INTEGRAL INEQUALITIES OF GRÜSS TYPE FOR OPERATOR
CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH
*-ALGEBRAS

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Abstract. In this paper we provide lower and upper bounds, or so called
Grüss type inequalities, in the order of the Hermitian Banach *
-algebra
the Čebyšev’s difference
\[ \int_0^1 p(\tau) f((1-\tau)a + \tau b) d\tau - \int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau)a + \tau b) d\tau \]
in the case that \( f(z) \) is analytic in \( G \), an open subset of \( C \), \( I \subseteq G \) is a real
interval and the function \( f(z) \) is operator convex on \( I \) while \( p : [0,1] \to \mathbb{R} \) is a
Lebesgue integrable function such that
\[ 0 \leq \int_0^\tau p(s) ds \leq \int_0^1 p(s) ds \text{ for all } \tau \in [0,1], \]
and \( a, b \in A \) with \( \sigma(a), \sigma(b) \subseteq I \).

1. Introduction

We need some preliminary concepts and facts about Banach *
-algebras.

Let \( A \) be a unital Banach *-algebra with unit 1. An element \( a \in A \) is called
selfadjoint if \( a^* = a \). \( A \) is called Hermitian if every selfadjoint element \( a \) in \( A \) has
real spectrum \( \sigma(a) \), namely \( \sigma(a) \subseteq \mathbb{R} \).

We say that an element \( a \) is nonnegative and write this as \( a \geq 0 \) if \( a^* = a \) and
\( \sigma(a) \subseteq [0,\infty) \). We say that \( a \) is positive and write \( a > 0 \) if \( a \geq 0 \) and \( 0 \notin \sigma(a) \).
Thus \( a > 0 \) implies that its inverse \( a^{-1} \) exists. Denote the set of all invertible
elements of \( A \) by \( \text{Inv} (A) \). If \( a, b \in \text{Inv} (A) \), then \( ab \in \text{Inv} (A) \) and
\( (ab)^{-1} = b^{-1}a^{-1} \). Also, saying that \( a \geq b \) means that \( a - b \geq 0 \) and, similarly \( a > b \) means that
\( a - b > 0 \).

The Shirali-Ford theorem asserts that if \( A \) is a unital Banach *
-algebra [11] (see
also [1, Theorem 41.5]), then
\[ a^*a \geq 0 \text{ for every } a \in A. \]

Based on this fact, Okayasu [10], Tanahashi and Uchiyama [12] proved the following
fundamental properties (see also [8]):

(i) If \( a, b \in A \), then \( a \geq 0, b \geq 0 \) imply \( a + b \geq 0 \) and \( a \geq 0 \) implies \( \alpha a \geq 0 \);
(ii) If \( a, b \in A \), then \( a > 0, b \geq 0 \) imply \( a + b > 0 \);
(iii) If \( a, b \in A \), then either \( a \geq b > 0 \) or \( a > b \geq 0 \) imply \( a > 0 \);
(iv) If \( a > 0 \), then \( a^{-1} > 0 \);

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(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [10] showed that the L"owner-Heinz inequality remains valid in a Hermitian unital Banach *-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ ($a \geq b$) implies that $a^p > b^p$ ($a^p \geq b^p$).

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma (a)$ and the fact that $\sigma (a)$ is a compact subset of $\mathbb{C}$ implies that $\inf \{z : z \in \sigma (a)\} > 0$ and $\sup \{z : z \in \sigma (a)\} < \infty$. Choose $\gamma$ to be close rectifiable curve in $\{\Re z > 0\}$, the right half open plane of the complex plane, such that $\sigma (a) \subset \text{ins } (\gamma)$, the inside of $\gamma$. Let $G$ be an open subset of $\mathbb{C}$ with $\sigma (a) \subset G$. If $f : G \rightarrow \mathbb{C}$ is analytic, we define an element $f(a)$ in $A$ by

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(z) (z - a)^{-1} \, dz,$$

where $\gamma$ is a close rectifiable curve such that $\sigma (a) \subset \text{ins } (\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of $\gamma$ and the Spectral Mapping Theorem (SMT)

$$\sigma (f(a)) = f(\sigma (a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z - a)^{-1} \, dz,$$

where $z^\alpha$ is the principal $\alpha$-power of $z$. Since $A$ is a Banach *-algebra, then $a^\alpha \in A$.

