GENERAL NORM INEQUALITIES OF TRAPEZOID TYPE FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s)\, ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x + ty)\, dt \right\|$$

in the case that $f : C \subset E \to F$ is *Fréchet differentiable* on the open and convex subset C of the Banach space E with values into another Banach space F, x, $y \in C$, $p : [0,1] \to \mathbb{C}$ is a Lebesgue integrable function and $\gamma \in \mathbb{C}$. Some particular cases of interest for different choices of γ are given. Applications for Banach algebras are also provided.

1. Introduction

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset if a normed vector space, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function f_u given by $t \mapsto f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative (directional derivative)* of f at a in the direction u. If $\nabla_a f(u)$ is defined and $\lambda \in \mathbb{R} \setminus \{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u) = \lambda \nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u.

Let E and F be normed vector spaces, and O be an open subset of F. A function $f : O \to F$ is called *Fréchet differentiable* at $x \in O$ if there exists a bounded linear operator $A : E \to F$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

If there exists such an operator A, it is unique, so we write $Df(x) = A$ and call it the *Fréchet derivative* of f at x.

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O \ni x \mapsto Df(x) \in \mathcal{B}(E, F)$ is continuous. A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a

1991 *Mathematics Subject Classification.* 46B20, 26D15, 47A63; 47A99.

Key words and phrases. Banach spaces, Norm inequalities, Midpoint inequalities, Banach algebras.

RGMIA Res. Rep. Coll. 23 (2020), Art. 26, 20 pp. Received 18/02/20

linear operation. If f is Fréchet differentiable at x, it is also Gâteaux differentiable there, and $\nabla_x f(u) = Df(x)(u)$ for all $u \in E$.

We say that the function $f : O \subset E \to F$ is *L-Lipschitzian* on O with the constant $L > 0$ if

$$\|f(x) - f(y)\| \le L \|x - y\| \text{ for all } x,\ y \in O.$$

In [13] we established among others the following *midpoint and trapezoid type inequalities* for L-Lipschitzian functions f on an open and convex subset C in E

$$\left\| \int_0^1 f((1-t)t + ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \le \frac{1}{4} L \|x - y\| \tag{1.1}$$

and

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)t + ty)\,dt \right\| \le \frac{1}{4} L \|x - y\| \tag{1.2}$$

for all x, $y \in C$. The constant $\frac{1}{4}$ is best possible in both inequalities (1.1) and (1.2).

For *Hermite-Hadamard's type inequalities*, namely

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(u)\,du \le \frac{f(a) + f(b)}{2}, \tag{1.3}$$

where $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$, see for instance [6], [7], [8], [20], [22], [23], [24], [26], [27], [28], [29], [30], [31], [32] and the references therein.

In the recent paper [14] we obtained among others the following weighted version of the trapezoid inequality (1.2).

Theorem 1. *Let $f : C \subset E \to F$ be a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$, namely $p(1-t) = p(t)$ for all $t \in [0,1]$. Then for all x, $y \in C$*

$$\begin{aligned}
&\left\| \left(\int_0^1 p(t)\,dt\right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty)\,dt \right\| \\
&\le \int_0^{1/2} \left(\int_t^{1/2} p(s)\,ds\right) \|Df((1-t)x + ty)(y-x)\|\,dt \\
&+ \int_{1/2}^1 \left(\int_{1/2}^t p(s)\,ds\right) \|Df((1-t)x + ty)(y-x)\|\,dt \\
&= \int_0^1 \left| \int_t^{1/2} p(s)\,ds \right| \|Df((1-t)x + ty)(y-x)\|\,dt \\
&=: B(f,p,x,y).
\end{aligned} \tag{1.4}$$

Moreover, we have the upper bounds

$$\begin{aligned}
B(f,p,x,y) &\le \frac{1}{2}\left(\int_0^1 p(s)\,ds\right) \int_0^1 \|Df((1-t)x + ty)(y-x)\|\,dt \\
&\le \frac{1}{2}\left(\int_0^1 p(s)\,ds\right) \sup_{t\in[0,1]} \|Df((1-t)x + ty)(y-x)\|,
\end{aligned} \tag{1.5}$$

$$B(f,p,x,y) \le \left(\frac{1}{2}\int_0^1 p(s)\,ds - \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right) tp(t)\,dt \right) \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \le \frac{1}{2}\left(\int_0^1 p(s)\,ds\right) \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \tag{1.6}$$

and

$$B(f,p,x,y) \le \left[\int_0^{1/2} \left(\int_t^{1/2} p(s)\,ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s)\,ds \right)^r dt \right]^{1/r} \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \le \left[\int_0^{1/2} \left(\int_t^{1/2} p(s)\,ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s)\,ds \right)^r dt \right]^{1/r} \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \tag{1.7}$$

for $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$.

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s)\,ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x+ty)\,dt \right\|$$

in the case that $f : C \subset E \to F$ is *Fréchet differentiable* on the open and convex subset C of the Banach space E with values into another Banach space F, $x,\ y \in C$, $p : [0,1] \to \mathbb{C}$ is a Lebesgue integrable function and $\gamma \in \mathbb{C}$. Some particular cases of interest for different choices of γ are given. Applications for Banach algebras are also provided.

2. Some Identities of Interest

Consider a function $f : C \subset E \to F$ that is defined on the open and convex set C. We have the following properties for the *auxiliary function*

$$\varphi_{(x,y)}(t) := f((1-t)x+ty),\ t\in[0,1],$$

where $x,\ y \in C$.

