INTEGRAL INEQUALITIES OF GRÜSS TYPE FOR OPERATOR CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we provide lower and upper bounds, or so called *Grüss type inequalities*, in the order of the Hermitian Banach $*$-algebra A for the *Čebyšev's difference*

$$\int_0^1 p(\tau) f((1-\tau)a+\tau b)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)a+\tau b)\,d\tau$$

in the case that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$, $I \subset G$ is a real interval and the function $f(z)$ is *operator convex* on I while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1],$$

and a, $b\in A$ with $\sigma(a)$, $\sigma(b)\subset I$.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \ge 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that a is *positive* and write $a > 0$ if $a \ge 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \ge b$ means that $a - b \ge 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [11] (see also [1, Theorem 41.5]), then

$$a^*a \ge 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [10], Tanahashi and Uchiyama [12] proved the following fundamental properties (see also [8]):

(i) If a, $b\in A$, then $a\ge 0$, $b\ge 0$ imply $a+b\ge 0$ and $\alpha\ge 0$ implies $\alpha a\ge 0$;
(ii) If a, $b\in A$, then $a>0$, $b\ge 0$ imply $a+b>0$;
(iii) If a, $b\in A$, then either $a\ge b>0$ or $a>b\ge 0$ imply $a>0$;
(iv) If $a>0$, then $a^{-1}>0$;

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(v) If $c>0$, then $0<b<a$ if and only if $cbc<cac$, also $0<b\leq a$ if and only if $cbc\leq cac$;
(vi) If $0<a<1$, then $1<a^{-1}$;
(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\leq a$, then $0<a^{-1}\leq b^{-1}$.

Okayasu [10] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b\in A$ and $p\in[0,1]$ then $a>b$ $(a\geq b)$ implies that $a^p>b^p$ $(a^p\geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re}z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz,$$

where γ is a close rectifiable curve such that $\sigma(a)\subset\operatorname{ins}(\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^\alpha:=\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a)^{-1}dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha\in A$. Moreover, since z^α is analytic in $\{\operatorname{Re}z>0\}$, then by (SMT) we have

$$\sigma(a^\alpha)=(\sigma(a))^\alpha=\{z^\alpha:z\in\sigma(a)\}\subset(0,\infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^\alpha\in A$ with $a^\alpha>0$ and $\left(a^2\right)^{1/2}=a$, [12, Lemma 6];
(ix) If $0<a\in A$ and α, $\beta\in\mathbb{R}$, then $a^\alpha a^\beta=a^{\alpha+\beta}$;
(x) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $(a^\alpha)^{-1}=\left(a^{-1}\right)^\alpha=a^{-\alpha}$;
(xi) If $0<a$, $b\in A$, α, $\beta\in\mathbb{R}$ and $ab=ba$, then $a^\alpha b^\beta=b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I\subset G$ assume that $f(z)\geq0$ for any $z\in I$. If $u\in A$ such that $\sigma(u)\subset I$, then by (SMT) we have

$$\sigma(f(u))=f(\sigma(u))\subset f(I)\subset[0,\infty)$$

meaning that $f(u)\geq0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I\subset G$, assume that $f(z)\geq g(z)$ for any $z\in I$. Then for any $u\in A$ with $\sigma(u)\subset I$ we have $f(u)\geq g(u)$ in the order of A.*

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let G be an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$, then by SMT the element $(1-t)a + tb \in A$ has the spectrum $\sigma((1-t)a+tb) \subset I$ for all $t \in [0,1]$. We say that an analytic function $f(z)$ in G is *operator convex* on I in the Hermitian Banach $*$-algebra A if

$$f((1-t)a+tb) \leq (1-t)f(a) + tf(b) \text{ in the order of } A \tag{1.1}$$

for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ and all $t \in [0,1]$.

In the recent paper [6] we obtained the following results:

Theorem 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have*

$$f(b) - f(a) \geq Df(a)(b-a) \tag{1.2}$$

in the order of A, where Df is the Fréchet derivative of f as a function of elements in the Hermitian Banach $*$-algebra A.

Let $f(z)$ be analytic in G, an open convex subset of $\mathbb{C}$ and $a, b \in A$ with $\sigma(a)$, $\sigma(b) \subset G$. Consider the auxiliary function $F_{(a,b)} : [0,1] \to A$ defined by

$$F_{(a,b)}(t) := f((1-t)a+tb).$$

The following characterization results also holds [6].

