SOME NEW INTEGRAL INEQUALITIES OF GRÜSS TYPE FOR OPERATOR CONVEX FUNCTIONS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we provide new *Grüss type inequalities* in the order of the Hermitian Banach $*$-algebra A for the *Čebyšev's difference*

$$\int_0^1 p(\tau) f((1-\tau)a+\tau b)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)a+\tau b)\, d\tau$$

in the case that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$, $I \subset G$ is a real interval and the function $f(z)$ is *operator convex* on I while $p:[0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau p(s)\, ds \leq \frac{1}{1-\tau}\int_\tau^1 p(s)\, ds \text{ for all } \tau \in (0,1)$$

and $a,\, b \in A$ with $\sigma(a),\, \sigma(b) \subset I$.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [11] (see also [1, Theorem 41.5]), then

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [10], Tanahashi and Uchiyama [12] proved the following fundamental properties (see also [8]):

(i) If $a,\, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a,\, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a,\, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;

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(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [10] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z) (z - a)^{-1} dz,$$

where γ is a close rectifiable curve such that $\sigma(a) \subset \operatorname{ins}(\gamma)$.

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [12, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $p(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq p(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq p(u)$ in the order of A.*

For some recent inequalities in Hermitian Banach $*$-algebras, see [3], [4] and [5].

Let G be an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$, then by SMT the element $(1-t)a + tb \in A$ has the spectrum $\sigma((1-t)a + tb) \subset I$ for all $t \in [0,1]$. We say that an analytic function $f(z)$ in G is *operator convex* on I in the Hermitian Banach $*$-algebra A if

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b) \text{ in the order of } A \tag{1.1}$$

for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ and all $t \in [0,1]$.

In the recent paper [6] we obtained the following results:

Theorem 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have*

$$f(b) - f(a) \geq Df(a)(b-a) \tag{1.2}$$

in the order of A, where Df is the Fréchet derivative of f as a function of elements in the Hermitian Banach $*$-algebra A.

Let $f(z)$ be analytic in G, an open convex subset of $\mathbb{C}$ and a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset G$. Consider the auxiliary function $F_{(a,b)} : [0,1] \to A$ defined by

$$F_{(a,b)}(t) := f((1-t)a + tb).$$

The following characterization result also holds:

Theorem 2. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. The function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A if and only if for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have for all t_2, $t_1 \in [0,1]$ with $t_1 < t_2$ that*

$$\begin{aligned} F'_{(a,b)}(t_2) &= Df((1-t_2)a + t_2 b)(b-a) \\ &\geq \frac{f((1-t_2)a + t_2 b) - f((1-t_1)a + t_1 b)}{t_2 - t_1} \\ &\geq Df((1-t_1)a + t_1 b)(b-a) = F'_{(a,b)}(t_1). \end{aligned} \tag{1.3}$$

We also have

$$Df(b)(b-a) \geq F'_{(a,b)}(t) = Df((1-t)a + tb)(b-a) \geq Df(a)(b-a) \tag{1.4}$$

for all $t \in (0,1)$.

It is well known that, if E is a Banach space and $p : [0,1] \to E$ is a continuous function, then p is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_0^1 p(t)\,dt$.

In the recent paper [7] we also obtained the following Féjer's type inequalities:

Theorem 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have*

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f((1-t)a + tb)\,dt - \left(\int_0^1 p(t)\,dt\right) f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\left(\int_0^1 \left|t - \frac{1}{2}\right| p(t)\,dt\right) [Df(b)(b-a) - Df(a)(b-a)]. \end{aligned} \tag{1.5}$$

In particular, for $p \equiv 1$ *we get*

$$0 \leq \int_0^1 f((1-t)a + tb)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.6}$$

We also have:

Theorem 4. *Assume that* $f(z)$ *is analytic in* G*, an open subset of* $\mathbb{C}$ *and* $I \subset G$ *a real interval. If the function* $f(z)$ *is operator convex on* I *in the Hermitian Banach* $*$*-algebra* A *and* $p : [0,1] \to [0,\infty)$ *is Lebesgue integrable and symmetric, namely* $p(1-t) = p(t)$ *for all* $t \in [0,1]$*, then for all* a*,* $b \in A$ *with* $\sigma(a)$*,* $\sigma(b) \subset I$ *we have*

$$0 \leq \left(\int_0^1 p(t)\,dt\right)\frac{f(a)+f(b)}{2} - \int_0^1 p(t) f((1-t)a+tb)\,dt \leq \frac{1}{2}\left(\int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right)p(t)\,dt\right)\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.7}$$

In particular, for $p \equiv 1$ *we get*

$$0 \leq \frac{f(a)+f(b)}{2} - \int_0^1 f((1-t)a+tb)\,dt \leq \frac{1}{8}\left[Df(b)(b-a) - Df(a)(b-a)\right]. \tag{1.8}$$

Motivated by the above results, in this paper we provide lower and upper bounds, so called *Grüss type inequalities*, in the order of the Hermitian Banach $*$-algebra A for the *Čebyšev's difference*

$$\int_0^1 p(\tau) f((1-\tau)a+\tau b)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)a+\tau b)\,d\tau$$

in the case that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$, $I \subset G$ a real interval and the function $f(z)$ is *operator convex* on I while $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds \leq \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau \in [0,1],$$

and a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 2. *Assume that* $f(z)$ *is analytic in* G*, an open subset of* $\mathbb{C}$ *and* $I \subset G$ *a real interval. If* $p : [0,1] \to [0,\infty)$ *is Lebesgue integrable, then for all* a*,* $b \in A$ *with* $\sigma(a)$*,* $\sigma(b) \subset I$ *we have for the auxiliary function* $F_{(a,b)}$ *that*

$$\int_0^1 p(\tau) F_{(a,b)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 F_{(a,b)}(\tau)\,d\tau = \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau}\right)F'_{(a,b)}(\tau)\,d\tau. \tag{2.1}$$

Proof. $F_{(a,b)}$ is obviously differentiable on $(0,1)$. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}
&\int_0^\tau tF'_{(a,b)}(t)\,dt+\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\\
&=\tau F_{(a,b)}(\tau)-\int_0^\tau F_{(a,b)}(t)\,dt-(\tau-1)\,F_{(a,b)}(\tau)-\int_\tau^1 F_{(a,b)}(t)\,dt\\
&=F_{(a,b)}(\tau)-\int_0^1 F_{(a,b)}(t)\,dt
\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $p(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}
&\int_0^1 p(\tau)\,F_{(a,b)}(\tau)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 F_{(a,b)}(t)\,dt\\
&=\int_0^1 p(\tau)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\tau+\int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\tau.
\end{aligned}\tag{2.2}$$

Using integration by parts, we get

$$\begin{aligned}
&\int_0^1 p(\tau)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\tau\\
&=\int_0^1\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)d\left(\int_0^\tau p(s)\,ds\right)\\
&=\left(\int_0^\tau p(s)\,ds\right)\left(\int_0^\tau tF'_{(a,b)}(t)\,dt\right)\Bigg|_0^1-\int_0^1\left(\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau\\
&=\left(\int_0^1 p(s)\,ds\right)\left(\int_0^1 tF'_{(a,b)}(t)\,dt\right)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau\\
&=\int_0^1\left(\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau\\
&=\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau F'_{(a,b)}(\tau)\,d\tau
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\int_0^1 p(\tau)\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\tau\\
&=\int_0^1\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)d\left(\int_0^\tau p(s)\,ds\right)\\
&=\left(\int_\tau^1 (t-1)\,F'_{(a,b)}(t)\,dt\right)\left(\int_0^\tau p(s)\,ds\right)\Bigg|_0^1\\
&+\int_0^1\left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,F'_{(a,b)}(\tau)\,d\tau\\
&=\int_0^1\left(\int_0^\tau p(s)\,ds\right)(\tau-1)\,F'_{(a,b)}(\tau)\,d\tau,
\end{aligned}\tag{2.4}$$

which proves the identity

$$\int_0^1 p(\tau) F_{(a,b)}(\tau) d\tau - \int_0^1 p(\tau) d\tau \int_0^1 F_{(a,b)}(\tau) d\tau \\ = \int_0^1 \left(\int_\tau^1 p(s) ds \right) \tau F'_{(a,b)}(\tau) d\tau + \int_0^1 \left(\int_0^\tau p(s) ds \right) (\tau - 1) F'_{(a,b)}(\tau) d\tau. \tag{2.5}$$

Now, observe that

$$\int_0^1 \left(\int_\tau^1 p(s) ds \right) \tau F'_{(a,b)}(\tau) d\tau + \int_0^1 \left(\int_0^\tau p(s) ds \right) (\tau - 1) F'_{(a,b)}(\tau) d\tau \\ = \int_0^1 \tau \left(\int_\tau^1 p(s) ds \right) F'_{(a,b)}(\tau) d\tau - \int_0^1 (1-\tau) \left(\int_0^\tau p(s) ds \right) F'_{(a,b)}(\tau) d\tau \\ = \int_0^1 \tau (1-\tau) \left(\frac{\int_\tau^1 p(s) ds}{1-\tau} - \frac{\int_0^\tau p(s) ds}{t} \right) F'_{(a,b)}(\tau) d\tau$$

and by (2.5) we obtain the desired equality (2.1). □

We have the following result:

Theorem 5. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p : [0,1] \to [0, \infty)$ is Lebesgue integrable and such that*

$$\frac{1}{\tau} \int_0^\tau p(s) ds \le \frac{1}{1-\tau} \int_\tau^1 p(s) ds \text{ for all } \tau \in (0,1), \tag{2.6}$$

then for all a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$ we have the inequalities

$$\left[\int_0^1 \left(\tau - \frac{1}{2} \right) p(\tau) d\tau \right] Df(a)(b-a) \\ \le \int_0^1 p(\tau) f((1-\tau) a + \tau b) d\tau - \int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau) a + \tau b) d\tau \\ \le \left[\int_0^1 \left(\tau - \frac{1}{2} \right) p(\tau) d\tau \right] Df(b)(b-a). \tag{2.7}$$

Proof. By the properties of $F_{(a,b)}$ from the above section, we have in the operator order that

$$F'_{(a,b)}(1-) \ge F'_{(a,b)}(\tau) \ge F'_{(a,b)}(0+) \tag{2.8}$$

for all $\tau \in (0,1)$.

Since

$$\frac{\int_\tau^1 p(s) ds}{1-\tau} - \frac{\int_0^\tau p(s) ds}{t} \ge 0$$

for all $\tau \in (0,1)$, hence

$$\tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)Df(b)(b-a)$$
$$\geq \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)F'_{(a,b)}(\tau)$$
$$\geq \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)Df(a)(b-a)$$

for all $\tau \in (0,1)$.

By taking the integral in this inequality, we get

$$\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau Df(b)(b-a) \tag{2.9}$$
$$\geq \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)F'_{(a,b)}(\tau)\,d\tau$$
$$\geq \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau Df(a)(b-a).$$

Since we also have

$$0 \leq \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right)d\tau$$
$$= \int_0^1 \left(\tau\int_\tau^1 p(s)\,ds-(1-\tau)\int_0^\tau p(s)\,ds\right)d\tau$$
$$= \int_0^1 \left(\tau\int_\tau^1 p(s)\,ds+\tau\int_0^\tau p(s)\,ds-\int_0^\tau p(s)\,ds\right)d\tau$$
$$= \int_0^1 \left(\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right)d\tau = \frac{1}{2}\int_0^1 p(s)\,ds-\int_0^1\left(\int_0^\tau p(s)\,ds\right)d\tau$$
$$= \frac{1}{2}\int_0^1 p(s)\,ds-\left[\left(\int_0^\tau p(s)\,ds\right)\tau\Big|_0^1-\int_0^1 p(\tau)\,\tau d\tau\right]$$
$$= \frac{1}{2}\int_0^1 p(s)\,ds-\int_0^1 p(s)\,ds+\int_0^1 p(\tau)\,\tau d\tau = \int_0^1 p(\tau)\left(\tau-\frac{1}{2}\right)d\tau,$$

then by employing Lemma 2 and the inequality (2.9) we obtain (2.7). □

Corollary 1. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p:[0,1]\to\mathbb{R}$ a monotonic nondecreasing function, then we have the inequalities (2.7).*

Proof. If $p:[0,1]\to\mathbb{R}$ is a monotonic nondecreasing function, then

$$\frac{1}{x}\int_0^x p(s)\,ds \leq p(x) \leq \frac{1}{1-x}\int_x^1 p(s)\,ds$$

for $x\in(0,1)$. Then by applying Theorem 5 we get the desired result. □

Corollary 2. *With the assumptions of Theorem 5 and if $Df(a)(b-a) \geq 0$ in the Hermitian Banach $*$-algebra A, then*

$$0 \leq \left[\int_0^1 \left(\tau - \frac{1}{2}\right) p(\tau)\, d\tau\right] Df(a)(b-a) \tag{2.10}$$
$$\leq \int_0^1 p(\tau) f((1-\tau)a + \tau b)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)a + \tau b)\, d\tau$$
$$\leq \left[\int_0^1 \left(\tau - \frac{1}{2}\right) p(\tau)\, d\tau\right] Df(b)(b-a).$$

If $p : [0,1] \to \mathbb{R}$ is *asymmetric* and Lebesgue integrable, then $\int_0^1 p(s)\, ds = 0$. If $\tau \in [0,1]$ then $\int_0^\tau p(s)\, ds + \int_\tau^1 p(s)\, ds = 0$, which implies that

$$\int_\tau^1 p(s)\, ds = -\int_0^\tau p(s)\, ds.$$

Corollary 3. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I \subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $p : [0,1] \to \mathbb{R}$ an asymmetric Lebesgue integrable function such that*

$$\int_0^\tau p(s)\, ds \leq 0 \text{ for all } \tau \in [0,1], \tag{2.11}$$

or, equivalently,

$$0 \leq \int_\tau^1 p(s)\, ds \text{ for all } \tau \in [0,1], \tag{2.12}$$

then for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ we have the inequalities

$$\left(\int_0^1 \tau p(\tau)\, d\tau\right) Df(a)(b-a) \leq \int_0^1 p(\tau) f((1-\tau)a + \tau b)\, d\tau \tag{2.13}$$
$$\leq \left(\int_0^1 \tau p(\tau)\, d\tau\right) Df(b)(b-a).$$

Proof. The condition

$$\frac{1}{\tau}\int_0^\tau p(s)\, ds \leq \frac{1}{1-\tau}\int_\tau^1 p(s)\, ds \text{ for all } \tau \in (0,1)$$

is thus equivalent to

$$\frac{1}{\tau}\int_0^\tau p(s)\, ds \leq -\frac{1}{1-\tau}\int_0^\tau p(s)\, ds$$

namely

$$\frac{1}{\tau}\int_0^\tau p(s)\, ds + \frac{1}{1-\tau}\int_0^\tau p(s)\, ds \leq 0,$$

which is equivalent to (2.11).

By utilising (2.7) we derive the desired result (2.13). □

If $q : [0,1] \to \mathbb{R}$ is integrable, then the function $p(s) = q(s) - q(1-s)$ is asymmetric. By the condition (2.11) we have

$$\int_0^\tau [q(s) - q(1-s)]\, ds \leq 0$$

namely

$$\int_0^{\tau} q(s)\,ds \le \int_0^{\tau} q(1-s)\,ds,\ \ \tau\in[0,1]. \tag{2.14}$$

If we put $u=1-s$, then

$$\int_0^{\tau} q(1-s)\,ds = \int_{1-\tau}^{1} q(s)\,ds$$

and we obtain

$$\int_0^{\tau} q(s)\,ds \le \int_{1-\tau}^{1} q(s)\,ds,\ \ \tau\in[0,1]. \tag{2.15}$$

We also have

$$\begin{aligned}\int_0^1 \tau p(\tau)\,d\tau &= \int_0^1 s\left[q(s)-q(1-s)\right]ds\\ &= \int_0^1 sq(s)\,ds - \int_0^1 (1-s)\,q(s)\,ds\\ &= \int_0^1 (2s-1)\,q(s)\,ds = 2\int_0^1 \left(s-\frac{1}{2}\right)q(s)\,ds\end{aligned}$$

and, for an integrable function $f:[0,1]\to A$ we have

$$\begin{aligned}\int_0^1 p(s)\,f(s)\,ds &= \int_0^1 \left[q(s)-q(1-s)\right]f(s)\,ds\\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(1-s)\,f(s)\,ds\\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(s)\,f(1-s)\,ds\\ &= \int_0^1 q(s)\left[f(s)-f(1-s)\right]ds.\end{aligned}$$

We can state:

Corollary 4. *Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$ and $I\subset G$ a real interval. If the function $f(z)$ is operator convex on I in the Hermitian Banach $*$-algebra A and $q:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that (2.14) holds, then we have the inequalities*

$$\begin{aligned}&\left(\int_0^1\left(\tau-\frac{1}{2}\right)q(\tau)\,d\tau\right)Df(a)(b-a)\\ &\le \frac{1}{2}\int_0^1 q(\tau)\left[f((1-\tau)a+\tau b)-f(\tau a+(1-\tau)b)\right]d\tau\\ &\le \left(\int_0^1\left(\tau-\frac{1}{2}\right)q(\tau)\,d\tau\right)Df(b)(b-a)\end{aligned} \tag{2.16}$$

for all $a, b\in A$ with $\sigma(a), \sigma(b)\subset I$.

3. Some Examples

Assume that $f(z)$ is analytic in G, an open subset of $\mathbb{C}$, $I \subset G$ a real interval and the function $f(z)$ is *operator convex* on I in the Hermitian Banach $*$-algebra A.

We consider the function $p(\tau) = \tau$, $\tau \in [0,1]$. Observe that

$$\int_0^1 \tau p(\tau)\, d\tau - \frac{1}{2}\int_0^1 p(\tau)\, d\tau = \int_0^1 \tau^2 d\tau - \frac{1}{2}\int_0^1 \tau d\tau = \frac{1}{12}.$$

Then by (2.7) we get for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$ that

$$\begin{aligned} &\frac{1}{12} Df(a)(b-a) \\ &\leq \int_0^1 \tau f((1-\tau)a + \tau b)\, d\tau - \frac{1}{2}\int_0^1 f((1-\tau)a + \tau b)\, d\tau \\ &\leq \frac{1}{12} Df(b)(b-a). \end{aligned} \tag{3.1}$$

For n a natural number, the function $p(\tau) = \left(\tau - \frac{1}{2}\right)^{2n+1}$, is increasing, then by (2.7)

$$\begin{aligned} &\left[\int_0^1 \tau \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau - \frac{1}{2}\int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau\right] Df(a)(b-a) \\ &\leq \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} f((1-\tau)a + \tau b)\, d\tau \\ &- \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau \int_0^1 f((1-\tau)a + \tau b)\, d\tau \\ &\leq \left[\int_0^1 \tau \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau - \frac{1}{2}\int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau\right] Df(b)(b-a), \end{aligned}$$

for all $a, b \in A$ with $\sigma(a), \sigma(b) \subset I$.

Observe that

$$\begin{aligned} &\int_0^1 \tau \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau - \frac{1}{2}\int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau \\ &= \int_0^1 \left(\tau - \frac{1}{2}\right)\left(\tau - \frac{1}{2}\right)^{2n+1} d\tau = \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+2} d\tau \\ &= \frac{2}{2n+3}\left(\frac{1}{2}\right)^{2n+3} = \frac{1}{(2n+3)\, 2^{2n+2}} \end{aligned}$$

and

$$\int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} d\tau = 0,$$

which gives

$$\begin{aligned} \frac{1}{(2n+3)\, 2^{2n+2}} Df(a)(b-a) &\leq \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1} f((1-\tau)a + \tau b)\, d\tau \\ &\leq \frac{1}{(2n+3)\, 2^{2n+2}} Df(b)(b-a) \end{aligned} \tag{3.2}$$

for all $a,\, b \in A$ with $\sigma(a)\,,\, \sigma(b) \subset I$ while n is a natural number.

The function $f(z) = z^{-1}$ is operator convex on $(0, \infty)$ in the Hermitian Banach $*$-algebra A, [6], and we have for $a,\, b > 0$ that

$$Df(a)(b-a) = -a^{-1}(b-a)\,a^{-1} \text{ and } Df(b)(b-a) = -b^{-1}(b-a)\,b^{-1}.$$

By making use of the inequality (3.1) we get

$$\begin{aligned} &\frac{1}{12}a^{-1}(a-b)\,a^{-1} \\ &\leq \int_0^1 \tau\left((1-\tau)\,a + \tau b\right)^{-1} d\tau - \frac{1}{2}\int_0^1 \left((1-\tau)\,a + \tau b\right)^{-1} d\tau \\ &\leq \frac{1}{12}b^{-1}(a-b)\,b^{-1}, \end{aligned} \tag{3.3}$$

while from (3.2) we derive

$$\begin{aligned} \frac{1}{(2n+3)\,2^{2n+2}}a^{-1}(a-b)\,a^{-1} &\leq \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1}\left((1-\tau)\,a + \tau b\right)^{-1} d\tau \\ &\leq \frac{1}{(2n+3)\,2^{2n+2}}b^{-1}(a-b)\,b^{-1} \end{aligned} \tag{3.4}$$

for all $a,\, b \in A$ with $a,\, b > 0$ and n is a natural number.

We observe that if $a \geq b > 0$, then $a^{-1}(a-b)\,a^{-1} \geq 0$ and by (3.4) we derive

$$\begin{aligned} 0 &\leq \frac{1}{(2n+3)\,2^{2n+2}}a^{-1}(a-b)\,a^{-1} \\ &\leq \int_0^1 \left(\tau - \frac{1}{2}\right)^{2n+1}\left((1-\tau)\,a + \tau b\right)^{-1} d\tau \leq \frac{1}{(2n+3)\,2^{2n+2}}b^{-1}(a-b)\,b^{-1}, \end{aligned} \tag{3.5}$$

where n is a natural number.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.

[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[3] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach ∗-algebras, *Oper. Matrices* **12** (2018), no. 4, 1009–1026. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 162. [http://rgmia.org/papers/v19/v19a162.pdf].

[4] S. S. Dragomir, Multiplicative inequalities for weighted geometric mean in Hermitian unital Banach *-algebras. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **112** (2018), no. 4, 1349–1365.

[5] S. S. Dragomir, Inequalities of Jensen's type for positive linear functionals on Hermitian unital Banach ∗-algebras, *Bull. Aust. Math. Soc.* (First published online 2020), page 1 of 11 doi:10.1017/S000497271900131X. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 142. 10 pp. [https://rgmia.org/papers/v19/v19a172.pdf].

[6] S. S. Dragomir, Inequalities of Hermite-Hadamard type for operator convex functions on Hermitian unital Banach ∗-algebras, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 17, 12 pp. [Online https://rgmia.org/papers/v23/v23a17.pdf].

[7] S. S. Dragomir, Inequalities of Féjer's type for operator convex functions on Hermitian unital Banach *-algebras, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 18, 12 pp. [Online https://rgmia.org/papers/v23/v23a18.pdf].

[8] B. Q. Feng, The geometric means in Banach ∗-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.

[9] G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, 1990.

[10] T. Okayasu, The Löwner-Heinz inequality in Banach $*$-algebra, *Glasgow Math. J.* **42** (2000), 243-246.
[11] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.
[12] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach $*$-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

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