Lemma 1. *Assume that the function* $f : C \subset E \to F$ *is Fréchet differentiable on the open and convex set* C. *Then for all* $x,\ y \in C$ *the auxiliary function* $\varphi_{(x,y)}$ *is differentiable on* $(0,1)$ *and*

$$\varphi'_{(x,y)}(t) = Df((1-t)x+ty)(y-x). \tag{2.1}$$

Also

$$\varphi'_{(x,y)}(0+) = Df(x)(y-x) \tag{2.2}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y-x). \tag{2.3}$$

Proof. Let $t\in(0,1)$ and $h\neq 0$ small enough such that $t+h\in(0,1)$. Then

$$\begin{aligned}(2.4)\qquad & \frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h}\\ &=\frac{f((1-t-h)x+(t+h)y)-f((1-t)x+ty)}{h}\\ &=\frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h}.\end{aligned}$$

Since f is Fréchet differentiable, hence by taking the limit over $h\to 0$ in (2.4) we get

$$\begin{aligned}\varphi'_{(x,y)}(t)&=\lim_{h\to 0}\frac{\varphi_{(x,y)}(t+h)-\varphi_{(x,y)}(t)}{h}\\ &=\lim_{h\to 0}\frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h}\\ &=Df((1-t)x+ty)(y-x),\end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned}\varphi'_{(x,y)}(0+)&=\lim_{h\to 0+}\frac{\varphi_{(x,y)}(h)-\varphi_{(x,y)}(0)}{h}\\ &=\lim_{h\to 0+}\frac{f((1-h)x+hy)-f(x)}{h}\\ &=\lim_{h\to 0+}\frac{f(x+h(y-x))-f(x)}{h}=Df(x)(y-x)\end{aligned}$$

since f is assumed to be Fréchet differentiable in x. This proves (2.2).

The equality (2.3) follows in a similar way. □

We have the following identity for the Riemann-Stieltjes integral:

Lemma 2. *Assume that the function $f:C\subset E\to F$ is Fréchet differentiable on the open and convex set C. Let $u:[0,1]\to\mathbb{C}$ be of bounded variation on $[0,1]$ and $s\in[0,1]$. Then for all $x,\ y\in C$ and any $\gamma,\ \mu\in\mathbb{C}$,*

$$\begin{aligned}(2.5)\qquad & [u(1)-\mu]\varphi_{(x,y)}(1)+[\gamma-u(0)]\varphi_{(x,y)}(0)+(\mu-\gamma)\varphi_{(x,y)}(s)\\ &-\int_0^1\varphi_{(x,y)}(t)\,du(t)\\ &=\int_0^s[u(t)-\gamma]\varphi'_{(x,y)}(t)\,dt+\int_s^1[u(t)-\mu]\varphi'_{(x,y)}(t)\,dt.\end{aligned}$$

In particular, for $\mu=\gamma$ we have

$$\begin{aligned}(2.6)\qquad & [u(1)-\gamma]\varphi_{(x,y)}(1)+[\gamma-u(0)]\varphi_{(x,y)}(0)-\int_0^1\varphi_{(x,y)}(t)\,du(t)\\ &=\int_0^1[u(t)-\gamma]\varphi'_{(x,y)}(t)\,dt.\end{aligned}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_0^s [u(t)-\gamma]\varphi'_{(x,y)}(t)\,dt = [u(s)-\gamma]\varphi_{(x,y)}(s) - [u(0)-\gamma]\varphi_{(x,y)}(0) - \int_0^s \varphi_{(x,y)}(t)\,du(t)$$

and

$$\int_s^1 [u(t)-\mu]\varphi'_{(x,y)}(t)\,dt = [u(1)-\mu]\varphi_{(x,y)}(1) - [u(s)-\mu]\varphi_{(x,y)}(s) - \int_s^1 \varphi_{(x,y)}(t)\,du(t)$$

for any $s\in[0,1]$.

If we add these two equalities, then we get

$$\begin{aligned}
&\int_0^s [u(t)-\gamma]\varphi'_{(x,y)}(t)\,dt + \int_s^1 [u(t)-\mu]\varphi'_{(x,y)}(t)\,dt \\
&= [u(1)-\mu]\varphi_{(x,y)}(1) + [\gamma-u(0)]\varphi_{(x,y)}(0) + [\mu-u(s)]\varphi_{(x,y)}(s) \\
&+ [u(s)-\gamma]\varphi_{(x,y)}(s) - \int_0^s \varphi_{(x,y)}(t)\,du(t) - \int_s^1 \varphi_{(x,y)}(t)\,du(t) \\
&= [u(1)-\mu]\varphi_{(x,y)}(1) + [\gamma-u(0)]\varphi_{(x,y)}(0) + (\mu-\gamma)\varphi_{(x,y)}(s) \\
&- \int_0^1 \varphi_{(x,y)}(t)\,du(t)
\end{aligned}$$

for any $s\in[0,1]$, which proves the desired equality (2.5). □

Remark 1. *From the equality (2.6) we have for $s\in[0,1]$ and $\gamma=u(s)$ that*

$$(2.7)\quad [u(1)-u(s)]\varphi_{(x,y)}(1) + [u(s)-u(0)]\varphi_{(x,y)}(0) - \int_0^1 \varphi_{(x,y)}(t)\,du(t) = \int_0^1 [u(t)-u(s)]\varphi'_{(x,y)}(t)\,dt$$

and, in particular

$$(2.8)\quad \left[u(1)-u\left(\frac{1}{2}\right)\right]\varphi_{(x,y)}(1) + \left[u\left(\frac{1}{2}\right)-u(0)\right]\varphi_{(x,y)}(0) - \int_0^1 \varphi_{(x,y)}(t)\,du(t) = \int_0^1 \left[u(t)-u\left(\frac{1}{2}\right)\right]\varphi'_{(x,y)}(t)\,dt.$$

Also, if $m\in[0,1]$ is such that $u(m)=\frac{u(0)+u(1)}{2}$, then from (2.7) we get

$$(2.9)\quad [u(1)-u(0)]\frac{\varphi_{(x,y)}(1)+\varphi_{(x,y)}(0)}{2} - \int_0^1 \varphi_{(x,y)}(t)\,du(t) = \int_0^1 [u(t)-u(m)]\varphi'_{(x,y)}(t)\,dt.$$

Now, if we take $\gamma = (1-\alpha) u(0) + \alpha u(1)$, $\alpha \in [0,1]$ *in (2.6), then we get*

$$[u(1) - u(0)] \left[(1-\alpha) \varphi_{(x,y)}(1) + \alpha \varphi_{(x,y)}(0) \right] - \int_0^1 \varphi_{(x,y)}(t)\, du(t) = \int_0^1 [u(t) - (1-\alpha) u(0) - \alpha u(1)] \varphi'_{(x,y)}(t)\, dt \tag{2.10}$$

and, in particular

$$[u(1) - u(0)] \frac{\varphi_{(x,y)}(1) + \varphi_{(x,y)}(0)}{2} - \int_0^1 \varphi_{(x,y)}(t)\, du(t) = \int_0^1 \left[u(t) - \frac{u(0) + u(1)}{2} \right] \varphi'_{(x,y)}(t)\, dt. \tag{2.11}$$

The case of weighted integrals is as follows:

Corollary 1. *Assume that the function* $f : C \subset E \to F$ *is Fréchet differentiable on the open and convex set* C. *Let* $p : [0,1] \to \mathbb{C}$ *be Lebesgue integrable on* $[0,1]$ *and* $s \in [0,1]$. *Then for all* x, $y \in C$ *and any* γ, $\mu \in \mathbb{C}$,

$$\begin{aligned} &\left(\int_0^1 p(s)\, ds - \mu \right) \varphi_{(x,y)}(1) + \gamma \varphi_{(x,y)}(0) + (\mu - \gamma) \varphi_{(x,y)}(s) \\ &- \int_0^1 p(t) \varphi_{(x,y)}(t)\, dt \\ &= \int_0^s \left(\int_0^t p(\tau)\, d\tau - \gamma \right) \varphi'_{(x,y)}(t)\, dt + \int_s^1 \left(\int_0^t p(\tau)\, d\tau - \mu \right) \varphi'_{(x,y)}(t)\, dt. \end{aligned} \tag{2.12}$$

In particular, for $\mu = \gamma$ *we have*

$$\left(\int_0^1 p(s)\, ds - \gamma \right) \varphi_{(x,y)}(1) + \gamma \varphi_{(x,y)}(0) - \int_0^1 p(t) \varphi_{(x,y)}(t)\, dt = \int_0^1 \left(\int_0^t p(\tau)\, d\tau - \gamma \right) \varphi'_{(x,y)}(t)\, dt. \tag{2.13}$$

The proof follows by Lemma 2 applied for the function $u : [0,1] \to \mathbb{C}$, $u(t) = \int_0^t p(s)\, ds$ that is absolutely continuous on $[0,1]$ and therefore of bounded variation and

$$\int_0^1 \varphi_{(x,y)}(t)\, du(t) = \int_0^1 p(t) \varphi_{(x,y)}(t)\, dt.$$

Remark 2. *With the assumptions of Corollary 1 and by utilising Remark 1 we get*

$$\left(\int_s^1 p(\tau)\, d\tau \right) \varphi_{(x,y)}(1) + \left(\int_0^s p(\tau)\, d\tau \right) \varphi_{(x,y)}(0) - \int_0^1 p(t) \varphi_{(x,y)}(t)\, dt = \int_0^1 \left(\int_s^t p(\tau)\, d\tau \right) \varphi'_{(x,y)}(t)\, dt \tag{2.14}$$

and, in particular

$$\left(\int_{1/2}^{1} p(\tau)\,d\tau\right)\varphi_{(x,y)}(1)+\left(\int_{0}^{1/2} p(\tau)\,d\tau\right)\varphi_{(x,y)}(0) \tag{2.15}$$
$$-\int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt$$
$$=\int_0^1\left(\int_{1/2}^{t} p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt.$$

Also, if $m\in[0,1]$ is such that $\int_0^m p(\tau)\,d\tau=\frac{1}{2}\int_0^1 p(\tau)\,d\tau$, then

$$\frac{\varphi_{(x,y)}(1)+\varphi_{(x,y)}(0)}{2}\int_0^1 p(\tau)\,d\tau-\int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt \tag{2.16}$$
$$=\int_0^1\left(\int_m^t p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt.$$

Now, for $\alpha\in[0,1]$ we get

$$\left[(1-\alpha)\,\varphi_{(x,y)}(1)+\alpha\varphi_{(x,y)}(0)\right]\int_0^1 p(\tau)\,d\tau-\int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt \tag{2.17}$$
$$=\int_0^1\left(\int_0^t p(\tau)\,d\tau-\alpha\int_0^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt$$

and, in particular

$$\frac{\varphi_{(x,y)}(1)+\varphi_{(x,y)}(0)}{2}\int_0^1 p(\tau)\,d\tau-\int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt \tag{2.18}$$
$$=\int_0^1\left(\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt.$$

3. Main Results

We have:

Theorem 2. *Assume that the function $f:C\subset E\to F$ is Fréchet differentiable on the open and convex set C. Let $p:[0,1]\to\mathbb{C}$ be Lebesgue integrable on $[0,1]$ and $s\in[0,1]$. Then for all x, $y\in C$ and any γ, $\mu\in\mathbb{C}$,*

$$\left\|\left(\int_0^1 p(s)\,ds-\mu\right)f(y)+\gamma f(x)+(\mu-\gamma)\,f((1-s)\,x+sy)\right. \tag{3.1}$$
$$\left.-\int_0^1 p(t)\,f((1-t)\,x+ty)\,dt\right\|$$
$$\le\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|\|Df((1-t)\,x+ty)(y-x)\|\,dt$$
$$+\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|\|Df((1-t)\,x+ty)(y-x)\|\,dt$$
$$=:B(f,p,x,y,\gamma,\mu).$$

In particular, for $\mu = \gamma$ we have

$$\begin{aligned}(3.2)\qquad & \left\| \left(\int_0^1 p(s)\,ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x + ty)\,dt \right\| \\ & \le \int_0^1 \left| \int_0^t p(\tau)\,d\tau - \gamma \right| \| Df((1-t)x+ty)(y-x) \| \,dt \\ & =: B(f,p,x,y,\gamma).\end{aligned}$$

Moreover, we have the upper bounds

$$(3.3)\qquad B(f,p,x,y,\gamma,\mu) \le \begin{cases} \max\left\{ \sup_{t\in[0,s]} \left| \int_0^t p(\tau)\,d\tau - \gamma \right|, \sup_{t\in[s,1]} \left| \int_0^t p(\tau)\,d\tau - \mu \right| \right\} \\ \times \left[\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \right] \\ \\ \left[\left(\int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right|^p dt \right) + \left(\int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right|^p dt \right) \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left[\int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right| dt + \int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right| dt \right] \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \end{cases}$$

and

$$(3.4)\qquad B(f,p,x,y,\gamma) \le \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau)\,d\tau - \gamma \right| \left[\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \right] \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau)\,d\tau - \gamma \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \quad p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau)\,d\tau - \gamma \right| dt \right] \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

Proof. We have by (2.12) that

$$\begin{aligned}(3.5)\qquad & \left(\int_0^1 p(s)\,ds - \mu \right) f(y) + \gamma f(x) + (\mu-\gamma) f((1-s)x+sy) \\ & - \int_0^1 p(t) f((1-t)x+ty)\,dt \\ & = \int_0^s \left(\int_0^t p(\tau)\,d\tau - \gamma \right) Df((1-t)x+ty)(y-x)\,dt \\ & + \int_s^1 \left(\int_0^t p(\tau)\,d\tau - \mu \right) Df((1-t)x+ty)(y-x)\,dt\end{aligned}$$

for all $x,\ y \in C$ and any $\gamma,\ \mu \in \mathbb{C}$.

By taking the norm in (3.5), we get

$$
\begin{aligned}
&\left\| \left(\int_0^1 p(s)\,ds - \mu \right) f(y) + \gamma f(x) + (\mu - \gamma) f((1-s)x + sy) \right. \\
&\left. - \int_0^1 p(t) f((1-t)x + ty)\,dt \right\| \\
&\le \left\| \int_0^s \left(\int_0^t p(\tau)\,d\tau - \gamma \right) Df((1-t)x + ty)(y-x)\,dt \right\| \\
&+ \left\| \int_s^1 \left(\int_0^t p(\tau)\,d\tau - \mu \right) Df((1-t)x + ty)(y-x)\,dt \right\| \\
&\le \int_0^s \left\| \left(\int_0^t p(\tau)\,d\tau - \gamma \right) Df((1-t)x + ty)(y-x) \right\| dt \\
&+ \int_s^1 \left\| \left(\int_0^t p(\tau)\,d\tau - \mu \right) Df((1-t)x + ty)(y-x) \right\| dt \\
&= \int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right| \| Df((1-t)x + ty)(y-x) \|\,dt \\
&+ \int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right| \| Df((1-t)x + ty)(y-x) \|\,dt \\
&= B(f, p, x, y, \gamma, \mu).
\end{aligned}
\tag{3.6}
$$

By using Hölder's inequality we have

$$
\begin{aligned}
&\int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right| \| Df((1-t)x + ty)(y-x) \|\,dt \\
&\le \begin{cases}
\sup_{t\in[0,s]} \left| \int_0^t p(\tau)\,d\tau - \gamma \right| \int_0^s \| Df((1-t)x + ty)(y-x) \|\,dt \\[2ex]
\left(\int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right|^p dt \right)^{1/p} \left(\int_0^s \| Df((1-t)x + ty)(y-x) \|^q\,dt \right)^{1/q} \\
p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex]
\sup_{t\in[0,s]} \| Df((1-t)x + ty)(y-x) \| \int_0^s \left| \int_0^t p(\tau)\,d\tau - \gamma \right| dt
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right| \| Df((1-t)x + ty)(y-x) \|\,dt \\
&\le \begin{cases}
\sup_{t\in[s,1]} \left| \int_0^t p(\tau)\,d\tau - \mu \right| \int_s^1 \| Df((1-t)x + ty)(y-x) \|\,dt \\[2ex]
\left(\int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right|^p dt \right)^{1/p} \left(\int_s^1 \| Df((1-t)x + ty)(y-x) \|^q\,dt \right)^{1/q} \\
p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex]
\sup_{t\in[s,1]} \| Df((1-t)x + ty)(y-x) \| \int_s^1 \left| \int_0^t p(\tau)\,d\tau - \mu \right| dt.
\end{cases}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&B(f,p,x,y,\gamma,\mu)\\
&\leq\begin{cases}
\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau-\gamma\right|\int_0^s\left\|Df((1-t)x+ty)(y-x)\right\|dt\\[2ex]
\left(\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|^p dt\right)^{1/p}\left(\int_0^s\left\|Df((1-t)x+ty)(y-x)\right\|^q dt\right)^{1/q}\\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\sup_{t\in[0,s]}\left\|Df((1-t)x+ty)(y-x)\right\|\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|dt
\end{cases}\\
&+\begin{cases}
\sup_{t\in[s,1]}\left|\int_0^t p(\tau)\,d\tau-\mu\right|\int_s^1\left\|Df((1-t)x+ty)(y-x)\right\|dt\\[2ex]
\left(\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|^p dt\right)^{1/p}\left(\int_s^1\left\|Df((1-t)x+ty)(y-x)\right\|^q dt\right)^{1/q}\\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\sup_{t\in[s,1]}\left\|Df((1-t)x+ty)(y-x)\right\|\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|dt.
\end{cases}\\
&\leq\begin{cases}
\max\left\{\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau-\gamma\right|,\sup_{t\in[s,1]}\left|\int_0^t p(\tau)\,d\tau-\mu\right|\right\}\\
\times\left[\int_0^s\left\|Df((1-t)x+ty)(y-x)\right\|dt+\int_s^1\left\|Df((1-t)x+ty)(y-x)\right\|dt\right]\\[2ex]
\left[\left(\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|^p dt\right)+\left(\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|^p dt\right)\right]^{1/p}\\
\times\left(\int_0^s\left\|Df((1-t)x+ty)(y-x)\right\|^q dt+\int_s^1\left\|Df((1-t)x+ty)(y-x)\right\|^q dt\right)^{1/q}\\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\max\left\{\sup_{t\in[0,s]}\left\|Df((1-t)x+ty)(y-x)\right\|,\right.\\
\left.\sup_{t\in[s,1]}\left\|Df((1-t)x+ty)(y-x)\right\|\right\}\\
\times\left[\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|dt+\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|dt\right]
\end{cases}\\
&=\begin{cases}
\max\left\{\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau-\gamma\right|,\sup_{t\in[s,1]}\left|\int_0^t p(\tau)\,d\tau-\mu\right|\right\}\\
\times\left[\int_0^1\left\|Df((1-t)x+ty)(y-x)\right\|dt\right]\\[2ex]
\left[\left(\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|^p dt\right)+\left(\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|^p dt\right)\right]^{1/p}\\
\times\left(\int_0^1\left\|Df((1-t)x+ty)(y-x)\right\|^q dt\right)^{1/q}\\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\left[\int_0^s\left|\int_0^t p(\tau)\,d\tau-\gamma\right|dt+\int_s^1\left|\int_0^t p(\tau)\,d\tau-\mu\right|dt\right]\\
\times\sup_{t\in[0,1]}\left\|Df((1-t)x+ty)(y-x)\right\|,
\end{cases}
\end{aligned}
$$

which proves the inequality (3.2). □

Corollary 2. *Assume that the function $f : C \subset E \to F$ is Fréchet differentiable on the open and convex set C. Let $p : [0,1] \to \mathbb{C}$ be Lebesgue integrable on $[0,1]$ and $s \in [0,1]$. Then for all $x,\ y \in C$ we have*

$$\left\| \left(\int_s^1 p(\tau)\,d\tau \right) f(y) + \left(\int_0^s p(\tau)\,d\tau \right) f(x) - \int_0^1 p(t) f((1-t)x+ty)\,dt \right\| \tag{3.7}$$
$$\leq \begin{cases} \sup_{t\in[0,1]} \left| \int_s^t p(\tau)\,d\tau \right| \left[\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \right] \\ \\ \left[\int_0^1 \left| \int_s^t p(\tau)\,d\tau \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \quad p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left[\int_0^1 \left| \int_s^t p(\tau)\,d\tau \right| dt \right] \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

In particular,

$$\left\| \left(\int_{\frac{1}{2}}^1 p(\tau)\,d\tau \right) f(y) + \left(\int_0^{\frac{1}{2}} p(\tau)\,d\tau \right) f(x) \right. \tag{3.8}$$
$$\left. - \int_0^1 p(t) f((1-t)x+ty)\,dt \right\|$$
$$\leq \begin{cases} \sup_{t\in[0,1]} \left| \int_{1/2}^t p(\tau)\,d\tau \right| \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau)\,d\tau \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau)\,d\tau \right| dt \right) \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

The proof follows by Theorem 2 on choosing $\gamma = \int_0^s p(\tau)\,d\tau,\ s \in [0,1]$.

Corollary 3. *Assume that the function $f : C \subset E \to F$ is Fréchet differentiable on the open and convex set C. Let $p : [0,1] \to \mathbb{C}$ be Lebesgue integrable on $[0,1]$. Then for all $x,\ y \in C$*

$$\left\| \left(\int_0^1 p(\tau)\,d\tau \right) [(1-\alpha) f(y) + \alpha f(x)] - \int_0^1 p(t) f((1-t)x+ty)\,dtdt \right\| \tag{3.9}$$
$$\leq \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau)\,d\tau - \alpha \int_0^1 p(\tau)\,d\tau \right| \\ \times \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau)\,d\tau - \alpha \int_0^1 p(\tau)\,d\tau \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q} \quad p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left| \int_0^t p(\tau)\,d\tau - \alpha \int_0^1 p(\tau)\,d\tau \right| dt \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \end{cases}$$

for all $\alpha \in [0,1]$.

In particular, we have the trapezoid type inequalities

$$\left\|\left(\int_0^1 p(\tau)\,d\tau\right)\frac{f(y)+f(x)}{2}-\int_0^1 p(t)\,f((1-t)x+ty)\,dtdt\right\| \tag{3.10}$$

$$\leq \begin{cases} \sup_{t\in[0,1]}\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right| \\ \times\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left[\int_0^1\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right|^p dt\right]^{1/p} \\ \times\left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q} \quad p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right|dt \\ \times\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

Remark 3. *Since the Fréchet derivative satisfies the condition*

$$\|Df(a)(b)\| \leq \|Df(a)\|\,\|b\|$$

for $a\in C$ *and* $b\in E$, *then for all* $x,\ y\in C$ *we also have the chain of inequalities*

$$\left\|\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\| \tag{3.11}$$

$$\leq \|y-x\|\times\begin{cases} \sup_{t\in[0,1]}\left|\int_s^t p(\tau)\,d\tau\right|\left[\int_0^1\|Df((1-t)x+ty)\|\,dt\right] \\ \\ \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right|^p dt\right]^{1/p} \\ \times\left(\int_0^1\|Df((1-t)x+ty)\|^q\,dt\right)^{1/q} \quad p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right|dt\right]\sup_{t\in[0,1]}\|Df((1-t)x+ty)\|. \end{cases}$$

In particular,

$$\left\|\left(\int_{\frac{1}{2}}^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^{\frac{1}{2}} p(\tau)\,d\tau\right)f(x)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\| \tag{3.12}$$

$$\leq \|y-x\|\times\begin{cases} \sup_{t\in[0,1]}\left|\int_{1/2}^t p(\tau)\,d\tau\right|\int_0^1\|Df((1-t)x+ty)\|\,dt \\ \\ \left[\int_0^1\left|\int_{1/2}^t p(\tau)\,d\tau\right|^p dt\right]^{1/p} \\ \times\left(\int_0^1\|Df((1-t)x+ty)\|^q\,dt\right)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^1\left|\int_{1/2}^t p(\tau)\,d\tau\right|dt\right)\sup_{t\in[0,1]}\|Df((1-t)x+ty)\|. \end{cases}$$

If $\alpha \in [0,1]$, then for all $x, y \in C$ we also have the chain of inequalities

$$\left\| \left(\int_0^1 p(\tau) d\tau \right) [(1-\alpha) f(y) + \alpha f(x)] - \int_0^1 p(t) f((1-t)x + ty) dtdt \right\| \tag{3.13}$$

$$\leq \|y-x\| \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x+ty)\| dt \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)\|^q dt \right)^{1/q}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|. \end{cases}$$

In particular, we have the trapezoid type inequalities

$$\left\| \left(\int_0^1 p(\tau) d\tau \right) \frac{f(y)+f(x)}{2} - \int_0^1 p(t) f((1-t)x + ty) dtdt \right\| \tag{3.14}$$

$$\leq \|y-x\| \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x+ty)\| dt \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right|^p dt \right]^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)\|^q dt \right)^{1/q} \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|. \end{cases}$$

4. Some Examples for Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [12].

Lemma 3. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$ we have*

$$\|f(y)-f(x)\|\le\|y-x\|\int_0^1 f_a'(\|(1-s)x+sy\|)\,ds. \tag{4.1}$$

We also have:

Lemma 4. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$ we have*

$$\|Df((1-t)x+ty)(y-x)\|\le\|y-x\|\,f_a'(\|(1-t)x+ty\|) \tag{4.2}$$

for all $t\in[0,1]$.

Proof. Let $\|u\|,\ \|v\|<R$. Then there exists $\delta>0$ such that $\|u+\varepsilon v\|<R$ for all $\varepsilon\in(-\delta,\delta)$ and by (4.1) we get

$$\|f(u+\varepsilon v)-f(u)\|\le\|u+\varepsilon v-u\|\int_0^1 f_a'(\|(1-s)u+s(u+\varepsilon v)\|)\,ds$$

namely

$$\begin{aligned}\left\|\frac{f(u+\varepsilon v)-f(u)}{\varepsilon}\right\|&\le\|v\|\int_0^1 f_a'(\|(1-s)u+s(u+\varepsilon v)\|)\,ds,\ \ \varepsilon\neq0\\&=\|v\|\int_0^1 f_a'(\|u+s\varepsilon v\|)\,ds\end{aligned}$$

and by taking $\varepsilon\to0$ we get, by the property of integral, that

$$\|Df(u)(v)\|\le\|v\|\,f_a'(\|u\|). \tag{4.3}$$

Now, if we take in (4.3) $u=(1-t)x+ty$ and $v=y-x$, then we get (4.2). □

We have the following result:

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Also, let $p:[0,1]\to\mathbb{C}$ be a Lebesgue integrable function on $[0,1]$. For any $x,\ y\in\mathcal{B}$ with $\|x\|,\ \|y\|<R$ we have*

$$\begin{aligned}&\left\|\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\| \\ &\le\|y-x\|\times\begin{cases}\sup_{t\in[0,1]}\left|\int_s^t p(\tau)\,d\tau\right|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\[2ex] \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right|^p dt\right]^{1/p}\\ \times\left(\int_0^1[f_a'(\|(1-t)x+ty\|)]^q\,dt\right)^{1/q}\ \ p,\ q>1,\ \frac1p+\frac1q=1\\[2ex] \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right|dt\right]\sup_{t\in[0,1]}f_a'(\|(1-t)x+ty\|).\end{cases}\end{aligned} \tag{4.4}$$

In particular,

$$\begin{aligned}(4.5)\quad & \left\|\left(\int_{\frac{1}{2}}^{1} p(\tau)\,d\tau\right) f(y)+\left(\int_{0}^{\frac{1}{2}} p(\tau)\,d\tau\right) f(x)\right.\\ & \left.-\int_0^1 p(t)\, f((1-t)x+ty)\,dt\right\|\\ & \leq \|y-x\|\times\begin{cases}\sup_{t\in[0,1]}\left|\int_{1/2}^t p(\tau)\,d\tau\right|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\ \\ \left[\int_0^1\left|\int_{1/2}^t p(\tau)\,d\tau\right|^p dt\right]^{1/p}\\ \times\left(\int_0^1 [f_a'(\|(1-t)x+ty\|)]^q\,dt\right)^{1/q} & p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \left(\int_0^1\left|\int_{1/2}^t p(\tau)\,d\tau\right| dt\right)\sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|)\,.\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}(4.6)\quad & \left\|\left(\int_0^1 p(\tau)\,d\tau\right)[(1-\alpha)f(y)+\alpha f(x)]-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|\\ & \leq\|y-x\|\times\begin{cases}\sup_{t\in[0,1]}\left|\int_0^t p(\tau)\,d\tau-\alpha\int_0^1 p(\tau)\,d\tau\right|\\ \times\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\ \\ \left[\int_0^1\left|\int_0^t p(\tau)\,d\tau-\alpha\int_0^1 p(\tau)\,d\tau\right|^p dt\right]^{1/p}\\ \times\left(\int_0^1[f_a'(\|(1-t)x+ty\|)]^q\,dt\right)^{1/q} & p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \int_0^1\left|\int_0^t p(\tau)\,d\tau-\alpha\int_0^1 p(\tau)\,d\tau\right| dt\\ \times\sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|)\end{cases}\end{aligned}$$

for all $\alpha\in[0,1]$.

In particular, we have the trapezoid type inequalities

$$\begin{aligned}(4.7)\quad & \left\|\left(\int_0^1 p(\tau)\,d\tau\right)\frac{f(y)+f(x)}{2}-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|\\ & \leq\|y-x\|\times\begin{cases}\sup_{t\in[0,1]}\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right|\\ \times\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\ \\ \left[\int_0^1\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right|^p dt\right]^{1/p}\\ \times\left(\int_0^1[f_a'(\|(1-t)x+ty\|)]^q\,dt\right)^{1/q} & p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \int_0^1\left|\int_0^t p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right| dt\\ \times\sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|)\,.\end{cases}\end{aligned}$$

The proof follows by Theorem 2 and Lemma 4.

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \ \lambda\in D(0,1); \tag{4.8}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1); \tag{4.9}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad \lambda\in\mathbb{C}, \tag{4.10}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad \lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad \lambda\in D(0,1);$$

$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad \lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1).$$

If we consider the exponential function $f(x) = \exp x$, then for $x, y \in \mathcal{B}$

$$\begin{aligned}(4.11)\qquad \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt &= \int_0^1 \exp(\|(1-t)x+ty\|)\,dt \\ &\le \int_0^1 \exp((1-t)\|x\|+t\|y\|)\,dt \\ &= \int_0^1 \exp(\|x\|+t(\|y\|-\|x\|))\,dt \\ &= \begin{cases} \frac{\exp\|y\|-\exp\|x\|}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\| \\ \exp\|x\| & \text{if } \|y\| = \|x\|. \end{cases} \\ &=: E_1(x,y).\end{aligned}$$

Also

$$\begin{aligned}(4.12)\qquad \int_0^1 [f_a'(\|(1-t)x+ty\|)]^q\,dt &= \int_0^1 \exp[q(\|(1-t)x+ty\|)]\,dt \\ &\le \int_0^1 \exp[q((1-t)\|x\|+t\|y\|)]\,dt \\ &= \begin{cases} \frac{\exp(q\|y\|)-\exp(q\|x\|)}{q(\|y\|-\|x\|)} & \text{if } \|y\| \neq \|x\| \\ \exp(q\|x\|) & \text{if } \|y\| = \|x\| \end{cases} \\ &=: E_q(x,y),\ \ q>1.\end{aligned}$$

Moreover, for $x, y \in \mathcal{B}$

$$\begin{aligned}(4.13)\qquad \sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|) &\le \sup_{t\in[0,1]} \exp((1-t)\|x\|+t\|y\|) \\ &= \max\{\|x\|,\|y\|\}.\end{aligned}$$

By making use of Theorem 3 and (4.11)-(4.13), we have for $p : [0,1] \to \mathbb{C}$, a Lebesgue integrable function on $[0,1]$ and any $x, y \in \mathcal{B}$ and $s \in [0,1]$, that

$$\begin{aligned}(4.14)\qquad &\left\| \left(\int_s^1 p(\tau)\,d\tau\right)\exp y + \left(\int_0^s p(\tau)\,d\tau\right)\exp x \right. \\ &\left. - \int_0^1 p(t)\exp((1-t)x+ty)\,dt \right\| \\ &\le \|y-x\| \times \begin{cases} \sup_{t\in[0,1]}\left|\int_s^t p(\tau)\,d\tau\right| E_1(x,y) \\ \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right|^p dt\right]^{1/p}[E_q(x,y)]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \left[\int_0^1\left|\int_s^t p(\tau)\,d\tau\right| dt\right]\max\{\|x\|,\|y\|\}. \end{cases}\end{aligned}$$