Theorem 2. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have for all t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ that*

$$\begin{aligned} F'_{(a,b)}(t_2) &= Df((1-t_2)a+t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a+t_2 b) - f((1-t_1)a+t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a+t_1 b)(b-a) = F'_{(a,b)}(t_1). \end{aligned} \tag{1.3}$$

We also have

$$Df(b)(b-a) \geq F'_{(a,b)}(t) = Df((1-t)a+tb)(b-a) \geq Df(a)(b-a) \tag{1.4}$$

for all $t \in (0,1)$.

It is well known that, if E is a Banach space and $g : [0,1] \to E$ is a continuous function, then g is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_0^1 g(t)\,dt$.

In the recent paper [7] we also obtained the following Féjer's type inequalities:

Theorem 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have*

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f((1-t)a+tb)\,dt - \left(\int_0^1 p(t)\,dt\right) f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right)[Df(b)(b-a) - Df(a)(b-a)]. \end{aligned} \tag{1.5}$$

In particular, for $p \equiv 1$ *we get*

$$0 \leq \int_0^1 f((1-t)a+tb)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.6}$$

We also have:

Theorem 4. *Assume that* $f(z)$ *is analytic in* G*, an open subset of* $\mathbb{C}$ *and* $I \subset G$ *a real interval. If the function* $f(z)$ *is operator convex on* I *in the Hermitian Banach* $*$*-algebra* A *and* $p : [0,1] \to [0,\infty)$ *is Lebesgue integrable and symmetric, then for all* $a, b \in A$ *with* $\sigma(a), \sigma(b) \subset I$ *we have*

$$0 \leq \left(\int_0^1 p(t)\,dt\right)\frac{f(a)+f(b)}{2} - \int_0^1 p(t) f((1-t)a+tb)\,dt \leq \frac{1}{2}\int_0^1 \left(\frac{1}{2} - \left|t-\frac{1}{2}\right|\right) p(t)\,dt \left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.7}$$

In particular, for $p \equiv 1$ *we get*

$$0 \leq \frac{f(a)+f(b)}{2} - \int_0^1 f((1-t)a+tb)\,dt \leq \frac{1}{8}\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.8}$$

Motivated by the above results, in this paper we provide lower and upper bounds, or so called *Grüss type inequalities*, in the order of the Hermitian Banach $*$-algebra A for the *Čebyšev's difference*

$$\int_0^1 p(\tau) f((1-\tau)a+\tau b)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)a+\tau b)\,d\tau$$

in the case that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$, $I \subset G$ a real interval and the function $f(z)$ is *operator convex* on I while $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1],$$

and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 2. *Assume that* $f(z)$ *is analytic in* G*, an open subset of* $\mathbb{C}$ *and* $I \subset G$ *a real interval. If* $p : [0,1] \to \mathbb{C}$ *is Lebesgue integrable, then for all* $a, b \in A$ *with* $\sigma(a), \sigma(b) \subset I$ *we have the following identity for the auxiliary function* $F_{(a,b)}$

$$\int_0^1 p(\tau) F_{(a,b)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 F_{(a,b)}(\tau)\,d\tau = \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau + \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(\tau-1) F'_{(a,b)}(\tau)\,d\tau. \tag{2.1}$$

Proof. The function $F_{(a,b)}$ is obviously differentiable on $(0,1)$. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}
&\int_0^\tau tF'_{(a,b)}(t)\,dt+\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt \\
&=\tau F_{(a,b)}(\tau)-\int_0^\tau F_{(a,b)}(t)\,dt-(\tau-1)\,F_{(a,b)}(\tau)-\int_\tau^1 F_{(a,b)}(t)\,dt \\
&=F_{(a,b)}(\tau)-\int_0^1 F_{(a,b)}(t)\,dt
\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $p(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}
&\int_0^1 p(\tau)\,F_{(a,b)}(\tau)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 F_{(a,b)}(t)\,dt \\
&=\int_0^1 p(\tau)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\tau+\int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\tau.
\end{aligned}\tag{2.2}$$

Using integration by parts, we derive

$$\begin{aligned}
&\int_0^1 p(\tau)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\tau \\
&=\int_0^1\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\left(\int_0^\tau p(s)\,ds\right) \\
&=\left(\int_0^\tau p(s)\,ds\right)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)\Bigg|_0^1-\int_0^1\left(\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau \\
&=\left(\int_0^1 p(s)\,ds\right)\left(\int_0^1 tF'_{(a,b)}(t)\,dt\right)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau \\
&=\int_0^1\left(\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau \\
&=\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\tau \\
&=\int_0^1\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\left(\int_0^\tau p(s)\,ds\right) \\
&=\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)\left(\int_0^\tau p(s)\,ds\right)\Bigg|_0^1 \\
&+\int_0^1\left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,F'_{(a,b)}(\tau)\,d\tau \\
&=\int_0^1\left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,F'_{(a,b)}(\tau)\,d\tau,
\end{aligned}$$

which proves the identity in (2.1). □

Theorem 5. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p:[0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{2.4}$$

then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have the following Grüss type inequalities

$$\begin{aligned} &\left[\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau\right] Df(a)(b-a) \\ &- \left[\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\right] Df(b)(b-a) \\ &\leq \int_0^1 p(\tau) f((1-\tau)a+\tau b)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)a+\tau b)\,d\tau \\ &\leq \left[\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau\right] Df(b)(b-a) \\ &- \left[\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\right] Df(a)(b-a) \end{aligned} \tag{2.5}$$

or, equivalently,

$$\begin{aligned} &\int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\,Df(b)(b-a) - p(s)\,Df(a)(b-a)]\,ds\right) d\tau \\ &\leq \int_0^1 p(\tau) f((1-\tau)a+\tau b)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)a+\tau b)\,d\tau \\ &\leq \int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\,Df(a)(b-a) - p(s)\,Df(b)(b-a)]\,ds\right) d\tau. \end{aligned} \tag{2.6}$$

Proof. We have for $F_{(a,b)}$ and $p:[0,1] \to \mathbb{R}$ a Lebesgue integrable function that

$$\begin{aligned} &\int_0^1 p(\tau) F_{(a,b)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 F_{(a,b)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)(\tau) F'_{(a,b)}(\tau)\,d\tau - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau) F'_{(a,b)}(\tau)\,d\tau. \end{aligned} \tag{2.7}$$

By the properties of $F_{(a,b)}$ from the above section, we have in the operator order that

$$\tau F'_{(a,b)}(1-) \geq \tau F'_{(a,b)}(\tau) \geq \tau F'_{(a,b)}(0+) \tag{2.8}$$

and

$$(1-\tau) F'_{(a,b)}(1-) \geq (1-\tau) F'_{(a,b)}(\tau) \geq (1-\tau) F'_{(a,b)}(0+) \tag{2.9}$$

for all $\tau \in (0,1)$.

From

$$\int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds = \int_0^\tau p(s)\,ds + \int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds \geq 0$ for all $\tau \in (0,1)$.

From (2.8) we obtain that

$$\left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(1-) \geq \left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(\tau) \geq \left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(0+)$$

and from (2.9) that

$$-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)F'_{(a,b)}(0+) \leq -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)F'_{(a,b)}(\tau) \leq -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)F'_{(a,b)}(1-)$$

all $\tau \in (0,1)$.

If we integrate these inequalities over $\tau \in [0,1]$ and add the obtained results, then we get

$$\begin{aligned}
&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau F'_{(a,b)}(1-) - \int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau F'_{(a,b)+}(0)\\
&\geq \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau - \int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)F'_{(a,b)}(\tau)\,d\tau\\
&\geq \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau F'_{(a,b)}(0+) - \int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau F'_{(a,b)}(1-).
\end{aligned}$$

By using the equality (2.1) we derive

$$\begin{aligned}
&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau F'_{(a,b)}(0+)\\
&- \int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau F'_{(a,b)}(1-)\\
&\leq \int_0^1 p(\tau)F_{(a,b)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau\int_0^1 F_{(a,b)}(\tau)\,d\tau\\
&\leq \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau F'_{(a,b)}(1-)\\
&- \int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau F'_{(a,b)}(0+),
\end{aligned} \tag{2.10}$$

and since $F'_{(a,b)}(1-) = Df(b)(b-a)$ and $F'_{(a,b)}(0+) = Df(b)(b-a)$ hence we obtain (2.5).