Moreover, since $z^\alpha$ is analytic in $\{\Re z > 0\}$, then by (SMT) we have

$$\sigma (a^\alpha) = (\sigma (a))^\alpha = \{z^\alpha : z \in \sigma (a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [12, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha + \beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xii) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma (u) \subset I$, then by (SMT) we have

$$\sigma (f(u)) = f(\sigma (u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of $A$.

Therefore, we can state the following fact that will be used to establish various inequalities in $A$, see also [3].

**Lemma 1.** Let $f(z)$ and $g(z)$ be analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma (u) \subset I$ we have $f(u) \geq g(u)$ in the order of $A$. 
For some recent inequalities in Hermitian Banach *-algebras, see [3], [4] and [5]. Let \( G \) be an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \), then by SMT the element \((1 - t) a + t b \in A \) has the spectrum \( \sigma ((1 - t) a + t b) \subset I \) for all \( t \in [0, 1] \). We say that an analytic function \( f(z) \) in \( G \) is operator convex on \( I \) in the Hermitian Banach *-algebra \( A \) if
\[
 f((1 - t) a + t b) \leq (1 - t) f(a) + t f(b) \quad \text{in the order of } A
\]
for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) and all \( t \in [0, 1] \).

In the recent paper [6] we obtained the following results:

**Theorem 1.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. The function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach *-algebra \( A \) if and only if for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) we have
\[
 (1.1) \quad f((1 - t) a + t b) \leq (1 - t) f(a) + t f(b) \quad \text{in the order of } A
\]
in the order of \( A \), where \( Df \) is the Fréchet derivative of \( f \) as a function of elements in the Hermitian Banach *-algebra \( A \).

Let \( f(z) \) be analytic in \( G \), an open convex subset of \( \mathbb{C} \) and \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \). Consider the auxiliary function \( F_{a,b} : [0, 1] \to A \) defined by
\[
 F_{a,b} (t) := f ((1 - t) a + t b).
\]

The following characterization results also holds [6].

**Theorem 2.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. The function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach *-algebra \( A \) if and only if for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) we have for all \( t_2, t_1 \in [0, 1] \) with \( t_1 < t_2 \) that
\[
 (1.3) \quad F_{a,b} (t_2) = Df ((1 - t_2) a + t_2 b) (b - a)
\]
\[
 \geq f((1 - t_2) a + t_2 b) - f ((1 - t_1) a + t_1 b)
\]
\[
 \frac{t_2 - t_1}{t_2 - t_1} Df ((1 - t_1) a + t_1 b) (b - a) = F_{a,b} (t_1).
\]

We also have
\[
 (1.4) \quad Df (b) (b - a) \geq F_{a,b} (t) = Df ((1 - t) a + t b) (b - a) \geq Df (a) (b - a)
\]
for all \( t \in (0, 1) \).

It is well known that, if \( E \) is a Banach space and \( g : [0, 1] \to E \) is a continuous function, then \( g \) is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by \( \int_0^1 g(t) \, dt \).

In the recent paper [7] we also obtained the following Féjer’s type inequalities:

**Theorem 3.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach *-algebra \( A \) and \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p(1-t) = p(t) \) for all \( t \in [0, 1] \), then for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) we have
\[
 (1.5) \quad 0 \leq \int_0^1 p(t) f((1 - t) a + t b) \, dt - \left( \int_0^1 p(t) \, dt \right) f\left(\frac{a + b}{2}\right)
\]
\[
 \leq \frac{1}{2} \left( \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt \right) \left[ Df(b)(b-a) - Df(a)(b-a) \right].
\]
In particular, for \( p \equiv 1 \) we get

\[ 0 \leq \int_{0}^{1} f ((1 - t) a + t b) \, dt - f \left( \frac{a + b}{2} \right) \]
\[ \leq \frac{1}{8} [Df (b)(b-a) - Df (a)(b-a)]. \]

We also have:

**Theorem 4.** Assume that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f (z) \) is operator convex on \( I \) in the Hermitian Banach \( * \)-algebra \( A \) and \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, then for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) we have

\[ 0 \leq \left( \int_{0}^{1} p (t) \, dt \right) \frac{f (a) + f (b)}{2} - \int_{0}^{1} p (t) f ((1 - t) a + t b) \, dt \]
\[ \leq \frac{1}{2} \int_{0}^{1} \left( \left| \frac{1}{2} - |t - \frac{1}{2}| \right| \right) p (t) \, dt [Df (b)(b-a) - Df (a)(b-a)]. \]

In particular, for \( p \equiv 1 \) we get

\[ 0 \leq \frac{f (a) + f (b)}{2} - \int_{0}^{1} f ((1 - t) a + t b) \, dt \]
\[ \leq \frac{1}{8} [Df (b)(b-a) - Df (a)(b-a)]. \]

Motivated by the above results, in this paper we provide lower and upper bounds, or so called Grüss type inequalities, in the order of the Hermitian Banach \( * \)-algebra \( A \) for the Čebyšev’s difference

\[ \int_{0}^{1} p (\tau) f ((1 - \tau) a + \tau b) \, d\tau - \int_{0}^{1} p (\tau) d\tau \int_{0}^{1} f ((1 - \tau) a + \tau b) \, d\tau \]

in the case that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \), \( I \subset G \) a real interval and the function \( f (z) \) is operator convex on \( I \) while \( p : [0, 1] \to \mathbb{R} \) is a Lebesgue integrable function such that

\[ 0 \leq \int_{0}^{\tau} p (s) \, ds \leq \int_{0}^{1} p (s) \, ds \text{ for all } \tau \in [0, 1], \]

and \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \).

2. **Main Results**

We start to the following identity that is of interest in itself as well:

**Lemma 2.** Assume that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If \( f : [0, 1] \to \mathbb{C} \) is Lebesgue integrable, then for all \( a, b \in A \) with \( \sigma (a), \sigma (b) \subset I \) we have the following identity for the auxiliary function \( F_{(a,b)} \)

\[ \int_{0}^{1} p (\tau) F_{(a,b)} (\tau) \, d\tau - \int_{0}^{1} p (\tau) d\tau \int_{0}^{1} F_{(a,b)} (\tau) \, d\tau \]
\[ = \int_{0}^{1} \left( \int_{0}^{\tau} p (s) \, ds \right) \tau F_{(a,b)} (\tau) \, d\tau + \int_{0}^{1} \left( \int_{0}^{\tau} p (s) \, ds \right) (\tau - 1) F_{(a,b)} (\tau) \, d\tau. \]
**Proof.** The function $F_{(a,b)}$ is obviously differentiable on $(0,1)$. Integrating by parts in the Bochner’s integral, we have

$$
\int_0^1 tF'_{(a,b)}(t) \, dt + \int_0^1 (t-1) F'_{(a,b)}(t) \, dt
$$

that holds for all $\tau \in [0,1]$.

If we multiply this identity by $p(\tau)$ and integrate over $\tau$ in $[0,1]$, then we get

$$
\int_0^1 p(\tau) F_{(a,b)}(\tau) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 F_{(a,b)}(t) \, dt
$$

