SOME NEW GRÜSS TYPE INTEGRAL INEQUALITIES FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. In this paper we obtain in this paper some upper bounds for the quantity

$$\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\|,$$

where f is a Fréchet differentiable function on the open and convex set C and $x,\ y \in C$, with $x \neq y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function on $[0,1]$. The case of weight p satisfying the condition

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds \le \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau \in (0,1)$$

is also analysed. Some applications for functions defined on Banach algebras are also given.

1. Introduction

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 1. *Let F be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho:\Omega\to[0,\infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\,dx = 1$. If $\alpha:\Omega\to\mathbb{K}$ is a Lebesgue integrable function such that there exists $\gamma,\ \Gamma\in\mathbb{K}$ with*

$$\left|\alpha(x) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}\left|\Gamma-\gamma\right| \tag{1.1}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-\alpha(x))\left(\overline{\alpha(x)}-\overline{\gamma}\right)\right] \ge 0 \tag{1.2}$$

for a.e. $x\in\Omega$, and $g:\Omega\to F$ is a Bochner measurable function such that $\rho\alpha g$ and ρg are Bochner integrable on Ω, then,

$$\begin{aligned} &\left\|\int_\Omega \rho(x)\alpha(x)g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx\int_\Omega\rho(x)g(x)\,dx\right\| \\ &\le \frac{1}{2}\left|\Gamma-\gamma\right|\int_\Omega\rho(x)\left\|g(x)-\int_\Omega\rho(y)g(y)\,dy\right\|dx. \end{aligned} \tag{1.3}$$

The constant $\frac{1}{2}$ in (1.3) is the best possible.

1991 *Mathematics Subject Classification.* 46B20, 26D15, 47A63; 47A99.

Key words and phrases. Banach spaces, Norm inequalities, Grüss integral inequalities, Banach algebras.

RGMIA Res. Rep. Coll. 23 (2020), Art. 23, 17 pp. Received 10/02/20

The following dual result also holds:

Theorem 2. *Let F and Ω, ρ be as above. If $g:\Omega\to X$ is Bochner measurable on Ω and there exist vector $v\in X$ and $r>0$ such that*

$$\|g(x)-v\|\le r \text{ for a.e. } x\in\Omega$$

and $\alpha:\Omega\to\mathbb{K}$ is a Lebesgue integrable function with $\rho\alpha g$, ρg Bochner integrable functions on Ω, then we have the sharp inequalities

$$\left\|\int_\Omega \rho(x)\alpha(x)g(x)\,dx-\int_\Omega\rho(x)\alpha(x)\,dx\int_\Omega\rho(x)g(x)\,dx\right\| \tag{1.4}$$
$$\le r\int_\Omega\rho(x)\left|\alpha(x)-\int_\Omega\rho(y)\alpha(y)\,dy\right|dx$$
$$\le r\left[\int_\Omega\rho(x)|\alpha(x)|^2\,dx-\left|\int_\Omega\rho(x)\alpha(x)\,dx\right|^2\right]^{\frac{1}{2}}.$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega=[0,1]$. Also let $\rho(t)=1$ and $g(t)=f((1-t)x+ty)$ for $t\in[0,1]$ and x, $y\in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f:C\subset E\to F$ is continuous on C and x, $y\in C$, $x\neq y$. If $p:[0,1]\to\mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma\in\mathbb{K}$ with*

$$\left|p(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma| \tag{1.5}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-p(t))\left(\overline{p(t)}-\overline{\gamma}\right)\right]\ge 0 \tag{1.6}$$

for a.e. $t\in[0,1]$, then,

$$\left\|\int_0^1 p(t)f((1-t)x+ty)\,dt-\int_0^1p(t)\,dt\int_0^1f((1-t)x+ty)\,dt\right\| \tag{1.7}$$
$$\le\frac{1}{2}|\Gamma-\gamma|\int_0^1\left\|f((1-t)x+ty)-\int_0^1f((1-s)x+sy)\,ds\right\|dt.$$

The constant $\frac{1}{2}$ in (1.7) is the best possible.