In particular,

$$\left\| \left(\int_{\frac{1}{2}}^{1} p(\tau)\, d\tau \right) f(y) + \left(\int_{0}^{\frac{1}{2}} p(\tau)\, d\tau \right) f(x) - \int_0^1 p(t) \exp((1-t)x + ty)\, dt \right\| \tag{4.15}$$
$$\leq \|y-x\| \times \begin{cases} \sup_{t\in[0,1]} \left| \int_{1/2}^t p(\tau)\, d\tau \right| E_1(x,y) \\ \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau)\, d\tau \right|^p dt \right]^{1/p} [E_q(x,y)]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau)\, d\tau \right| dt \right) \max\{\|x\|, \|y\|\}. \end{cases}$$

Also,

$$\left\| \left(\int_0^1 p(\tau)\, d\tau \right) [(1-\alpha)\exp(y) + \alpha \exp(x)] - \int_0^1 p(t) \exp((1-t)x + ty)\, dtdt \right\| \tag{4.16}$$
$$\leq \|y-x\| \times \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau)\, d\tau - \alpha \int_0^1 p(\tau)\, d\tau \right| E_1(x,y) \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau)\, d\tau - \alpha \int_0^1 p(\tau)\, d\tau \right|^p dt \right]^{1/p} (E_q(x,y))^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left| \int_0^t p(\tau)\, d\tau - \alpha \int_0^1 p(\tau)\, d\tau \right| dt \max\{\|x\|, \|y\|\}, \end{cases}$$

for all $\alpha \in [0,1]$.

In particular, we have the trapezoid type inequalities

$$\left\| \left(\int_0^1 p(\tau)\, d\tau \right) \frac{\exp(y) + \exp(x)}{2} - \int_0^1 p(t) \exp((1-t)x + ty)\, dtdt \right\| \tag{4.17}$$
$$\leq \|y-x\| \times \begin{cases} \sup_{t\in[0,1]} \left| \int_0^t p(\tau)\, d\tau - \frac{1}{2} \int_0^1 p(\tau)\, d\tau \right| E_1(x,y) \\ \\ \left[\int_0^1 \left| \int_0^t p(\tau)\, d\tau - \frac{1}{2} \int_0^1 p(\tau)\, d\tau \right|^p dt \right]^{1/p} [E_q(x,y)]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left| \int_0^t p(\tau)\, d\tau - \frac{1}{2} \int_0^1 p(\tau)\, d\tau \right| dt \max\{\|x\|, \|y\|\}. \end{cases}$$

The interested reader may apply the above inequalities for the other functions listed in (4.8)-(4.10). The details are omitted.

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[3] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[4] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[5] L. Ciurdariu, A note concerning several Hermite-Hadamard inequalities for different types of convex functions. *Int. J. Math. Anal.* **6** (2012), no. 33-36, 1623–1639.

[6] R. Coleman, *Calculus on Normed Vector Spaces*, Springer, 2012.

[7] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications. *J. Math. Anal. Appl.* **245** (2000), no. 2, 489–501.

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.[Online http://rgmia.org/monographs/hermite_hadamard.html].

[9] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[10] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[11] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[12] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* No. **29** (2015), 61–83.

[13] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications. *Kodai Math. J.* **39** (2016), no. 1, 227–251.

[14] S. S. Dragomir, Weighted norm inequalities of trapezoid type for Frechet differentiable functions in Banach spaces, *RGMIA Res. Rep. Coll.* **23** (2020), Art. 22, 19 pp. [Online https://rgmia.org/papers/v23/v23a22.pdf].

[15] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[16] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[17] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[18] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[19] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[20] Y. Feng and W. Zhao, Refinement of Hermite-Hadamard inequality. *Far East J. Math. Sci.* **68** (2012), no. 2, 245–250.

[21] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.

[22] X. Gao, A note on the Hermite-Hadamard inequality. *J. Math. Inequal.* **4** (2010), no. 4, 587–591.

[23] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, Hermite-Hadamard type and Fejér type inequalities for general weights (I). *J. Inequal. Appl.* **2013**, 2013:170.

[24] U. S. Kırmacı and R. Dikici, On some Hermite-Hadamard type inequalities for twice differentiable mappings and applications. *Tamkang J. Math.* **44** (2013), no. 1, 41–51.

[25] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.

[26] M. Muddassar, M. I. Bhatti and M. Iqbal, Some new s-Hermite-Hadamard type inequalities for differentiable functions and their applications. *Proc. Pakistan Acad. Sci.* **49** (2012), no. 1, 9–17.

[27] M. Matić and J. Pečarić, Note on inequalities of Hadamard's type for Lipschitzian mappings. *Tamkang J. Math.* **32** (2001), no. 2, 127–130.

[28] M. Z. Sarikaya, On new Hermite Hadamard Fejér type integral inequalities. *Stud. Univ. Babeş-Bolyai Math.* **57** (2012), no. 3, 377–386.

[29] S. Wąsowicz and A. Witkowski, On some inequality of Hermite-Hadamard type. *Opuscula Math.* **32** (2012), no. 3, 591–600.

[30] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. *J. Funct. Spaces Appl.* **2012**, Art. ID 980438, 14 pp.

[31] G. Zabandan, A. Bodaghi and A. Kılıçman, The Hermite-Hadamard inequality for r-convex functions. *J. Inequal. Appl.* **2012**, 2012:215, 8 pp.

[32] C.-J. Zhao, W.-S. Cheung and X.-Y. Li, On the Hermite-Hadamard type inequalities. *J. Inequal. Appl.* **2013**, 2013:228.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa