24) \quad B_2(f, x, y) \leq \frac{1}{8} \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt$$

$$(2.25) \quad B_2(f, p, x, y) \leq \frac{1}{24} \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|$$

and

$$(2.26) \quad B_2(f, p, x, y) \leq \frac{1}{8(2r+1)^{1/r}} \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}$$

for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.

If $Df : C \rightarrow \mathcal{B}(E, F)$ is norm convex on C , then by Theorem 2 for $p \equiv 1$ we get

$$(2.27) \quad \left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) dt \right\| \\ \leq \frac{1}{8} \|y-x\| [\|Df(x)\| + \|Df(y)\|],$$

for all $x, y \in C$.

Also, if $p(t) = |t - \frac{1}{2}|$, $t \in [0, 1]$ and $Df : C \rightarrow \mathcal{B}(E, F)$ is norm convex on C , then by (2.16) we get

$$(2.28) \quad \left\| \frac{f(x) + f(y)}{8} - \int_0^1 \left| t - \frac{1}{2} \right| f((1-t)x + ty) dt \right\| \\ \leq \frac{1}{48} \|y - x\| [\|Df(x)\| + \|Df(y)\|],$$

for all $x, y \in C$

3. SOME EXAMPLES FOR BANACH ALGEBRAS

Let \mathcal{B} be an algebra. An *algebra norm* on \mathcal{B} is a map $\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*.

We assume that the Banach algebra is *unital*, this means that \mathcal{B} has an identity 1 and that $\|1\| = 1$.

Let \mathcal{B} be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of \mathcal{B} is denoted by $\text{Inv}\mathcal{B}$. If $a, b \in \text{Inv}\mathcal{B}$ then $ab \in \text{Inv}\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [12].

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$(3.1) \quad \|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f'_a(\|(1-s)x + sy\|) ds.$$

We also have:

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$(3.2) \quad \|Df((1-t)x + ty)(y - x)\| \leq \|y - x\| f'_a(\|(1-t)x + ty\|)$$

for all $t \in [0, 1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (3.1) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f'_a(\|(1-s)u + s(u + \varepsilon v)\|) ds$$

namely

$$\begin{aligned} \left\| \frac{f(u + \varepsilon v) - f(u)}{\varepsilon} \right\| &\leq \|v\| \int_0^1 f'_a(\|(1-s)u + s(u + \varepsilon v)\|) ds, \quad \varepsilon \neq 0 \\ &= \|v\| \int_0^1 f'_a(\|u + s\varepsilon v\|) ds \end{aligned}$$

and by taking $\varepsilon \rightarrow 0$ we get, by the property of integral, that

$$(3.3) \quad \|Df(u)(v)\| \leq \|v\| f'_a(\|u\|).$$

Now, if we take in (3.3) $u = (1-t)x + ty$ and $v = y - x$, then we get (3.2). \square

We have the following result:

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. Also, let $p : [0, 1] \rightarrow [0, \infty)$ be a Lebesgue integrable and symmetric function on $[0, 1]$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\begin{aligned} (3.4) \quad &\left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ &\leq \|y - x\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) f'_a(\|(1-t)x + ty\|) dt \\ &\quad + \|y - x\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) f'_a(\|(1-t)x + ty\|) dt \\ &= \|y - x\| \int_0^{1/2} \left| \int_t^{1/2} p(s) ds \right| f'_a(\|(1-t)x + ty\|) dt \\ &=: C(f, p, x, y). \end{aligned}$$

Moreover, we have the upper bounds

$$\begin{aligned} (3.5) \quad B(f, p, x, y) &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \int_0^1 f'_a(\|(1-t)x + ty\|) dt \\ &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \sup_{t \in [0, 1]} f'_a(\|(1-t)x + ty\|), \end{aligned}$$

$$\begin{aligned} (3.6) \quad B(f, p, x, y) &\leq \|y - x\| \left(\frac{1}{2} \int_0^1 p(s) ds - \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) p(t) t dt \right) \\ &\quad \times \sup_{t \in [0, 1]} f'_a(\|(1-t)x + ty\|) \\ &\leq \frac{1}{2} \|y - x\| \int_0^1 p(s) ds \sup_{t \in [0, 1]} f'_a(\|(1-t)x + ty\|) \end{aligned}$$

and

$$\begin{aligned}
(3.7) \quad & B(f, p, x, y) \\
& \leq \|y - x\| \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\
& \quad \times \left(\int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt \right)^{1/q} \\
& \leq \|y - x\| \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\
& \quad \times \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|).
\end{aligned}$$

The proof follows by Theorem 1 and Lemma 3.

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
(3.8) \quad & \left\| \left(\int_0^1 p(s) ds \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\
& \leq \frac{1}{2} \|y - x\| [f'_a(\|x\|) + f'_a(\|y\|)] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt.
\end{aligned}$$

Proof. Since f'_a is monotonic nondecreasing and convex, then for $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have by (3.4)

$$\begin{aligned}
(3.9) \quad & \left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\
& \leq \|y - x\| \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) [(1-t)f'_a(\|x\|) + tf'_a(\|y\|)] dt \\
& \quad + \|y - x\| \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) [(1-t)f'_a(\|x\|) + tf'_a(\|y\|)] dt.
\end{aligned}$$

By making use of a similar argument to the one in the proof of Theorem 2, we get from (3.9) the desired result (3.8). The details are omitted. \square

Remark 2. *If we take $p \equiv 1$ in (3.8), then we get*

$$\begin{aligned}
(3.10) \quad & \left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) dt \right\| \\
& \leq \frac{1}{8} \|y - x\| [f'_a(\|x\|) + f'_a(\|y\|)],
\end{aligned}$$

while for $p(t) = |t - \frac{1}{2}|$, $t \in [0, 1]$ we get

$$\begin{aligned}
(3.11) \quad & \left\| \frac{f(x) + f(y)}{8} - \int_0^1 \left| t - \frac{1}{2} \right| f((1-t)x + ty) dt \right\| \\
& \leq \frac{1}{48} \|y - x\| [f'_a(\|x\|) + f'_a(\|y\|)]
\end{aligned}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$.

As some natural examples that are useful for applications, we can point out that, if

$$(3.12) \quad \begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \quad \lambda \in D(0, 1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \quad \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \quad \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1); \end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$(3.13) \quad \begin{aligned} f_a(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1); \\ g_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \quad \lambda \in \mathbb{C}; \\ h_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \quad \lambda \in \mathbb{C}; \\ l_a(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1). \end{aligned}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$(3.14) \quad \begin{aligned} \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \quad \lambda \in \mathbb{C}, \\ \frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\ \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \quad \lambda \in D(0, 1); \\ \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1) \\ {}_2F_1(\alpha, \beta, \gamma, \lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \quad \alpha, \beta, \gamma > 0, \\ &\lambda \in D(0, 1). \end{aligned}$$

From (3.8) we derive the inequalities

$$(3.15) \quad \left\| \left(\int_0^1 p(s) ds \right) \frac{\exp(x) + \exp(y)}{2} - \int_0^1 p(t) \exp((1-t)x + ty) dt \right\| \\ \leq \frac{1}{2} \|y - x\| [\exp(\|x\|) + \exp(\|y\|)] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt,$$

$$(3.16) \quad \left\| \left(\int_0^1 p(s) ds \right) \frac{\sin(x) + \sin(y)}{2} - \int_0^1 p(t) \sin((1-t)x + ty) dt \right\| \\ \leq \frac{1}{2} \|y - x\| [\cosh(\|x\|) + \cosh(\|y\|)] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt.$$

and

$$(3.17) \quad \left\| \left(\int_0^1 p(s) ds \right) \frac{\cos(x) + \cos(y)}{2} - \int_0^1 p(t) \cos((1-t)x + ty) dt \right\| \\ \leq \frac{1}{2} \|y - x\| [\sinh(\|x\|) + \sinh(\|y\|)] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt$$

for all $x, y \in \mathcal{B}$.

If $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$, then we also have

$$(3.18) \quad \left\| \left(\int_0^1 p(s) ds \right) \frac{(1 \pm x)^{-1} + (1 \pm y)^{-1}}{2} \right. \\ \left. - \int_0^1 p(t) (1 \pm [(1-t)x + ty])^{-1} dt \right\| \\ \leq \frac{1}{2} \|y - x\| \left[(1 - \|x\|)^{-2} + (1 - \|y\|)^{-2} \right] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt$$

and

$$(3.19) \quad \left\| \left(\int_0^1 p(s) ds \right) \frac{\ln[(1 \pm x)^{-1}] + \ln[(1 \pm y)^{-1}]}{2} \right. \\ \left. - \int_0^1 p(t) \ln[(1 \pm [(1-t)x + ty])^{-1}] dt \right\| \\ \leq \frac{1}{2} \|y - x\| \left[(1 - \|x\|)^{-1} + (1 - \|y\|)^{-1} \right] \int_0^1 \left(\frac{1}{2} - \left| \frac{1}{2} - t \right| \right) p(t) dt.$$

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