If there exists a vector v and $r>0$ such that

$$\|f((1-t)x+ty)-v\|\le r \text{ for a.e. } t\in[0,1] \tag{1.8}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\left\|\int_0^1 q(t)f((1-t)x+ty)\,dt-\int_0^1q(t)\,dt\int_0^1f((1-t)x+ty)\,dt\right\| \tag{1.9}$$
$$\le r\int_0^1\left|q(t)-\int_0^1q(s)\,ds\right|dt\le r\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}.$$

We observe that, if there exists two vectors $z,\ w\in F$ such that

$$\left\|f\left((1-t)x+ty\right)-\frac{z+w}{2}\right\|\leq\frac{1}{2}\left\|w-z\right\|\text{ for a.e. }t\in[0,1]\tag{1.10}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\begin{aligned}&\left\|\int_0^1 q(t)f\left((1-t)x+ty\right)dt-\int_0^1 q(t)\,dt\int_0^1 f\left((1-t)x+ty\right)dt\right\|\\&\leq\frac{1}{2}\left\|w-z\right\|\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt\\&\leq\frac{1}{2}\left\|w-z\right\|\left[\int_0^1\left|q(t)\right|^2dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}.\end{aligned}\tag{1.11}$$

For some recent results on Grüss' type inequalities, see [1]-[5] and [7]-[16].

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset if a normed vector space, f a real-valued function defined on O, $a\in O$ and u a nonzero element of E. The function f_u given by $t\mapsto f(a+tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0):=\lim_{t\to0}\frac{f(a+tu)-f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative (directional derivative)* of f at a in the direction u. If $\nabla_a f(u)$ is defined and $\lambda\in\mathbb{R}\backslash\{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u)=\lambda\nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u.

Let E and F be normed vector spaces, and O be an open subset of F. A function $f:O\to F$ is called *Fréchet differentiable* at $x\in O$ if there exists a bounded linear operator $A:E\to F$ such that

$$\lim_{\|h\|\to0}\frac{\|f(x+h)-f(x)-Ah\|}{\|h\|}=0.$$

If there exists such an operator A, it is unique, so we write $Df(x)=A$ and call it the *Fréchet derivative* of f at x.

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O\ni x\mapsto Df(x)\in\mathcal{B}(E,F)$ is continuous. A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a linear operation. If f is Fréchet differentiable at x, it is also Gâteaux differentiable there, and $\nabla_x f(u)=Df(x)(u)$ for all $u\in E$.

Motivated by the above results, we obtain in this paper some upper bounds for the quantity

$$\left\|\int_0^1 p(t)f\left((1-t)x+ty\right)dt-\int_0^1 p(t)\,dt\int_0^1 f\left((1-t)x+ty\right)dt\right\|,$$

where f is a Fréchet differentiable function on the open and convex set C and x, $y\in C$, with $x\neq y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function on $[0,1]$. A particular case of interest is when the weight p satisfies the condition

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds \le \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1). \tag{1.12}$$

Some applications for functions defined on Banach algebras are also given.

2. Main Results

We need the following preliminary result:

Lemma 1. *Let f be a Fréchet differentiable function on the open and convex set C and $x, y \in C$, with $x \neq y$. Then the auxiliary function $\varphi_{(x,y)}(t) := f((1-t)x + ty)$, $t \in [0,1]$, is differentiable on $(0,1)$ and*

$$\varphi'_{(x,y)}(t) = Df((1-t)x+ty)(y-x). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(x,y)}(0+) = Df(x)(y-x) \tag{2.2}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y-x). \tag{2.3}$$

Proof. For the sake of completeness, we give here a short proof. Let $t\in(0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h} \\ &= \frac{f((1-t-h)x+(t+h)y)-f((1-t)x+ty)}{h} \\ &= \frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h}. \end{aligned} \tag{2.4}$$

Since f is Fréchet differentiable hence by taking the limit over $h\to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) &= \lim_{h\to 0}\frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h} \\ &= \lim_{h\to 0}\frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h} \\ &= Df((1-t)x+ty)(y-x), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) &= \lim_{h\to 0+}\frac{\varphi_{(x,y)}(h)-\varphi_{(x,y)}(0)}{h} \\ &= \lim_{h\to 0+}\frac{f((1-h)x+hy)-f(x)}{h} \\ &= \lim_{h\to 0+}\frac{f(x+h(y-x))-f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x. This proves (2.2).

