Abstract. In this paper we provide new Grüss type inequalities in the order of the Hermitian Banach ∗-algebra $A$ for the Čebyšev’s difference
\[
\int_0^1 p(\tau) f((1-\tau)a + \tau b) d\tau - \int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau)a + \tau b) d\tau
\]
in the case that $f(z)$ is analytic in $G$, an open subset of $\mathbb{C}$, $I \subset G$ is a real interval and the function $f(z)$ is operator convex on $I$ while $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that
\[
\frac{1}{\tau} \int_0^\tau p(s) ds \leq \frac{1}{1-\tau} \int_\tau^1 p(s) ds \text{ for all } \tau \in (0,1)
\]
and $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$.

1. Introduction

We need some preliminary concepts and facts about Banach ∗-algebras.

Let $A$ be a unital Banach ∗-algebra with unit 1. An element $a \in A$ is called selfadjoint if $a^* = a$. $A$ is called Hermitian if every selfadjoint element $a$ in $A$ has real spectrum $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element $a$ is nonnegative and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that $a$ is positive and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse $a^{-1}$ exists. Denote the set of all invertible elements of $A$ by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The Shirali-Ford theorem asserts that if $A$ is a unital Banach ∗-algebra [11] (see also [1, Theorem 41.5]), then
\[
a^* a \geq 0 \text{ for every } a \in A.
\]

Based on this fact, Okayasu [10], Tanahashi and Uchiyama [12] proved the following fundamental properties (see also [8]):

(i) If $a, b \in A$, then $a \geq 0, b \geq 0$ imply $a + b \geq 0$ and $a \geq 0$ implies $aa \geq 0$;
(ii) If $a, b \in A$, then $a > 0, b > 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;

1991 Mathematics Subject Classification. 47A63, 47A30, 15A60, 26D15, 26D10.
Key words and phrases. Hermitian unital Banach ∗-algebra, Grüss type inequalities, Operator convex functions.
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [10] showed that the \textit{Löwner-Heinz inequality} remains valid in a Hermitian unital Banach $\ast$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ ($a \geq b$) implies that $a^p > b^p$ ($a^p \geq b^p$).

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma (a)$ and the fact that $\sigma (a)$ is a compact subset of $\mathbb{C}$ implies that $\inf \{z : z \in \sigma (a)\} > 0$ and $\sup \{z : z \in \sigma (a)\} < \infty$. Choose $\gamma$ to be close rectifiable curve in $\{\Re z > 0\}$, the right half open plane of the complex plane, such that $\sigma (a) \subset \text{ins} (\gamma)$, the inside of $\gamma$. Let $G$ be an open subset of $\mathbb{C}$ with $\sigma (a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f (a)$ in $A$ by

$$f (a) := \frac{1}{2\pi i} \int_{\gamma} f (z) (z - a)^{-1} dz,$$

where $\gamma$ is a close rectifiable curve such that $\sigma (a) \subset \text{ins} (\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f (a)$ does not depend on the choice of $\gamma$ and the Spectral Mapping Theorem (SMT)

$$\sigma (f (a)) = f (\sigma (a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z - a)^{-1} dz,$$

where $z^\alpha$ is the principal $\alpha$-power of $z$. Since $A$ is a Banach $\ast$-algebra, then $a^\alpha \in A$. Moreover, since $z^\alpha$ is analytic in $\{\Re z > 0\}$, then by (SMT) we have

$$\sigma (a^\alpha) = (\sigma (a))^\alpha = \{z^\alpha : z \in \sigma (a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [12, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha + \beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f (\cdot)$ is analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f (z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma (u) \subset I$, then by (SMT) we have

$$\sigma (f (u)) = f (\sigma (u)) \subset f (I) \subset [0, \infty)$$

meaning that $f (u) \geq 0$ in the order of $A$.

Therefore, we can state the following fact that will be used to establish various inequalities in $A$, see also [3].

**Lemma 1.** Let $f (z)$ and $p (z)$ be analytic in $G$, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f (z) \geq p (z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma (u) \subset I$ we have $f (u) \geq p (u)$ in the order of $A$. 
For some recent inequalities in Hermitian Banach \(*\)-algebras, see [3], [4] and [5].

Let \( G \) be an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \), then by SMT the element \((1 - t)a + tb \in A\) has the spectrum \( \sigma((1 - t)a + tb) \subset I\) for all \( t \in [0, 1] \). We say that an analytic function \( f(z) \) in \( G \) is \textit{operator convex} on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) if
\[
(1.1) \quad f((1 - t)a + tb) \leq (1 - t)f(a) + tf(b) \quad \text{in the order of} \ A
\]
for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) and all \( t \in [0, 1] \).

In the recent paper [6] we obtained the following results:

**Theorem 1.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. The function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) if and only if for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have
\[
(1.2) \quad f(b) - f(a) \geq Df(a)(b - a)
\]
in the order of \( A \), where \( Df \) is the Fréchet derivative of \( f \) as a function of elements in the Hermitian Banach \(*\)-algebra \( A \).

Let \( f(z) \) be analytic in \( G \), an open convex subset of \( \mathbb{C} \) and \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset G \). Consider the auxiliary function \( F_{a,b} : [0, 1] \rightarrow A \) defined by
\[
F_{a,b}(t) := f((1 - t)a + tb).
\]

The following characterization result also holds:

**Theorem 2.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. The function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) if and only if for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have for all \( t_2, t_1 \in [0, 1] \) with \( t_1 < t_2 \) that
\[
(1.3) \quad F_{a,b}'(t_2) = Df((1 - t_2)a + t_2b)(b - a)
\]
\[
\geq \frac{f((1 - t_2)a + t_2b) - f((1 - t_1)a + t_1b)}{t_2 - t_1}
\]
\[
\geq Df((1 - t_1)a + t_1b)(b - a) = F_{a,b}'(t_1).
\]

We also have
\[
(1.4) \quad Df(b)(b - a) \geq F_{a,b}'(t) = Df((1 - t)a + tb)(b - a) \geq Df(a)(b - a)
\]
for all \( t \in (0, 1) \).

It is well known that, if \( E \) is a Banach space and \( p : [0, 1] \rightarrow E \) is a continuous function, then \( p \) is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by \( \int_0^1 p(t)\, dt \).

In the recent paper [7] we also obtained the following Féjer’s type inequalities:

**Theorem 3.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) and \( p : [0, 1] \rightarrow [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p(1 - t) = p(t) \) for all \( t \in [0, 1] \), then for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have
\[
(1.5) \quad 0 \leq \int_0^1 p(t)\, f((1 - t)a + tb)\, dt - \left( \int_0^1 p(t)\, dt \right) f\left( \frac{a + b}{2} \right)
\]
\[
\leq \frac{1}{2} \left( \int_0^1 \left| t - \frac{1}{2} \right| p(t)\, dt \right) \left[ Df(b)(b - a) - Df(a)(b - a) \right].
\]
In particular, for \( p \equiv 1 \) we get

\[
0 \leq \int_0^1 f ((1 - t) a + t b) \, dt - f \left( \frac{a + b}{2} \right)
\]

\[
\leq \frac{1}{8} \left[ Df (b) (b - a) - Df (a) (b - a) \right].
\]

We also have:

**Theorem 4.** Assume that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f (z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) and \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p (1 - t) = p (t) \) for all \( t \in [0, 1] \), then for all \( a, b \in A \) with \( \sigma (a) \), \( \sigma (b) \subset I \) we have

\[
0 \leq \left( \int_0^1 p (t) \, dt \right) \frac{f (a) + f (b)}{2} - \int_0^1 p (t) f ((1 - t) a + t b) \, dt
\]

\[
\leq \frac{1}{2} \left( \int_0^1 \left( \frac{1}{2} - t \right) \frac{1}{2} \right) p (t) \, dt \left[ Df (b) (b - a) - Df (a) (b - a) \right].
\]