If we change the variable $\alpha = 1-\tau$, then we have

$$\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau = \int_0^1\left(\int_{1-\alpha}^1 p(s)\,ds\right)(1-\alpha)\,d\alpha.$$

Also by the change of variable $u = 1-s$, we get

$$\int_{1-\alpha}^1 p(s)\,ds = \int_0^\alpha p(1-u)\,du,$$

which implies that

$$\int_0^1 \left(\int_\tau^1 p(s)\, ds \right) \tau d\tau = \int_0^1 \left(\int_0^\tau p(1-s)\, ds \right) (1-\tau)\, d\tau.$$

Therefore

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\, ds \right) \tau d\tau F'_{(a,b)}(1-) - \int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(0+) \\
&= \int_0^1 \left(\int_0^\tau p(1-s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(1-) \\
&- \int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(0+) \\
&= \int_0^1 (1-\tau) \left(\int_0^\tau \left[p(1-s) F'_{(a,b)}(1-) - p(s) F'_{(a,b)+}(0+) \right] ds \right) d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\, ds \right) \tau d\tau F'_{(a,b)}(0+) - \int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(1-) \\
&= \int_0^1 \left(\int_0^\tau p(1-s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(0+) \\
&- \int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau F'_{(a,b)}(1-) \\
&= \int_0^1 (1-\tau) \left(\int_0^\tau \left[p(1-s) F'_{(a,b)}(0+) - p(s) F'_{(a,b)}(1-) \right] ds \right) d\tau,
\end{aligned}$$

and by (2.10) we get (2.6). □

We say that the function $p : [0,1] \to \mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t) = p(t) \text{ for all } t \in [0,1].$$

Corollary 1. *With the assumptions of Theorem 5 and, in addition, if* $p : [0,1] \to \mathbb{R}$ *is a symmetric function on* $[0,1]$*, then we have*

$$\begin{aligned}
&-\frac{1}{2} \left(\int_0^1 p(\tau)\, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)] \\
&\le -\left(\int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)] \\
&\le \int_0^1 p(\tau) f((1-\tau)a + \tau b)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)a + \tau b)\, d\tau \\
&\le \left(\int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)] \\
&\le \frac{1}{2} \left(\int_0^1 p(\tau)\, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)].
\end{aligned} \tag{2.11}$$

Proof. Since p is symmetric, then $p(1-s) = p(s)$ for all $s \in [0,1]$ and by (2.6) we get

$$\int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau \left[F'_{(a,b)}(0+) - F'_{(a,b)}(1-) \right]$$
$$\leq \int_0^1 p(\tau) F_{(a,b)}(\tau)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 F_{(a,b)}(\tau)\, d\tau$$
$$\leq \left[F'_{(a,b)}(1-) - F'_{(a,b)}(0+) \right] \int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau,$$

which is equivalent to the second and third inequalities (2.11).

Since $0 \leq \int_0^\tau p(s)\, ds \leq \int_0^1 p(\tau)\, d\tau$, hence

$$\int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau \leq \int_0^1 p(\tau)\, d\tau \int_0^1 (1-\tau)\, d\tau = \frac{1}{2} \int_0^1 p(\tau)\, d\tau$$

and the last part of (2.11) is proved. □

Remark 1. *If the function p is nonnegative and symmetric then the inequality (2.11) holds true.*

3. Some Examples

In the following, we assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval and the function $f(z)$ is *operator convex* on I in the Hermitian Banach $*$-algebra A and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$.

If we consider the weight $p : [0,1] \to [0,\infty)$, $p(s) = \left| s - \frac{1}{2} \right|$, then

$$\int_0^1 \left(\int_0^\tau p(s)\, ds \right) (1-\tau)\, d\tau$$
$$= \int_0^1 \left(\int_0^\tau \left| s - \frac{1}{2} \right| ds \right) (1-\tau)\, d\tau = \int_0^{\frac{1}{2}} \left(\int_0^\tau \left| s - \frac{1}{2} \right| ds \right) (1-\tau)\, d\tau$$
$$+ \int_{\frac{1}{2}}^1 \left(\int_0^\tau \left| s - \frac{1}{2} \right| ds \right) (1-\tau)\, d\tau = \int_0^{\frac{1}{2}} \left(\int_0^\tau \left(\frac{1}{2} - s \right) ds \right) (1-\tau)\, d\tau$$
$$+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - s \right) ds + \int_{\frac{1}{2}}^\tau \left(s - \frac{1}{2} \right) \right) (1-\tau)\, d\tau$$
$$= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau - \frac{\tau^2}{2} \right) (1-\tau)\, d\tau$$
$$+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - s \right) ds + \int_{\frac{1}{2}}^\tau \left(s - \frac{1}{2} \right) ds \right) (1-\tau)\, d\tau.$$

We have

$$\int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau - \frac{\tau^2}{2} \right) (1-\tau)\, d\tau = \frac{1}{2} \int_0^{\frac{1}{2}} (1-\tau)\, \tau\, (1-\tau)\, d\tau$$
$$= \frac{1}{2} \int_0^{\frac{1}{2}} (1-\tau)^2\, \tau d\tau = \frac{11}{384}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(\int_{0}^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^{\tau}\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau\\
&=\int_{\frac{1}{2}}^{1}\left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^{2}\right)(1-\tau)\,d\tau\\
&=\frac{1}{8}\int_{\frac{1}{2}}^{1}(1-\tau)\,d\tau+\frac{1}{2}\int_{\frac{1}{2}}^{1}\left(\tau-\frac{1}{2}\right)^{2}(1-\tau)\,d\tau=\frac{7}{384}.
\end{aligned}$$

Therefore

$$\int_{0}^{1}\left(\int_{0}^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{64}.$$

Since $\int_0^1\left|\tau-\frac{1}{2}\right|d\tau=\frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_{0}^{1}\left(\int_{0}^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{16}.$$

Utilising (2.11) for symmetric weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, we get

$$\begin{aligned}
&-\frac{3}{16}\left[Df(b)(b-a)-Df(a)(b-a)\right]\\
&\leq 4\int_{0}^{1}\left|\tau-\frac{1}{2}\right|f((1-\tau)a+\tau b)\,d\tau-\int_{0}^{1}f((1-\tau)a+\tau b)\,d\tau\\
&\leq\frac{3}{16}\left[Df(b)(b-a)-Df(a)(b-a)\right],
\end{aligned}\tag{3.1}$$

where f is an operator convex function on I and $a,b\in A$ with $\sigma(a),\sigma(b)\subset I$.

Consider now the symmetric function $p(s)=(1-s)s$, $s\in[0,1]$. Then

$$\int_{0}^{\tau}p(s)\,ds=\int_{a}^{\tau}(1-s)\,sds=-\frac{1}{6}\tau^{2}(2\tau-3),\ \ \tau\in[0,1]$$

and

$$\int_{0}^{1}\left(\int_{0}^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=-\frac{1}{6}\int_{0}^{1}\tau^{2}(2\tau-3)(1-\tau)\,d\tau=\frac{1}{40}.$$

Also

$$\int_{0}^{1}p(\tau)\,d\tau=\int_{0}^{1}(1-\tau)\,\tau d\tau=\frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_{0}^{1}\left(\int_{0}^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{20}$$

and by (2.11) we obtain

$$\begin{aligned}
&-\frac{3}{20}\left[Df(b)(b-a)-Df(a)(b-a)\right]\\
&\leq 6\int_{0}^{1}(1-\tau)\,\tau f((1-\tau)a+\tau b)\,d\tau-\int_{0}^{1}f((1-\tau)a+\tau b)\,d\tau\\
&\leq\frac{3}{20}\left[Df(b)(b-a)-Df(a)(b-a)\right],
\end{aligned}\tag{3.2}$$

where f is an operator convex function and $a,b\in A$ with $\sigma(a),\sigma(b)\subset I$.

The function $f(z)=z^{-1}$ is operator convex on $(0,\infty)$ in the Hermitian Banach $*$-algebra A, [6], and we have for $a,\ b>0$ that

$$Df(a)(b-a)=-a^{-1}(b-a)a^{-1} \text{ and } Df(b)(b-a)=-b^{-1}(b-a)b^{-1}.$$

If we write the inequalities (2.11) for this function, then we get

$$\begin{aligned}
(3.3)\qquad & -\frac{1}{2}\left(\int_0^1 p(\tau)\,d\tau\right)\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right] \\
&\leq -\left(\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\right)\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right] \\
&\leq \int_0^1 p(\tau)\left((1-\tau)a+\tau b\right)^{-1}d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1\left((1-\tau)a+\tau b\right)^{-1}d\tau \\
&\leq \left(\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\right)\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right] \\
&\leq \frac{1}{2}\left(\int_0^1 p(\tau)\,d\tau\right)\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right],
\end{aligned}$$

where p satisfies the condition (2.4).

If in (3.3) we take $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then we get

$$\begin{aligned}
(3.4)\qquad & -\frac{3}{20}\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right] \\
&\leq 4\int_0^1\left|\tau-\frac{1}{2}\right|\left((1-\tau)a+\tau b\right)^{-1}d\tau-\int_0^1\left((1-\tau)a+\tau b\right)^{-1}d\tau \\
&\leq \frac{3}{20}\left[a^{-1}(b-a)a^{-1}-b^{-1}(b-a)b^{-1}\right]
\end{aligned}$$

for $a,\ b>0$.

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