The equality (2.3) follows in a similar way. □

Our main result is incorporated in the following Grüss' type inequality:

Theorem 3. *Let f be a Fréchet differentiable function on the open and convex set C and x, $y \in C$, with $x \neq y$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function, then we have the inequality*

$$\left\| \int_0^1 p(\tau) f((1-\tau)x + \tau y)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)x + \tau y)\, d\tau \right\| \tag{2.5}$$
$$\leq \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \|Df((1-\tau)x + \tau y)(y-x)\|\, d\tau$$
$$\leq \frac{1}{4} \int_0^1 \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \|Df((1-\tau)x + \tau y)(y-x)\|\, d\tau.$$

Proof. We also need the following identity that is of interest in itself:

Let f be an operator convex function on I and x, $y \in C$, with $x \neq y$. If f is a Fréchet differentiable function on the open and convex set C and x, $y \in C$ and $p : [0,1] \to \mathbb{C}$ is a Lebesgue integrable function, then we have the equality

$$\int_0^1 p(\tau) f((1-\tau)x + \tau y)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)x + \tau y)\, d\tau \tag{2.6}$$
$$= \int_0^1 \tau(1-\tau) \left(\frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right) Df((1-\tau)x + \tau y)(y-x)\, d\tau.$$

Integrating by parts in the Bochner's integral, we have

$$\int_0^\tau t\varphi'_{(x,y)}(t)\, dt + \int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\, dt$$
$$= \tau\varphi_{(x,y)}(\tau) - \int_0^\tau \varphi_{(x,y)}(t)\, dt - (\tau - 1)\varphi_{(x,y)}(\tau) - \int_\tau^1 \varphi_{(x,y)}(t)\, dt$$
$$= \varphi_{(x,y)}(\tau) - \int_0^1 \varphi_{(x,y)}(t)\, dt$$

that holds for all $\tau \in [0,1]$ .If we multiply this identity by $p(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 \varphi_{(x,y)}(t)\, dt \tag{2.7}$$
$$= \int_0^1 p(\tau) \left(\int_0^\tau t\varphi'_{(x,y)}(t)\, dt \right) d\tau + \int_0^1 p(\tau) \left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\, dt \right) d\tau.$$

Using integration by parts, we get

$$\int_0^1 p(\tau) \left(\int_0^\tau t\varphi'_{(x,y)}(t)\, dt \right) d\tau \tag{2.8}$$
$$= \int_0^1 \left(\int_0^\tau t\varphi'_{(x,y)}(t)\, dt \right) d\left(\int_0^\tau p(s)\, ds \right)$$
$$= \left(\int_0^\tau p(s)\, ds \right) \left(\int_0^\tau t\varphi'_{(x,y)}(t)\, dt \right) \Big|_0^1$$
$$- \int_0^1 \left(\int_0^\tau p(s)\, ds \right) \tau\varphi'_{(x,y)}(\tau)\, d\tau$$

$$
\begin{aligned}
&= \left(\int_0^1 p(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 p(s)\,ds - \int_0^\tau p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau p(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau p(s)\,ds\right)\Bigg|_0^1 \\
&+ \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau,
\end{aligned} \tag{2.9}
$$

which proves the identity

$$
\begin{aligned}
&\int_0^1 p(\tau)\,\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&+ \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau.
\end{aligned} \tag{2.10}
$$

Now, observe that

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau + \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \tau\left(\int_\tau^1 p(s)\,ds\right)\varphi'_{(x,y)}(\tau)\,d\tau - \int_0^1 (1-\tau)\left(\int_0^\tau p(s)\,ds\right)\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\varphi'_{(x,y)}(\tau)\,d\tau
\end{aligned}
$$

and by (2.10) we obtain the desired equality (2.6).

If we take the norm in the equality (2.6), then we have

$$\begin{aligned}
&\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \\
&\leq \int_0^1 \left\| \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right) Df((1-\tau)x+\tau y)(y-x) \right\| d\tau \\
&= \int_0^1 \tau(1-\tau)\left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \left\| Df((1-\tau)x+\tau y)(y-x) \right\| d\tau
\end{aligned}$$

and since $\tau(1-\tau) \leq \frac{1}{4}$ for $\tau \in [0,1]$, hence (2.5) is proved. □

Corollary 2. *With the assumptions of Theorem 3 and if*

$$\sup_{\tau\in[0,1]} \left\| Df((1-\tau)x+\tau y)(y-x) \right\| < \infty,$$

then

$$\begin{aligned}
(2.11)\quad &\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \\
&\leq \sup_{\tau\in[0,1]} \left\| Df((1-\tau)x+\tau y)(y-x) \right\| \\
&\times \int_0^1 \tau(1-\tau)\left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| d\tau \\
&\leq \sup_{\tau\in[0,1]} \left\| Df((1-\tau)x+\tau y)(y-x) \right\| \\
&\times \begin{cases}
\frac{1}{4}\int_0^1 \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right| d\tau \\[2ex]
[\beta(r+1,r+1)]^{1/r} \left(\int_0^1 \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right|^q d\tau \right)^{1/q} \\
\text{where } r,\ q>1,\ \frac{1}{r}+\frac{1}{q}=1; \\[2ex]
\frac{1}{6}\sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right|,
\end{cases}
\end{aligned}$$

where β *is the Beta function*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ x,\ y>0.$$

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
(2.12)\quad &\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \\
&\leq \sup_{\tau\in[0,1]} \left| \tau\int_0^1 p(s)\,ds - \int_0^\tau p(s)\,ds \right| \int_0^1 \left\| Df((1-\tau)x+\tau y)(y-x) \right\| d\tau \\
&\leq \frac{1}{4}\sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \int_0^1 \left\| Df((1-\tau)x+\tau y)(y-x) \right\| d\tau.
\end{aligned}$$

Proof. We have

$$\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau$$

$$\leq \sup_{\tau\in[0,1]}\left[\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\right]\int_0^1 \|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau$$

$$= \sup_{\tau\in[0,1]}\left[\left|\tau\int_\tau^1 p(s)\,ds-(1-\tau)\int_0^\tau p(s)\,ds\right|\right]\int_0^1 \|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau$$

$$= \sup_{\tau\in[0,1]}\left|\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right|\int_0^1 \|Df((1-\tau)x+\tau y)(y-x)\|\,d\tau.$$

Also

$$\sup_{\tau\in[0,1]}\left[\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\right]\leq \frac{1}{4}\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|$$

and the inequalities in (2.12) are proved. □

When more restrictions are imposed on the weight function p one can obtain a simpler bound as follows:

Corollary 4. *With the assumptions of Corollary 2 and if $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that*

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\leq \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1), \tag{2.13}$$

then we have

$$\left\|\int_0^1 p(\tau)f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.14}$$
$$\leq \sup_{\tau\in[0,1]}\|Df((1-\tau)x+\tau y)(y-x)\|\int_0^1\left(\tau-\frac{1}{2}\right)p(\tau)\,d\tau.$$

Proof. From (2.11) we have

$$\left\|\int_0^1 p(\tau)f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{2.15}$$
$$\leq \sup_{\tau\in[0,1]}\|Df((1-\tau)x+\tau y)(y-x)\|$$
$$\times\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|d\tau$$
$$= \sup_{\tau\in[0,1]}\|Df((1-\tau)x+\tau y)(y-x)\|$$
$$\times\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right)d\tau.$$

Since, we have the equality

$$
\begin{aligned}
0 &\leq \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right)d\tau \\
&= \int_0^1 \left(\tau\int_\tau^1 p(s)\,ds-(1-\tau)\int_0^\tau p(s)\,ds\right)d\tau \\
&= \int_0^1 \left(\tau\int_\tau^1 p(s)\,ds+\tau\int_0^\tau p(s)\,ds-\int_0^\tau p(s)\,ds\right)d\tau \\
&= \int_0^1 \left(\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right)d\tau \\
&= \frac{1}{2}\int_0^1 p(s)\,ds-\int_0^1\left(\int_0^\tau p(s)\,ds\right)d\tau \\
&= \frac{1}{2}\int_0^1 p(s)\,ds-\left[\left(\int_0^\tau p(s)\,ds\right)\tau\Big|_0^1-\int_0^1 p(\tau)\,\tau d\tau\right] \\
&= \frac{1}{2}\int_0^1 p(s)\,ds-\int_0^1 p(s)\,ds+\int_0^1 p(\tau)\,\tau d\tau \\
&= \int_0^1 p(\tau)\,\tau d\tau-\frac{1}{2}\int_0^1 p(s)\,ds=\int_0^1 p(\tau)\left(\tau-\frac{1}{2}\right)d\tau,
\end{aligned}
$$

hence by (2.15) we get (2.14). □

Remark 1. *If p is continuous and monotonic nondecreasing, then by the midpoint theorem for the Riemann integral, we have for $\tau\in(0,1)$ that*

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds=p(u)\leq p(v)=\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds$$

for some $0<u<\tau<v<1$, which shows that the condition (2.13) is valid.

Remark 2. *Let f be a Fréchet differentiable function on the open and convex set C and $x,\ y\in C$, with $x\neq y$ and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function. Since the Fréchet derivative satisfies the condition*

$$\|Df(a)(b)\|\leq\|Df(a)\|\,\|b\|$$

for $a\in C$ and $b\in E$, then we also have the chain of inequalities

$$
\begin{aligned}
&\left\|\int_0^1 p(\tau)f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \\
&\leq\|y-x\|\int_0^1\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right| \\
&\times\|Df((1-\tau)x+\tau y)\|\,d\tau \\
&\leq\frac{1}{4}\|y-x\|\int_0^1\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right| \\
&\times\|Df((1-\tau)x+\tau y)\|\,d\tau.
\end{aligned}
\tag{2.16}
$$

If

$$\sup_{\tau\in[0,1]}\|Df((1-\tau)x+\tau y)\|<\infty,$$

then

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \tag{2.17}$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)\|$$
$$\times \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| d\tau$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)\|$$
$$\times \begin{cases} \frac{1}{4}\int_0^1 \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right| d\tau \\ \\ [\beta(r+1,r+1)]^{1/r} \left(\int_0^1 \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right|^q d\tau \right)^{1/q} \\ \text{where } r,\ q>1,\ \frac{1}{r}+\frac{1}{q}=1; \\ \\ \frac{1}{6}\sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)ds}{1-\tau} - \frac{\int_0^\tau p(s)ds}{\tau} \right|, \end{cases}$$

where β is the Beta function.

We also have

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \tag{2.18}$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \left| \tau\int_0^1 p(s)\,ds - \int_0^\tau p(s)\,ds \right|$$
$$\times \int_0^1 \|Df((1-\tau)x+\tau y)\|\,d\tau$$
$$\leq \frac{1}{4}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right|$$
$$\times \int_0^1 \|Df((1-\tau)x+\tau y)\|\,d\tau.$$

Moreover, if p satisfies the condition (2.13), then

$$\left\| \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right\| \tag{2.19}$$
$$\leq \|y-x\| \sup_{\tau\in[0,1]} \|Df((1-\tau)x+\tau y)\| \int_0^1 \left(\tau-\frac{1}{2}\right) p(\tau)\,d\tau$$

for $x,\ y\in C$, with $x\neq y$.

3. Some Examples for Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\|\,\|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}\mathcal{B}$. If $a, b \in \text{Inv}\mathcal{B}$ then $ab \in \text{Inv}\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [8].

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-s)x + sy\|)\, ds. \tag{3.1}$$

We also have:

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|Df((1-\tau)x + \tau y)(y - x)\| \leq \|y - x\| f_a'(\|(1-\tau)x + \tau y\|) \tag{3.2}$$

for all $\tau \in [0,1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (3.1) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds$$

namely

$$\begin{aligned} \left\| \frac{f(u + \varepsilon v) - f(u)}{\varepsilon} \right\| &\leq \|v\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds, \ \varepsilon \neq 0 \\ &= \|v\| \int_0^1 f_a'(\|u + s\varepsilon v\|)\, ds \end{aligned}$$

and by taking $\varepsilon \to 0$ we get, by the property of integral, that

$$\|Df(u)(v)\| \leq \|v\| f_a'(\|u\|). \tag{3.3}$$

Now, if we take in (3.3) $u = (1-\tau)x + \tau y$ and $v = y - x$, then we get (3.2). □

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If*

$p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function, then

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.4}$$
$$\le \|y-x\|\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|$$
$$\times f_a'(\|(1-\tau)x+\tau y\|)\,d\tau$$
$$\le \frac{1}{4}\|y-x\|\int_0^1\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right| f_a'(\|(1-\tau)x+\tau y\|)\,d\tau.$$

for any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$.

The proof follows by Theorem 3 and Lemma 3.
From Corollary 3 we also have:

Corollary 5. *Withe the assumptions of Theorem 4 we have*

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.5}$$
$$\le \frac{1}{4}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\int_0^1 f_a'(\|(1-\tau)x+\tau y\|)\,d\tau.$$

Moreover, we have the upper bounds

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.6}$$
$$\le \frac{1}{4}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|$$
$$\times\begin{cases}\frac{f_a(\|x\|)-f_a(\|y\|)}{\|x\|-\|y\|} & \text{if } \|x\|\neq\|y\|,\\[2ex] f_a'(\|x\|) & \text{if } \|x\|\neq\|y\|\end{cases}$$

and

$$\left\|\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right\| \tag{3.7}$$
$$\le \frac{1}{8}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|$$
$$\times\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right]$$
$$\le \frac{1}{4}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}$$

for any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$.

Proof. Since f_a' is increasing, hence

$$\begin{aligned}\int_0^1 f_a'\left(\left\|(1-\tau)x+\tau y\right\|\right)d\tau &\leq \int_0^1 f_a'\left((1-\tau)\left\|x\right\|+\tau\left\|y\right\|\right)d\tau \\ &= \begin{cases} \frac{f_a(\|x\|)-f_a(\|y\|)}{\|x\|-\|y\|} \text{ if } \|x\|\neq\|y\|, \\ \\ f_a'\left(\|x\|\right) \text{ if } \|x\|\neq\|y\|. \end{cases}\end{aligned}$$

Also, because f_a' is convex, then by Hermite-Hadamard inequality, we derive that

$$\begin{aligned}&\int_0^1 f_a'\left(\left\|(1-\tau)x+\tau y\right\|\right)d\tau \\ &\leq \frac{1}{2}\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'\left(\|x\|\right)+f_a'\left(\|y\|\right)}{2}\right] \\ &\leq \frac{f_a'\left(\|x\|\right)+f_a'\left(\|y\|\right)}{2},\end{aligned}$$

which proves (3.7). □

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n = \ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n} = \cos\lambda,\ \lambda\in\mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1} = \sin\lambda,\ \lambda\in\mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty}(-1)^n\lambda^n = \frac{1}{1+\lambda},\ \lambda\in D(0,1); \end{aligned} \tag{3.8}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_a(\lambda) &= \sum_{n=1}^{\infty}\frac{1}{n}\lambda^n = \ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \\ g_a(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n} = \cosh\lambda,\ \lambda\in\mathbb{C}; \\ h_a(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh\lambda,\ \lambda\in\mathbb{C}; \\ l_a(\lambda) &= \sum_{n=0}^{\infty}\lambda^n = \frac{1}{1-\lambda},\ \lambda\in D(0,1). \end{aligned} \tag{3.9}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.10}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1).$$

Further we assume that $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function on $[0,1]$.

If we write the inequalities (3.6) and (3.7) for the exponential function $f(x) = \exp x$, then we get

$$\begin{aligned} &\left\|\int_0^1 p(\tau)\exp((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau\int_0^1 \exp((1-\tau)x+\tau y)\,d\tau\right\| \\ &\le \frac{1}{4}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau}\right| \\ &\qquad \times \begin{cases} \frac{\exp(\|x\|)-\exp(\|y\|)}{\|x\|-\|y\|} & \text{if } \|x\|\neq\|y\|, \\ \exp(\|x\|) & \text{if } \|x\|\neq\|y\| \end{cases} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\left\|\int_0^1 p(\tau)\exp((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau\int_0^1 \exp((1-\tau)x+\tau y)\,d\tau\right\| \\ &\le \frac{1}{8}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau}\right| \\ &\quad\times\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right) + \frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right] \\ &\le \frac{1}{4}\|y-x\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau}\right| \\ &\qquad \times \frac{\exp(\|x\|)+\exp(\|y\|)}{2} \end{aligned} \tag{3.12}$$

for all $x,\, y \in \mathcal{B}$.

If we write the inequalities (3.6) and (3.7) for the trigonometric function $f(x) = \sin x$, then we get

$$\begin{aligned} &\left\| \int_0^1 p(\tau) \sin((1-\tau)x + \tau y)\, d\tau \right. \\ &\qquad \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \sin((1-\tau)x + \tau y)\, d\tau \right\| \\ &\qquad \le \frac{1}{4} \|y - x\| \sup_{\tau \in [0,1]} \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\qquad\qquad \times \begin{cases} \frac{\sinh(\|x\|) - \sinh(\|y\|)}{\|x\| - \|y\|} \text{ if } \|x\| \neq \|y\|, \\ \\ \cosh(\|x\|) \text{ if } \|x\| \neq \|y\| \end{cases} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\left\| \int_0^1 p(\tau) \sin((1-\tau)x + \tau y)\, d\tau \right. \\ &\qquad \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \sin((1-\tau)x + \tau y)\, d\tau \right\| \\ &\qquad \le \frac{1}{8} \|y - x\| \sup_{\tau \in [0,1]} \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\qquad \times \left[\cosh\left(\left\| \frac{x+y}{2} \right\| \right) + \frac{\cosh(\|x\|) + \cosh(\|y\|)}{2} \right] \\ &\qquad \le \frac{1}{4} \|y - x\| \sup_{\tau \in [0,1]} \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\qquad\qquad \times \frac{\cosh(\|x\|) + \cosh(\|y\|)}{2} \end{aligned} \tag{3.14}$$

for all $x,\, y \in \mathcal{B}$.

Also, if we write the inequalities (3.6) and (3.7) for the functions $f(x) = (1 \pm x)^{-1}$, then we get

$$\begin{aligned} &\left\| \int_0^1 p(\tau) \left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right. \\ &\qquad \left. - \int_0^1 p(\tau)\, d\tau \int_0^1 \left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \\ &\qquad \le \frac{1}{4} \|y - x\| \sup_{\tau \in [0,1]} \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\qquad\qquad \times \begin{cases} \frac{1}{(1-\|x\|)(1-\|y\|)} \text{ if } \|x\| \neq \|y\|, \\ \\ \frac{1}{(1-\|x\|)^2} \text{ if } \|x\| \neq \|y\| \end{cases} \end{aligned} \tag{3.15}$$

and

$$\left\| \int_0^1 p(\tau)\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 \left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \tag{3.16}$$
$$\leq \frac{1}{8}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \times \left[\left(1 - \left\| \frac{x+y}{2} \right\|\right)^{-2} + \frac{(1-\|x\|)^{-2} + (1-\|y\|)^{-2}}{2} \right]$$
$$\leq \frac{1}{4}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \times \frac{(1-\|x\|)^{-2} + (1-\|y\|)^{-2}}{2}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$.

Finally, by using the logarithmic functions $f(x) = \ln(1 \pm x)^{-1}$ and the inequalities (3.6) and (3.7), we obtain

$$\left\| \int_0^1 p(\tau)\ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \tag{3.17}$$
$$\leq \frac{1}{4}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \times \begin{cases} \frac{\ln(1-\|x\|)^{-1} - \ln(1-\|y\|)^{-1}}{\|x\|-\|y\|} & \text{if } \|x\| \neq \|y\|, \\ \frac{1}{(1-\|x\|)} & \text{if } \|x\| \neq \|y\| \end{cases}$$

and

$$\left\| \int_0^1 p(\tau)\ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 \ln\left[1 \pm ((1-\tau)x + \tau y)\right]^{-1} d\tau \right\| \tag{3.18}$$
$$\leq \frac{1}{8}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \times \left[\left(1 - \left\| \frac{x+y}{2} \right\|\right)^{-1} + \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2} \right]$$

$$\leq \frac{1}{4}\|y-x\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \times \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2}$$

for any x, $y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$.

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