In particular, for \( p \equiv 1 \) we get

\[
0 \leq \frac{f (a) + f (b)}{2} - \int_0^1 f ((1 - t) a + t b) \, dt
\]

\[
\leq \frac{1}{8} \left[ Df (b) (b - a) - Df (a) (b - a) \right].
\]

Motivated by the above results, in this paper we provide lower and upper bounds, so called Grüss type inequalities, in the order of the Hermitian Banach \(*\)-algebra \( A \) for the Čebyšev’s difference

\[
\int_0^1 p (t) f ((1 - t) a + t b) \, dt - \int_0^1 p (t) \, dt \int_0^1 f ((1 - t) a + t b) \, dt
\]

in the case that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \), \( I \subset G \) a real interval and the function \( f (z) \) is operator convex on \( I \) while \( p : [0, 1] \to \mathbb{R} \) is a Lebesgue integrable function such that

\[
\frac{1}{\tau} \int_0^\tau p (s) \, ds \leq \frac{1}{1 - \tau} \int_\tau^1 p (s) \, ds \text{ for all } \tau \in [0, 1],
\]

and \( a, b \in A \) with \( \sigma (a) \), \( \sigma (b) \subset I \).

2. **Main Results**

We start to the following identity that is of interest in itself as well:

**Lemma 2.** Assume that \( f (z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable, then for all \( a, b \in A \) with \( \sigma (a) \), \( \sigma (b) \subset I \) we have for the auxiliary function \( F_{(a,b)}(t) \) that

\[
\int_0^1 p (t) F_{(a,b)} (t) \, dt - \int_0^1 p (t) \, dt \int_0^1 F_{(a,b)} (t) \, dt
\]

\[
= \int_0^1 \tau (1 - \tau) \left( \int_\tau^1 \frac{p (s) \, ds}{1 - \tau} - \int_0^\tau \frac{p (s) \, ds}{\tau} \right) F'_{(a,b)} (\tau) \, d\tau.
\]
Proof. $F_{(a,b)}$ is obviously differentiable on $(0, 1)$. Integrating by parts in the Bochner’s integral, we have

$$
\int_0^\tau tF'_{(a,b)}(t)\,dt + \int_\tau^1 (t-1)F'_{(a,b)}(t)\,dt
= \tau F_{(a,b)}(\tau) - \int_0^\tau F_{(a,b)}(t)\,dt - (\tau - 1)F_{(a,b)}(\tau) - \int_\tau^1 F_{(a,b)}(t)\,dt
= F_{(a,b)}(\tau) - \int_0^1 F_{(a,b)}(t)\,dt
$$

that holds for all $\tau \in [0, 1]$.

If we multiply this identity by $p(\tau)$ and integrate over $\tau$ in $[0, 1]$, then we get

$$
\int_0^1 p(\tau) F_{(a,b)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 F_{(a,b)}(t)\,dt
= \int_0^1 p(\tau) \left( \int_0^\tau tF'_{(a,b)}(t)\,dt \right)\,d\tau + \int_0^1 p(\tau) \left( \int_\tau^1 (t-1)F'_{(a,b)}(t)\,dt \right)\,d\tau.
$$

Using integration by parts, we get

$$
\int_0^1 p(\tau) \left( \int_0^\tau tF'_{(a,b)}(t)\,dt \right)\,d\tau
= \int_0^1 \left( \int_0^\tau tF'_{(a,b)}(t)\,dt \right)\,d \left( \int_0^\tau p(s)\,ds \right)
= \left( \int_0^1 p(s)\,ds \right) \left( \int_0^\tau tF'_{(a,b)}(t)\,dt \right)\bigg|_0^1 - \int_0^1 \left( \int_0^\tau p(s)\,ds \right) \tau F'_{(a,b)}(\tau)\,d\tau
$$