Using integration by parts, we derive

$$
\int_0^1 p(\tau) \left( \int_0^\tau tF'_{(a,b)}(t) \, dt \right) d\tau
$$

and

$$
\int_0^1 p(\tau) \left( \int_\tau^1 (t-1) F'_{(a,b)}(t) \, dt \right) d\tau
$$

which proves the identity in (2.1).
Theorem 5. Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) and \( p : [0,1] \to \mathbb{R} \) is a Lebesgue integrable function such that

\[
0 \leq \int_0^\tau p(s) \, ds \leq \int_0^1 p(s) \, ds \quad \text{for all} \quad \tau \in [0,1],
\]

then for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have the following Grüss type inequalities

\[
\left\lfloor \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau \, d\tau \right\rfloor Df(a)(b-a) - \left\lfloor \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) \, d\tau \right\rfloor Df(b)(b-a)
\]

\[
\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 f((1-\tau)a + \tau b) \, d\tau
\]

\[
\leq \left\lfloor \int_0^1 \left( \int_0^1 p(s) \, ds \right) \tau \, d\tau \right\rfloor Df(b)(b-a) - \left\lfloor \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) (1-\tau) \, d\tau \right\rfloor Df(a)(b-a)
\]

or, equivalently,

\[
\int_0^1 (1-\tau) \left( \int_\tau^1 [p(1-s) Df(b)(b-a) - p(s) Df(a)(b-a)] \, ds \right) \, d\tau
\]

\[
\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 f((1-\tau)a + \tau b) \, d\tau
\]

\[
\leq \int_0^1 (1-\tau) \left( \int_0^1 [p(1-s) Df(a)(b-a) - p(s) Df(b)(b-a)] \, ds \right) \, d\tau.
\]

Proof. We have for \( F_{(a,b)} \) and \( p : [0,1] \to \mathbb{R} \) a Lebesgue integrable function that

\[
\int_0^1 p(\tau) \, d\tau F_{(a,b)}(\tau) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 F_{(a,b)}(\tau) \, d\tau
\]

\[
= \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) (\tau) F'_{(a,b)}(\tau) \, d\tau - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) F'_{(a,b)}(\tau) \, d\tau.
\]

By the properties of \( F_{(a,b)} \) from the above section, we have in the operator order that

\[
\tau F'_{(a,b)} (1-) \geq \tau F'_{(a,b)} (\tau) \geq \tau F'_{(a,b)} (0+)
\]

and

\[
(1-\tau) F'_{(a,b)} (1-) \geq (1-\tau) F'_{(a,b)} (\tau) \geq (1-\tau) F'_{(a,b)} (0+)
\]

for all \( \tau \in (0,1) \).

From

\[
\int_0^\tau p(s) \, ds \leq \int_0^1 p(s) \, ds = \int_0^\tau p(s) \, ds + \int_\tau^1 p(s) \, ds,
\]

we get that \( \int_\tau^1 p(s) \, ds \geq 0 \) for all \( \tau \in (0,1) \).
From (2.8) we obtain that
\[
\left( \int_\tau^1 p(s) \, ds \right) \tau F'_{(a,b)} (1-) \geq \left( \int_\tau^1 p(s) \, ds \right) \tau F'_{(a,b)} (\tau)
\]
\[
\geq \left( \int_\tau^1 p(s) \, ds \right) \tau F'_{(a,b)} (0+)
\]
and from (2.9) that
\[
- \left( \int_0^\tau p(s) \, ds \right) (1- \tau) F'_{(a,b)} (0+) \leq - \left( \int_0^\tau p(s) \, ds \right) (1- \tau) F'_{(a,b)} (\tau)
\]
\[
\leq - \left( \int_0^\tau p(s) \, ds \right) (1- \tau) F'_{(a,b)} (1-)
\]
all \( \tau \in (0,1) \).

If we integrate these inequalities over \( \tau \in [0,1] \) and add the obtained results, then we get
\[
\int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau d\tau F'_{(a,b)} (1-) - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1- \tau) \, d\tau F'_{(a,b)} (0+) \\
\geq \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau F'_{(a,b)} (\tau) \, d\tau - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1- \tau) \, F'_{(a,b)} (\tau) \, d\tau \\
\geq \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau d\tau F'_{(a,b)} (0+) - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1- \tau) \, d\tau F'_{(a,b)} (1-) .
\]

By using the equality (2.1) we derive
\[
(2.10) \quad \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau d\tau F'_{(a,b)} (0+) \\
- \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1- \tau) \, d\tau F'_{(a,b)} (1-) \\
\leq \int_0^1 p(\tau) F_{(a,b)} (\tau) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 F_{(a,b)} (\tau) \, d\tau \\
\leq \int_0^1 \left( \int_\tau^1 p(s) \, ds \right) \tau d\tau F'_{(a,b)} (1-) \\
- \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1- \tau) \, d\tau F'_{(a,b)} (0+) ,
\]

and since \( F'_{(a,b)} (1-) = Df(b)(b-a) \) and \( F'_{(a,b)} (0+) = Df(b)(b-a) \) hence we obtain (2.5).

If we change the variable \( \alpha = 1-\tau \), then we have
\[
\int_0^1 \left( \int_a^1 p(s) \, ds \right) \tau d\tau = \int_0^1 \left( \int_{1-\alpha}^1 p(s) \, ds \right) (1- \alpha) \, d\alpha .
\]

Also by the change of variable \( u = 1-s \), we get
\[
\int_0^1 p(s) \, ds = \int_0^1 p(1-u) \, du ,
\]
which implies that
\[
\int_0^1 \left( \int_0^1 p(s) \, ds \right) \tau \, d\tau = \int_0^1 \left( \int_0^\tau p(1-s) \, ds \right) (1-\tau) \, d\tau.
\]

Therefore
\[
\int_0^1 \left( \int_0^1 p(s) \, ds \right) \tau \, d\tau \mathcal{F}'_{(a,b)} (1-) - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (0+)
\]
\[
= \int_0^1 \left( \int_0^\tau p(1-s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (1-)
\]
\[
- \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (0+)
\]
\[
= \int_0^1 (1-\tau) \left( \int_0^\tau \left[ p(1-s) \mathcal{F}'_{(a,b)} (1-) - p(s) \mathcal{F}'_{(a,b)} (0+) \right] \, ds \right) \, d\tau,
\]
and
\[
\int_0^1 \left( \int_0^1 p(s) \, ds \right) \tau \, d\tau \mathcal{F}'_{(a,b)} (0+) - \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (1-)
\]
\[
= \int_0^1 \left( \int_0^\tau p(1-s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (0+)
\]
\[
- \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1-\tau) \, d\tau \mathcal{F}'_{(a,b)} (1-)
\]
\[
= \int_0^1 (1-\tau) \left( \int_0^\tau \left[ p(1-s) \mathcal{F}'_{(a,b)} (0+) - p(s) \mathcal{F}'_{(a,b)} (1-) \right] \, ds \right) \, d\tau,
\]
and by (2.10) we get (2.6). \hfill \square

We say that the function \( p : [0,1] \to \mathbb{R} \) is symmetric on \([0,1]\) if
\[
p(1-t) = p(t) \text{ for all } t \in [0,1].
\]

**Corollary 1.** With the assumptions of Theorem 5 and, in addition, if \( p : [0,1] \to \mathbb{R} \) is a symmetric function on \([0,1]\), then we have

\[
(2.11) \quad - \frac{1}{2} \left( \int_0^1 p(\tau) \, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)]
\]
\[
\leq - \left( \int_0^1 \left( \int_0^{\tau} p(s) \, ds \right) (1-\tau) \, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)]
\]
\[
\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 f((1-\tau)a + \tau b) \, d\tau
\]
\[
\leq \left( \int_0^1 \left( \int_0^{\tau} p(s) \, ds \right) (1-\tau) \, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)]
\]
\[
\leq \frac{1}{2} \left( \int_0^1 p(\tau) \, d\tau \right) [Df(b)(b-a) - Df(a)(b-a)].
\]
Proof. Since $p$ is symmetric, then $p(1 - s) = p(s)$ for all $s \in [0, 1]$ and by (2.6) we get

$$
\int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1 - \tau) \, d\tau \left[ F'_{(a,b)}(0+) - F'_{(a,b)}(1-) \right]
\leq \int_0^1 p(\tau) F_{(a,b)}(\tau) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 F_{(a,b)}(\tau) \, d\tau
\leq \left[ F_{(a,b)}(1-) - F_{(a,b)}(0+) \right] \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1 - \tau) \, d\tau,
$$

which is equivalent to the second and third inequalities (2.11).

Since $0 \leq \int_0^\tau p(s) \, ds \leq \int_0^1 p(\tau) \, d\tau$, hence

$$
\int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1 - \tau) \, d\tau \leq \int_0^1 p(\tau) \, d\tau \int_0^1 (1 - \tau) \, d\tau = \frac{1}{2} \int_0^1 p(\tau) \, d\tau
$$

and the last part of (2.11) is proved.

Remark 1. If the function $p$ is nonnegative and symmetric then the inequality (2.11) holds true.

3. Some Examples

In the following, we assume that $f(z)$ is analytic in $G$, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval and the function $f(z)$ is operator convex on $I$ in the Hermitian Banach $*$-algebra $A$ and $a, b \in A$ with $\sigma(a), \sigma(b) \in I$.

If we consider the weight $p : [0, 1] \to [0, \infty)$, $p(s) = |s - \frac{1}{2}|$, then

$$
\int_0^1 \left( \int_0^\tau p(s) \, ds \right) (1 - \tau) \, d\tau
= \int_0^1 \left( \int_0^\tau |s - \frac{1}{2}| \, ds \right) (1 - \tau) \, d\tau = \int_0^{\frac{1}{2}} \left( \int_0^\tau |s - \frac{1}{2}| \, ds \right) (1 - \tau) \, d\tau
+ \int_0^1 \left( \int_{\frac{1}{2}}^\tau |s - \frac{1}{2}| \, ds \right) (1 - \tau) \, d\tau
+ \int_0^1 \left( \int_{\frac{1}{2}}^\tau \left( \frac{1}{2} - s \right) \, ds + \int_\frac{1}{2}^\tau \left( s - \frac{1}{2} \right) \, ds \right) (1 - \tau) \, d\tau
= \int_0^{\frac{1}{2}} \left( \frac{1}{2} \tau - \frac{\tau^2}{2} \right) (1 - \tau) \, d\tau
+ \int_0^{\frac{1}{2}} \left( \int_{\frac{1}{2}}^\tau \left( \frac{1}{2} - s \right) \, ds + \int_\frac{1}{2}^\tau \left( s - \frac{1}{2} \right) \, ds \right) (1 - \tau) \, d\tau.
$$

We have

$$
\int_0^{\frac{1}{2}} \left( \frac{1}{2} \tau - \frac{\tau^2}{2} \right) (1 - \tau) \, d\tau = \frac{1}{2} \int_0^{\frac{1}{2}} (1 - \tau)^2 \tau \, d\tau = \frac{11}{384}
$$
and
\[\int_{\frac{1}{2}}^{1} \left( \int_{0}^{\frac{1}{2}} \left( \frac{1}{2} - s \right) ds + \int_{\frac{1}{2}}^{1} \left( s - \frac{1}{2} \right) ds \right) \left( 1 - \tau \right) d\tau \]
\[= \int_{\frac{1}{2}}^{1} \left( \frac{1}{8} + \frac{1}{2} \left( \tau - \frac{1}{2} \right)^2 \right) \left( 1 - \tau \right) d\tau \]
\[= \frac{1}{8} \int_{\frac{1}{2}}^{1} \left( 1 - \tau \right) d\tau + \frac{1}{2} \int_{\frac{1}{2}}^{1} \left( \tau - \frac{1}{2} \right)^2 \left( 1 - \tau \right) d\tau = \frac{7}{384}.\]

Therefore
\[\int_{0}^{1} \left( \int_{0}^{\tau} p(s) ds \right) \left( 1 - \tau \right) d\tau = \frac{3}{64}.\]

Since \(\int_{0}^{1} \left| \tau - \frac{1}{2} \right| d\tau = \frac{1}{4}\), hence
\[\frac{1}{\int_{0}^{1} p(\tau) d\tau} \int_{0}^{1} \left( \int_{0}^{\tau} p(s) ds \right) \left( 1 - \tau \right) d\tau = \frac{3}{16}.\]

Utilising (2.11) for symmetric weight \(p : [0, 1] \rightarrow [0, \infty), \ p(s) = |s - \frac{1}{2}|\), we get
\[(3.1) \quad - \frac{3}{16} [Df (b) (b - a) - Df (a) (b - a)]
\leq 4 \int_{0}^{1} \left| \tau - \frac{1}{2} \right| f \left( (1 - \tau) a + \tau b \right) d\tau - \int_{0}^{1} f \left( (1 - \tau) a + \tau b \right) d\tau
\leq \frac{3}{16} [Df (b) (b - a) - Df (a) (b - a)],\]

where \(f\) is an operator convex function on \(I\) and \(a, b \in A\) with \(\sigma (a), \sigma (b) \subseteq I\).

Consider now the symmetric function \(p(s) = (1 - s) s, s \in [0, 1]\). Then
\[\int_{0}^{\tau} p(s) ds = \int_{a}^{\tau} (1 - s) s ds = -\frac{1}{6} \tau^2 (2\tau - 3), \ \tau \in [0, 1]\]
and
\[\int_{0}^{1} \left( \int_{0}^{\tau} p(s) ds \right) \left( 1 - \tau \right) d\tau = -\frac{1}{6} \int_{0}^{1} \tau^2 (2\tau - 3) (1 - \tau) d\tau = \frac{1}{40}.\]

Also
\[\int_{0}^{1} p(\tau) d\tau = \int_{0}^{1} (1 - \tau) \tau d\tau = \frac{1}{6}\]
and
\[\frac{1}{\int_{0}^{1} p(\tau) d\tau} \int_{0}^{1} \left( \int_{0}^{\tau} p(s) ds \right) \left( 1 - \tau \right) d\tau = \frac{3}{20}.\]

and by (2.11) we obtain
\[(3.2) \quad - \frac{3}{20} [Df (b) (b - a) - Df (a) (b - a)]
\leq 6 \int_{0}^{1} (1 - \tau) \tau f \left( (1 - \tau) a + \tau b \right) d\tau - \int_{0}^{1} f \left( (1 - \tau) a + \tau b \right) d\tau
\leq \frac{3}{20} [Df (b) (b - a) - Df (a) (b - a)],\]

where \(f\) is an operator convex function and \(a, b \in A\) with \(\sigma (a), \sigma (b) \subseteq I\).
The function \( f (z) = z^{-1} \) is operator convex on \((0, \infty)\) in the Hermitian Banach *-algebra \(A\), [6], and we have for \(a, b > 0\) that
\[
Df (a) (b - a) = -a^{-1} (b - a) a^{-1} \quad \text{and} \quad Df (b) (b - a) = -b^{-1} (b - a) b^{-1}.
\]

If we write the inequalities (2.11) for this function, then we get
\[
\frac{1}{2} \int_0^1 p (\tau) d\tau \left[a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1}\right]
\leq - \left(\int_0^1 \left(\int_0^\tau p (s) ds\right) (1 - \tau) d\tau\right) \left[a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1}\right]
\leq \int_0^1 p (\tau) \left( (1 - \tau) a + \tau b \right)^{-1} d\tau - \int_0^1 p (\tau) d\tau \int_0^1 \left( (1 - \tau) a + \tau b \right)^{-1} d\tau
\leq \frac{1}{2} \left(\int_0^1 p (\tau) d\tau\right) \left[a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1}\right],
\]
where \(p\) satisfies the condition (2.4).

If in (3.3) we take \( p : [0, 1] \to [0, \infty) \), \( p (s) = |s - \frac{1}{2}| \), then we get
\[
\frac{1}{20} \left[a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1}\right]
\leq 4 \int_0^1 \left|\tau - \frac{1}{2}\right| \left( (1 - \tau) a + \tau b \right)^{-1} d\tau - \int_0^1 \left( (1 - \tau) a + \tau b \right)^{-1} d\tau
\leq \frac{3}{20} \left[a^{-1} (b - a) a^{-1} - b^{-1} (b - a) b^{-1}\right]
\]
for \(a, b > 0\).

References


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