and

$$
\int_0^1 p(\tau) \left( \int_\tau^1 (t-1)F'_{(a,b)}(t)\,dt \right)\,d\tau
= \int_0^1 \left( \int_\tau^1 (t-1)F'_{(a,b)}(t)\,dt \right)\,d \left( \int_0^\tau p(s)\,ds \right)
= \left( \int_0^1 (t-1)F'_{(a,b)}(t)\,dt \right) \left( \int_0^\tau p(s)\,ds \right)\bigg|_0^1
+ \int_0^1 \left( \int_0^\tau p(s)\,ds \right) (\tau - 1) F'_{(a,b)}(\tau)\,d\tau
= \int_0^1 \left( \int_0^\tau p(s)\,ds \right) (\tau - 1) F'_{(a,b)}(\tau)\,d\tau,
$$

that completes the proof.
which proves the identity

\[
(2.5) \quad \int_0^1 p(\tau) F_{(a,b)}(\tau) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 F_{(a,b)}(\tau) \, d\tau
= \int_0^1 \left[ \int_\tau^1 p(s) \, ds \right] \tau F'_{(a,b)}(\tau) \, d\tau + \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (\tau - 1) F'_{(a,b)}(\tau) \, d\tau.
\]

Now, observe that

\[
\int_0^1 \left[ \int_\tau^1 p(s) \, ds \right] \tau F'_{(a,b)}(\tau) \, d\tau + \int_0^1 \left( \int_0^\tau p(s) \, ds \right) (\tau - 1) F'_{(a,b)}(\tau) \, d\tau
= \int_0^1 \tau \left( \int_0^\tau p(s) \, ds \right) F'_{(a,b)}(\tau) \, d\tau - \int_0^1 (1 - \tau) \left( \int_0^\tau p(s) \, ds \right) F'_{(a,b)}(\tau) \, d\tau
= \int_0^1 (\tau - 1) \left( \int_0^\tau p(s) \, ds \, d\tau - \int_0^\tau p(s) \, ds \, \frac{1}{1 - \tau} \right) F'_{(a,b)}(\tau) \, d\tau
\]

and by (2.5) we obtain the desired equality (2.1). \qed

We have the following result:

**Theorem 5.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) and \( p : [0,1] \to [0, \infty) \) is Lebesgue integrable and such that

\[
(2.6) \quad \frac{1}{\tau} \int_0^\tau p(s) \, ds \leq \frac{1}{1 - \tau} \int_\tau^1 p(s) \, ds \text{ for all } \tau \in (0,1),
\]

then for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) we have the inequalities

\[
(2.7) \quad \left[ \int_0^1 \left( \tau - \frac{1}{2} \right) p(\tau) \, d\tau \right] Df(a)(b-a)
\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 f((1-\tau)a + \tau b) \, d\tau
\leq \left[ \int_0^1 \left( \tau - \frac{1}{2} \right) p(\tau) \, d\tau \right] Df(b)(b-a).
\]

**Proof.** By the properties of \( F_{(a,b)} \) from the above section, we have in the operator order that

\[
(2.8) \quad F'_{(a,b)}(1-) \geq F'_{(a,b)}(\tau) \geq F'_{(a,b)}(0+)
\]

for all \( \tau \in (0,1) \).

Since

\[
\frac{\int_\tau^1 p(s) \, ds}{1 - \tau} - \frac{\int_0^\tau p(s) \, ds}{t} \geq 0
\]
for all \( \tau \in (0, 1) \), hence
\[
\tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) F(b) \, (b - a)
\]
\[
\geq \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) F_{(a,b)}(\tau)
\]
\[
\geq \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) F(a) \, (b - a)
\]

for all \( \tau \in (0, 1) \).

By taking the integral in this inequality, we get
\[
\int_{0}^{1} \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) \, d\tau Df(b) \, (b - a)
\]
\[
\geq \int_{0}^{1} \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) F_{(a,b)}(\tau) \, d\tau
\]
\[
\geq \int_{0}^{1} \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) \, d\tau Df(a) \, (b - a).
\]

Since we also have
\[
0 \leq \int_{0}^{1} \tau (1 - \tau) \left( \int_{\tau}^{1} \frac{p(s) \, ds}{1 - \tau} - \int_{0}^{\tau} \frac{p(s) \, ds}{t} \right) \, d\tau
\]
\[
= \int_{0}^{1} \left( \tau \int_{\tau}^{1} p(s) \, ds - (1 - \tau) \int_{0}^{\tau} p(s) \, ds \right) \, d\tau
\]
\[
= \int_{0}^{1} \left( \tau \int_{\tau}^{1} p(s) \, ds + \tau \int_{0}^{\tau} p(s) \, ds - \int_{0}^{\tau} p(s) \, ds \right) \, d\tau
\]
\[
= \int_{0}^{1} \left( \tau \int_{0}^{1} p(s) \, ds - \tau \int_{0}^{\tau} p(s) \, ds \right) \, d\tau = \frac{1}{2} \int_{0}^{1} p(s) \, ds - \int_{0}^{1} \left( \tau \int_{0}^{\tau} p(s) \, ds \right) \, d\tau
\]
\[
= \frac{1}{2} \int_{0}^{1} p(s) \, ds - \left[ \left( \int_{0}^{\tau} p(s) \, ds \right) \bigg|_{0}^{1} - \int_{0}^{1} p(\tau) \, d\tau \right]
\]
\[
= \frac{1}{2} \int_{0}^{1} p(s) \, ds - \int_{0}^{1} p(s) \, ds + \int_{0}^{1} p(\tau) \, d\tau = \int_{0}^{1} p(\tau) \left( \tau - \frac{1}{2} \right) \, d\tau,
\]
then by employing Lemma 2 and the inequality (2.9) we obtain (2.7).

**Corollary 1.** Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \) and \( I \subset G \) a real interval. If the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \(*\)-algebra \( A \) and \( p : [0, 1] \to \mathbb{R} \) a monotonic nondecreasing function, then we have the inequalities (2.7).

**Proof.** If \( p : [0, 1] \to \mathbb{R} \) is a monotonic nondecreasing function, then
\[
\frac{1}{x} \int_{x}^{\infty} p(s) \, ds \leq p(x) \leq \frac{1}{1 - x} \int_{x}^{1} p(s) \, ds
\]
for \( x \in (0, 1) \). Then by applying Theorem 5 we get the desired result. 

\( \square \)
Corollary 2. With the assumptions of Theorem 5 and if $Df(a)(b-a) \geq 0$ in the Hermitian Banach *-algebra $A$, then

$$0 \leq \left[ \int_0^1 \left( \tau - \frac{1}{2} \right) p(\tau) \, d\tau \right] Df(a)(b-a)$$

$$\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau - \int_0^1 p(\tau) \, d\tau \int_0^1 f((1-\tau)a + \tau b) \, d\tau$$

$$\leq \left[ \int_0^1 \left( \tau - \frac{1}{2} \right) p(\tau) \, d\tau \right] Df(b)(b-a).$$

If $p : [0, 1] \to \mathbb{R}$ is asymmetric and Lebesgue integrable, then $\int_0^1 p(s) \, ds = 0$. If $\tau \in [0, 1]$ then $\int_0^\tau p(s) \, ds + \int_\tau^1 p(s) \, ds = 0$, which implies that

$$\int_0^\tau p(s) \, ds = -\int_0^\tau p(s) \, ds.$$

Corollary 3. Assume that $f(z)$ is analytic in $G$, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on $I$ in the Hermitian Banach *-algebra $A$ and $p : [0, 1] \to \mathbb{R}$ an asymmetric Lebesgue integrable function such that

$$\int_0^1 p(s) \, ds \leq 0 \text{ for all } \tau \in [0, 1],$$

or, equivalently,

$$0 \leq \int_0^1 p(s) \, ds \text{ for all } \tau \in [0, 1],$$

then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have the inequalities

$$\int_0^\tau \tau p(\tau) \, d\tau \leq \int_0^1 p(\tau) f((1-\tau)a + \tau b) \, d\tau$$

$$\leq \left( \int_0^1 \tau p(\tau) \, d\tau \right) Df(b)(b-a).$$

Proof. The condition

$$\frac{1}{\tau} \int_0^\tau p(s) \, ds \leq \frac{1}{1-\tau} \int_\tau^1 p(s) \, ds$$

for all $\tau \in (0, 1)$ is thus equivalent to

$$\frac{1}{\tau} \int_0^\tau p(s) \, ds \leq -\frac{1}{1-\tau} \int_0^\tau p(s) \, ds$$

namely

$$\frac{1}{\tau} \int_0^\tau p(s) \, ds + \frac{1}{1-\tau} \int_0^\tau p(s) \, ds \leq 0,$$

which is equivalent to (2.11).

By utilising (2.7) we derive the desired result (2.13).

If $q : [0, 1] \to \mathbb{R}$ is integrable, then the function $p(s) = q(s) - q(1-s)$ is asymmetric. By the condition (2.11) we have

$$\int_0^\tau [q(s) - q(1-s)] \, ds \leq 0$$
namely

\[(2.14) \quad \int_0^\tau q(s)\,ds \leq \int_0^\tau q(1 - s)\,ds, \quad \tau \in [0, 1].\]

If we put \(u = 1 - s\), then

\[\int_0^\tau q(1 - s)\,ds = \int_{1-\tau}^{1} q(s)\,ds\]

and we obtain

\[(2.15) \quad \int_0^\tau q(s)\,ds \leq \int_{1-\tau}^{1} q(s)\,ds, \quad \tau \in [0, 1].\]

We also have

\[
\int_0^1 \tau p(\tau)\,d\tau = \int_0^1 s[q(s) - q(1 - s)]\,ds
\]

\[= \int_0^1 sq(s)\,ds - \int_0^1 (1 - s)q(s)\,ds\]

\[= \int_0^1 (2s - 1)q(s)\,ds = 2\int_0^1 \left(s - \frac{1}{2}\right)q(s)\,ds\]

and, for an integrable function \(f : [0, 1] \to A\) we have

\[
\int_0^1 p(s) f(s)\,ds = \int_0^1 [q(s) - q(1 - s)] f(s)\,ds
\]

\[= \int_0^1 q(s) f(s)\,ds - \int_0^1 q(1 - s) f(s)\,ds\]

\[= \int_0^1 q(s) f(s)\,ds - \int_0^1 q(s) f(1 - s)\,ds\]

\[= \int_0^1 q(s) [f(s) - f(1 - s)]\,ds.\]

We can state:

**Corollary 4.** Assume that \(f(z)\) is analytic in \(G\), an open subset of \(\mathbb{C}\) and \(I \subseteq G\) a real interval. If the function \(f(z)\) is operator convex on \(I\) in the Hermitian Banach \(*\)-algebra \(A\) and \(q : [0, 1] \to \mathbb{R}\) is a Lebesgue integrable function such that (2.14) holds, then we have the inequalities

\[(2.16) \quad \left(\int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau\right) Df(a)(b - a)\]

\[\leq \frac{1}{2} \int_0^1 q(\tau)[f((1 - \tau)a + \tau b) - f(\tau a + (1 - \tau)b)]\,d\tau\]

\[\leq \left(\int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau\right) Df(b)(b - a)\]

for all \(a, b \in A\) with \(\sigma(a), \sigma(b) \subseteq I\).
3. Some Examples

Assume that \( f(z) \) is analytic in \( G \), an open subset of \( \mathbb{C} \), \( I \subset G \) a real interval and the function \( f(z) \) is operator convex on \( I \) in the Hermitian Banach \( * \)-algebra \( A \).

We consider the function \( p(\tau) = \tau, \tau \in [0,1] \). Observe that

\[
\int_0^1 \tau p(\tau) \, d\tau - \frac{1}{2} \int_0^1 p(\tau) \, d\tau = \int_0^1 \tau^2 \, d\tau - \frac{1}{2} \int_0^1 \tau \, d\tau = \frac{1}{12}.
\]

Then by (2.7) we get for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \) that

\[
\frac{1}{12} Df(a) (b - a) \
\leq \int_0^1 \tau f((1 - \tau) a + \tau b) \, d\tau - \frac{1}{2} \int_0^1 f((1 - \tau) a + \tau b) \, d\tau \\
\leq \frac{1}{12} Df(b) (b - a).
\]

For \( n \) a natural number, the function \( p(\tau) = (\tau - \frac{1}{2})^{2n+1} \), is increasing, then by (2.7)

\[
\left[ \int_0^1 \tau \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau - \frac{1}{2} \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau \right] Df(a) (b - a) \\
\leq \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} f((1 - \tau) a + \tau b) \, d\tau \\
- \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau \int_0^1 f((1 - \tau) a + \tau b) \, d\tau \\
\leq \left[ \int_0^1 \tau \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau - \frac{1}{2} \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau \right] Df(b) (b - a),
\]

for all \( a, b \in A \) with \( \sigma(a), \sigma(b) \subset I \).

Observe that

\[
\int_0^1 \tau \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau - \frac{1}{2} \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau \\
= \int_0^1 \left( \tau - \frac{1}{2} \right) \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau = \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+2} \, d\tau \\
= \frac{2}{2n+3} \left( \frac{1}{2} \right)^{2n+3} = \frac{1}{(2n+3)^{2n+2}}
\]

and

\[
\int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} \, d\tau = 0,
\]

which gives

\[
\frac{1}{(2n+3)^{2n+2}} Df(a) (b - a) \leq \int_0^1 \left( \tau - \frac{1}{2} \right)^{2n+1} f((1 - \tau) a + \tau b) \, d\tau \\
\leq \frac{1}{(2n+3)^{2n+2}} Df(b) (b - a)
\]
for all \(a, b \in A\) with \(\sigma (a), \sigma (b) \subset I\) while \(n\) is a natural number.

The function \(f (z) = z^{-1}\) is operator convex on \((0, \infty)\) in the Hermitian Banach *-algebra \(A\), [6], and we have for \(a, b > 0\) that

\[
Df (a) (b - a) = -a^{-1} (b - a) a^{-1} \quad \text{and} \quad Df (b) (b - a) = -b^{-1} (b - a) b^{-1}.
\]

By making use of the inequality (3.1) we get

\[
\frac{1}{12} a^{-1} (a - b) a^{-1} \leq \int_0^1 \tau ((1 - \tau) a + \tau b)^{-1} d\tau - \frac{1}{2} \int_0^1 ((1 - \tau) a + \tau b)^{-1} d\tau
\]

\[
\leq \frac{1}{12} b^{-1} (a - b) b^{-1},
\]

while from (3.2) we derive

\[
\frac{1}{(2n + 3) 2^{2n+2}} a^{-1} (a - b) a^{-1} \leq \int_0^1 \left( \tau - \frac{1}{2} \right) ^{2n+1} ((1 - \tau) a + \tau b)^{-1} d\tau
\]

\[
\leq \frac{1}{(2n + 3) 2^{2n+2}} b^{-1} (a - b) b^{-1}
\]

for all \(a, b \in A\) with \(a, b > 0\) and \(n\) is a natural number.

We observe that if \(a \geq b > 0\), then \(a^{-1} (a - b) a^{-1} \geq 0\) and by (3.4) we derive

\[
0 \leq \frac{1}{(2n + 3) 2^{2n+2}} a^{-1} (a - b) a^{-1}
\]

\[
\leq \int_0^1 \left( \tau - \frac{1}{2} \right) ^{2n+1} ((1 - \tau) a + \tau b)^{-1} d\tau \leq \frac{1}{(2n + 3) 2^{2n+2}} b^{-1} (a - b) b^{-1},
\]

where \(n\) is a natural number.

**References**


1*Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.*

*E-mail address: sever.dragomir@vu.edu.au*

*URL: http://rgmia.org/dragomir*

2*DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa*