GENERAL NORM INEQUALITIES OF MIDPOINT TYPE FOR FRÉCHET DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. In this paper we establish some upper bounds for the quantity

$$\left\| (1-\alpha)\left\{ \left(\int_s^1 p(\tau)\, d\tau \right) f(y) + \left(\int_0^s p(\tau)\, d\tau \right) f(x) \right\} \right.$$
$$\left. +\alpha \left(\int_0^1 p(\tau)\, d\tau \right) f((1-s)x+sy) - \int_0^1 p(t) f((1-t)x+ty)\, dt \right\|$$

in the case that $f: C \subset E \to F$ is *Fréchet differentiable* on the open and convex subset C of the Banach space E with values into another Banach space F, $x, y \in C$, $p: [0,1] \to \mathbb{C}$ is a Lebesgue integrable function and $\alpha \in [0,1]$. Some particular cases of interest for different choices of α and s are given. Applications for Banach algebras are also provided.

1. Introduction

We recall some facts about differentiation of functions between normed vector spaces, [6].

Let O be an open subset if a normed vector space, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function f_u given by $t \mapsto f(a+tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0) := \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the *Gâteaux derivative (directional derivative)* of f at a in the direction u. If $\nabla_a f(u)$ is defined and $\lambda \in \mathbb{R} \setminus \{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u) = \lambda \nabla_a f(u)$. The function f is *Gâteaux differentiable* at a if $\nabla_a f(u)$ exists for all directions u.

Let E and F be normed vector spaces, and O be an open subset of F. A function $f: O \to F$ is called *Fréchet differentiable* at $x \in O$ if there exists a bounded linear operator $A: E \to F$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

If there exists such an operator A, it is unique, so we write $Df(x) = A$ and call it the *Fréchet derivative* of f at x.

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O \ni x \mapsto Df(x) \in \mathcal{B}(E,F)$ is continuous. A function Fréchet

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differentiable at a point is continuous at that point. Fréchet differentiation is a linear operation. If f is Fréchet differentiable at x, it is also Gâteaux differentiable there, and $\nabla_x f(u) = Df(x)(u)$ for all $u \in E$.

We say that the function $f : O \subset E \to F$ is *L-Lipschitzian* on O with the constant $L > 0$ if

$$\|f(x) - f(y)\| \le L\|x - y\| \text{ for all } x,\ y \in O.$$

In [14] we established among others the following *midpoint and trapezoid type inequalities* for L-Lipschitzian functions f on an open and convex subset C in E

$$\left\|\int_0^1 f((1-t)t + ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \le \frac{1}{4}L\|x-y\| \tag{1.1}$$

and

$$\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)t + ty)\,dt\right\| \le \frac{1}{4}L\|x-y\| \tag{1.2}$$

for all x, $y \in C$. The constant $\frac{1}{4}$ is best possible in both inequalities (1.1) and (1.2).

For *Hermite-Hadamard's type inequalities*, namely

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(u)\,du \le \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$, see for instance [6], [7], [8], [22], [24], [25], [26], [28], [29], [30], [31], [32], [33], [34] and the references therein.

In the recent paper [15] we obtained, among others, the following midpoint weighted inequality:

Theorem 1. *Let $f : C \subset E \to F$ be Fréchet differentiable on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$. Then for all x, $y \in C$*

$$\begin{aligned}
&\left\|\int_0^1 p(t) f((1-t)x + ty)\,dt - \left(\int_0^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right)\right\| \\
&\le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)\|Df((1-t)x+ty)(y-x)\|\,dt \\
&+ \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\|Df((1-t)x+ty)(y-x)\|\,dt \\
&=: B(f,p,x,y).
\end{aligned} \tag{1.4}$$

Moreover, we have the upper bounds

$$\begin{aligned}
B(f,p,x,y) &\le \frac{1}{2}\left(\int_0^1 p(s)\,ds\right)\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\
&\le \frac{1}{2}\left(\int_0^1 p(s)\,ds\right)\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|,
\end{aligned} \tag{1.5}$$

$$B(f,p,x,y) \leq \left[\frac{1}{2}\int_0^1 p(s)\,ds - \int_0^1 \operatorname{sgn}\left(t-\frac{1}{2}\right)\left(\int_0^t p(s)\,ds\right)dt\right] \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\| \leq \frac{1}{2}\left(\int_0^1 p(s)\,ds\right)\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\| \tag{1.6}$$

and

$$B(f,p,x,y) \leq \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q} \leq \left[\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)^r dt + \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)^r dt\right]^{1/r} \times \sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\| \tag{1.7}$$

for $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$.

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left\|(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)\right\} +\alpha\left(\int_0^1 p(\tau)\,d\tau\right)f((1-s)x+sy)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|$$

in the case that $f:C\subset E\to F$ is a *Fréchet differentiable* function on the open and convex subset C of the Banach space E with values into another Banach space F, $x,\ y\in C$, $p:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function and $\alpha,\ s\in[0,1]$. Some particular cases of interest for different choices of α and s are given. Applications for Banach algebras are also provided.

2. Some Identities of Interest

Consider a function $f:C\subset E\to F$ that is defined on the open and convex set C. We have the following properties for the *auxiliary function*

$$\varphi_{(x,y)}(t) := f((1-t)x+ty),\ t\in[0,1],$$

where $x,\ y\in C$.

Lemma 1. *Assume that the function* $f:C\subset E\to F$ *is Fréchet differentiable on the open and convex set* C. *Then for all* $x,\ y\in C$ *the auxiliary function* $\varphi_{(x,y)}$ *is differentiable on* $(0,1)$ *and*

$$\varphi'_{(x,y)}(t) = Df((1-t)x+ty)(y-x). \tag{2.1}$$

Also

$$\varphi'_{(x,y)}(0+) = Df(x)(y-x) \tag{2.2}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y-x). \tag{2.3}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \frac{f((1-t-h)x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}. \end{aligned} \tag{2.4}$$

Since f is Fréchet differentiable, hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) &= \lim_{h\to 0} \frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \\ &= Df((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(x,y)}(h) - \varphi_{(x,y)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)x + hy) - f(x)}{h} \\ &= \lim_{h\to 0+} \frac{f(x + h(y-x)) - f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x. This proves (2.2).

The equality (2.3) follows in a similar way. □

We have the following identity for the Riemann-Stieltjes integral:

Lemma 2. *Assume that the function $f : C \subset E \to F$ is Fréchet differentiable on the open and convex set C. Let $u : [0,1] \to \mathbb{C}$ be of bounded variation on $[0,1]$ and $s \in [0,1]$. Then for all x, $y \in C$ and any γ, $\mu \in \mathbb{C}$,*

$$\begin{aligned} &[u(1) - \mu]\varphi_{(x,y)}(1) + [\gamma - u(0)]\varphi_{(x,y)}(0) + (\mu - \gamma)\varphi_{(x,y)}(s) \\ &- \int_0^1 \varphi_{(x,y)}(t)\,du(t) \\ &= \int_0^s [u(t) - \gamma]\varphi'_{(x,y)}(t)\,dt + \int_s^1 [u(t) - \mu]\varphi'_{(x,y)}(t)\,dt. \end{aligned} \tag{2.5}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\begin{aligned} \int_0^s [u(t) - \gamma]\varphi'_{(x,y)}(t)\,dt &= [u(s) - \gamma]\varphi_{(x,y)}(s) - [u(0) - \gamma]\varphi_{(x,y)}(0) \\ &\quad - \int_0^s \varphi_{(x,y)}(t)\,du(t) \end{aligned}$$

and

$$\int_s^1 [u(t)-\mu]\,\varphi'_{(x,y)}(t)\,dt = [u(1)-\mu]\,\varphi_{(x,y)}(1) - [u(s)-\mu]\,\varphi_{(x,y)}(s) - \int_s^1 \varphi_{(x,y)}(t)\,du(t)$$

for any $s \in [0,1]$.

If we add these two equalities, we get

$$\begin{aligned}
&\int_0^s [u(t)-\gamma]\,\varphi'_{(x,y)}(t)\,dt + \int_s^1 [u(t)-\mu]\,\varphi'_{(x,y)}(t)\,dt \\
&= [u(1)-\mu]\,\varphi_{(x,y)}(1) + [\gamma - u(0)]\,\varphi_{(x,y)}(0) + [\mu - u(s)]\,\varphi_{(x,y)}(s) \\
&+ [u(s)-\gamma]\,\varphi_{(x,y)}(s) - \int_0^s \varphi_{(x,y)}(t)\,du(t) - \int_s^1 \varphi_{(x,y)}(t)\,du(t) \\
&= [u(1)-\mu]\,\varphi_{(x,y)}(1) + [\gamma - u(0)]\,\varphi_{(x,y)}(0) + (\mu-\gamma)\,\varphi_{(x,y)}(s) \\
&- \int_0^1 \varphi_{(x,y)}(t)\,du(t)
\end{aligned}$$

for any $s \in [0,1]$, which proves the desired equality (2.5). □

If in (2.5) we take $\gamma = \alpha u(0) + (1-\alpha)\,u(s)$ and $\mu = (1-\alpha)\,u(s) + \alpha u(1)$ where $s \in [0,1]$ and $\alpha \in [0,1]$, then we get

$$\begin{aligned}
&(1-\alpha)\left\{[u(1)-u(s)]\,\varphi_{(x,y)}(1) + [u(s)-u(0)]\,\varphi_{(x,y)}(0)\right\} \\
&+ \alpha\,[u(1)-u(0)]\,\varphi_{(x,y)}(s) - \int_0^1 \varphi_{(x,y)}(t)\,du(t) \\
&= \int_0^s [u(t) - \alpha u(0) - (1-\alpha)\,u(s)]\,\varphi'_{(x,y)}(t)\,dt \\
&+ \int_s^1 [u(t) - (1-\alpha)\,u(s) - \alpha u(1)]\,\varphi'_{(x,y)}(t)\,dt.
\end{aligned} \tag{2.6}$$

If in this equality, we take $\alpha = 1$, then we get the *Montgomery type identity*

$$\begin{aligned}
&[u(1)-u(0)]\,\varphi_{(x,y)}(s) - \int_0^1 \varphi_{(x,y)}(t)\,du(t) \\
&= \int_0^s [u(t)-u(0)]\,\varphi'_{(x,y)}(t)\,dt + \int_s^1 [u(t)-u(1)]\,\varphi'_{(x,y)}(t)\,dt,
\end{aligned} \tag{2.7}$$

for $s \in [0,1]$, which was obtained for the first time in the scalar case by the author in [9].

If in (2.6) we take $\alpha=\frac{1}{2}$, then we get

$$\begin{aligned}(2.8)\qquad &\frac{1}{2}\left\{[u(1)-u(s)]\varphi_{(x,y)}(1)+[u(s)-u(0)]\varphi_{(x,y)}(0)\right\}\\ &+\frac{1}{2}[u(1)-u(0)]\varphi_{(x,y)}(s)-\int_0^1\varphi_{(x,y)}(t)\,du(t)\\ &=\int_0^s\left[u(t)-\frac{u(0)+u(s)}{2}\right]\varphi'_{(x,y)}(t)\,dt\\ &+\int_s^1\left[u(t)-\frac{u(s)+u(1)}{2}\right]\varphi'_{(x,y)}(t)\,dt.\end{aligned}$$

If in (2.6) we take $\alpha=0$, we get

$$\begin{aligned}(2.9)\qquad &\left\{[u(1)-u(s)]\varphi_{(x,y)}(1)+[u(s)-u(0)]\varphi_{(x,y)}(0)\right\}-\int_0^1\varphi_{(x,y)}(t)\,du(t)\\ &=\int_0^1[u(t)-u(s)]\varphi'_{(x,y)}(t)\,dt.\end{aligned}$$

The case of weighted integrals is as follows:

Corollary 1. *Assume that the function $f:C\subset E\to F$ is Fréchet differentiable on the open and convex set C. Let $p:[0,1]\to\mathbb{C}$ be Lebesgue integrable on $[0,1]$ and $s\in[0,1]$. Then for all x, $y\in C$ and $\alpha\in[0,1]$ we have*

$$\begin{aligned}(2.10)\qquad &(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(1)+\left(\int_0^s p(\tau)\,d\tau\right)\varphi_{(x,y)}(0)\right\}\\ &+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(s)-\int_0^1\varphi_{(x,y)}(t)\,p(t)\,dt\\ &=\int_0^s\left(\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt\\ &+\int_s^1\left(\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt.\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.11)\qquad &\left(\int_0^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(s)-\int_0^1\varphi_{(x,y)}(t)\,p(t)\,dt\\ &=\int_0^s\left(\int_0^t p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt-\int_s^1\left(\int_t^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt,\end{aligned}$$

$$\begin{aligned}(2.12)\qquad &\frac{1}{2}\left\{\left(\int_s^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(1)+\left(\int_0^s p(\tau)\,d\tau\right)\varphi_{(x,y)}(0)\right\}\\ &+\frac{1}{2}\left(\int_0^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(s)-\int_0^1\varphi_{(x,y)}(t)\,p(t)\,dt\\ &=\int_0^s\left(\frac{1}{2}\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt\\ &+\int_s^1\left(\int_s^t p(\tau)\,d\tau-\frac{1}{2}\int_s^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt\end{aligned}$$

and

$$\left(\int_s^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(1)+\left(\int_0^s p(\tau)\,d\tau\right)\varphi_{(x,y)}(0)-\int_0^1\varphi_{(x,y)}(t)\,p(t)\,dt \tag{2.13}$$
$$=\int_s^1\left(\int_s^t p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt-\int_0^s\left(\int_t^s p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt.$$

Proof. By the identity (2.6) for $u(t)=\int_0^t p(\tau)\,d\tau$ we derive

$$(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(1)+\left(\int_0^s p(\tau)\,d\tau\right)\varphi_{(x,y)}(0)\right\}$$
$$+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)\varphi_{(x,y)}(s)-\int_0^1\varphi_{(x,y)}(t)\,p(t)\,dt$$

$$=\int_0^s\left[\int_0^t p(\tau)\,d\tau-(1-\alpha)\int_0^s p(\tau)\,d\tau\right]\varphi'_{(x,y)}(t)\,dt$$
$$+\int_s^1\left[\int_0^t p(\tau)\,d\tau-(1-\alpha)\int_0^s p(\tau)\,d\tau-\alpha\int_0^1 p(\tau)\,d\tau\right]\varphi'_{(x,y)}(t)\,dt$$
$$=\int_0^s\left(\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt$$
$$+\int_s^1\left(\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right)\varphi'_{(x,y)}(t)\,dt,$$

namely, the identity (2.10). □

3. Main Results

The following result provides an error approximation of the weighted integral by a three points rule:

Theorem 2. *Assume that the function $f:C\subset E\to F$ is Fréchet differentiable on the open and convex set C. Let $p:[0,1]\to\mathbb{C}$ be Lebesgue integrable on $[0,1]$ and $s\in[0,1]$. Then for all $x,\ y\in C$ and any $\alpha\in[0,1]$ we have*

$$\left\|(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)\right\}\right. \tag{3.1}$$
$$\left.+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)f((1-s)x+sy)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|$$
$$\leq\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|\|Df((1-t)x+ty)(y-x)\|\,dt$$
$$+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\|Df((1-t)x+ty)(y-x)\|\,dt$$
$$=:B(f,p,x,y,\alpha,s).$$

Moreover, we have the bounds

$$B(f,p,x,y,\alpha,s) \tag{3.2}$$

$$\leq \begin{cases} \max\left\{\sup_{t\in[0,s]}\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|,\right. \\ \left.\sup_{t\in[s,1]}\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\right\} \\ \times\int_0^1\|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|^p dt+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times\left(\int_0^1\|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}, p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|dt+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|dt\right) \\ \times\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

Proof. From the identity (2.10) we have

$$\begin{aligned}(3.3)\qquad &(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)\right\}\\ &+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)f((1-s)x+sy)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\\ &=\int_0^s\left(\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\,dt\\ &+\int_s^1\left(\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\,dt\end{aligned}$$

for all $x,\,y\in C$ and any $\alpha\in[0,1]$.

By taking the norm in (3.3) and using the integral's properties we get

$$\begin{aligned}(3.4)\qquad &\left\|(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)\right\}\right.\\ &\left.+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)f((1-s)x+sy)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|\\ &\leq\left\|\int_0^s\left(\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\,dt\right\|\\ &+\left\|\int_s^1\left(\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\,dt\right\|\\ &\leq\int_0^s\left\|\left(\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\right\|dt\\ &+\int_s^1\left\|\left(\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right)Df((1-t)x+ty)(y-x)\right\|dt\\ &=\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|\|Df((1-t)x+ty)(y-x)\|\,dt\\ &+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\|Df((1-t)x+ty)(y-x)\|\,dt\\ &=B(f,p,x,y,\alpha,s),\end{aligned}$$

which proves the inequality (3.1).

By Hölder's integral inequality we have

$$\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt$$
$$\leq \begin{cases} \sup_{t\in[0,s]} \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| \\ \times \int_0^s \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times \left(\int_0^s \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},\ p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[0,s]} \|Df((1-t)x+ty)(y-x)\| \end{cases}$$

and

$$\int_s^1 \left|\int_s^t p(\tau)\,d\tau - \alpha \int_s^1 p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt$$
$$\leq \begin{cases} \sup_{t\in[s,1]} \left|\int_s^t p(\tau)\,d\tau - \alpha \int_s^1 p(\tau)\,d\tau\right| \\ \times \int_s^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_s^1 \left|\int_s^t p(\tau)\,d\tau - \alpha \int_s^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times \left(\int_s^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},\ p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_s^1 \left|\int_s^t p(\tau)\,d\tau - \alpha \int_s^1 p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[s,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

Therefore, by summing these two inequalities we get

$$\begin{aligned} &\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ &+ \int_s^1 \left|\int_s^t p(\tau)\,d\tau - \alpha \int_s^1 p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt \end{aligned} \tag{3.5}$$

$$\leq \begin{cases} \sup_{t\in[0,s]} \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| \\ \times \int_0^s \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times \left(\int_0^s \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},\ p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^s \left|\alpha \int_0^s p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[0,s]} \|Df((1-t)x+ty)(y-x)\| \end{cases}$$

$$
+\begin{cases}
\sup_{t\in[s,1]}\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right| \\
\times\int_s^1\|Df((1-t)x+ty)(y-x)\|\,dt \\
\\
\left(\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\
\times\left(\int_s^1\|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},\ p,q>1,\frac{1}{p}+\frac{1}{q}=1 \\
\\
\left(\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right| dt\right) \\
\times\sup_{t\in[s,1]}\|Df((1-t)x+ty)(y-x)\|.
\end{cases}
$$

Now, observe that

$$
\begin{aligned}
&\sup_{t\in[0,s]}\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right| \\
&\times\int_0^s\|Df((1-t)x+ty)(y-x)\|\,dt \\
&+\sup_{t\in[s,1]}\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right| \\
&\times\int_s^1\|Df((1-t)x+ty)(y-x)\|\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\le\max\left\{\sup_{t\in[0,s]}\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|,\sup_{t\in[s,1]}\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\right\} \\
&\times\int_0^1\|Df((1-t)x+ty)(y-x)\|\,dt,
\end{aligned}
$$

which proves the first bound in (3.2).

By Hölder's discrete inequality we have

$$
\begin{aligned}
&\left(\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\
&\times\left(\int_0^s\|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\
&\times\left(\int_s^1\|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}
\end{aligned}
$$

$$\leq \left(\int_0^s \left| \alpha \int_0^s p(\tau)\, d\tau - \int_t^s p(\tau)\, d\tau \right|^p dt + \int_s^1 \left| \int_s^t p(\tau)\, d\tau - \alpha \int_s^1 p(\tau)\, d\tau \right|^p dt \right)^{1/p}$$
$$\times \left(\int_0^s \| Df((1-t)x+ty)(y-x) \|^q dt + \int_s^1 \| Df((1-t)x+ty)(y-x) \|^q dt \right)^{1/q}$$
$$= \left(\int_0^s \left| \alpha \int_0^s p(\tau)\, d\tau - \int_t^s p(\tau)\, d\tau \right|^p dt + \int_s^1 \left| \int_s^t p(\tau)\, d\tau - \alpha \int_s^1 p(\tau)\, d\tau \right|^p dt \right)^{1/p}$$
$$\times \left(\int_0^1 \| Df((1-t)x+ty)(y-x) \|^q dt \right)^{1/q},$$

which proves the second bound in (3.2).

Also,

$$\left(\int_0^s \left| \alpha \int_0^s p(\tau)\, d\tau - \int_t^s p(\tau)\, d\tau \right| dt \right)$$
$$\times \sup_{t\in[0,s]} \| Df((1-t)x+ty)(y-x) \|$$
$$+ \left(\int_s^1 \left| \int_s^t p(\tau)\, d\tau - \alpha \int_s^1 p(\tau)\, d\tau \right| dt \right)$$
$$\times \sup_{t\in[s,1]} \| Df((1-t)x+ty)(y-x) \|$$

$$\leq \left(\int_0^s \left| \alpha \int_0^s p(\tau)\, d\tau - \int_t^s p(\tau)\, d\tau \right| dt + \int_s^1 \left| \int_s^t p(\tau)\, d\tau - \alpha \int_s^1 p(\tau)\, d\tau \right| dt \right)$$
$$\leq \max \left\{ \sup_{t\in[0,s]} \| Df((1-t)x+ty)(y-x) \|, \sup_{t\in[s,1]} \| Df((1-t)x+ty)(y-x) \| \right\}$$
$$= \left(\int_0^s \left| \alpha \int_0^s p(\tau)\, d\tau - \int_t^s p(\tau)\, d\tau \right| dt + \int_s^1 \left| \int_s^t p(\tau)\, d\tau - \alpha \int_s^1 p(\tau)\, d\tau \right| dt \right)$$
$$\times \sup_{t\in[0,1]} \| Df((1-t)x+ty)(y-x) \|,$$

which proves the last bound in (3.2). □

We have the following particular inequalities of interest:

Corollary 2. *With the assumptions of Theorem 1 we have the Ostrowski type inequality*

$$\left\| \left(\int_0^1 p(\tau)\, d\tau \right) f((1-s)x+sy) - \int_0^1 p(t) f((1-t)x+ty)\, dt \right\| \tag{3.6}$$
$$\leq \int_0^s \left| \int_0^t p(\tau)\, d\tau \right| \| Df((1-t)x+ty)(y-x) \| \, dt$$
$$+ \int_s^1 \left| \int_t^1 p(\tau)\, d\tau \right| \| Df((1-t)x+ty)(y-x) \| \, dt$$
$$=: B_1(f,p,x,y,s).$$

Moreover, we have the bounds

$$B_1(f,p,x,y,s) \leq \begin{cases} \max\left\{\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau\right|, \sup_{t\in[s,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\} \\ \times \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^s \left|\int_0^t p(\tau)\,d\tau\right|^p dt + \int_s^1 \left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^s \left|\int_0^t p(\tau)\,d\tau\right| dt + \int_s^1 \left|\int_t^1 p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases} \tag{3.7}$$

If we take $s = 1/2$ in Corollary 2, then we obtain the midpoint inequalities

$$\begin{aligned} &\left\|\left(\int_0^1 p(\tau)\,d\tau\right) f\left(\frac{x+y}{2}\right) - \int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\| \\ &\leq \int_0^{1/2} \left|\int_0^t p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ &+ \int_{1/2}^1 \left|\int_t^1 p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ &=: B_1(f,p,x,y). \end{aligned} \tag{3.8}$$

Moreover, we have the bounds

$$B_1(f,p,x,y) \leq \begin{cases} \max\left\{\sup_{t\in[0,1/2]}\left|\int_0^t p(\tau)\,d\tau\right|, \sup_{t\in[1/2,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\} \\ \times \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^{1/2} \left|\int_0^t p(\tau)\,d\tau\right|^p dt + \int_{1/2}^1 \left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^{1/2} \left|\int_0^t p(\tau)\,d\tau\right| dt + \int_{1/2}^1 \left|\int_t^1 p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases} \tag{3.9}$$

Corollary 3. *With the assumptions of Theorem 1 we have the trapezoid type inequality*

$$\begin{aligned}(3.10)\quad &\left\|\left(\int_s^1 p(\tau)\,d\tau\right) f(y) + \left(\int_0^s p(\tau)\,d\tau\right) f(x) \right.\\ &\left. - \int_0^1 p(t) f((1-t)x+ty)\,dt\right\| \\ &\le \int_0^1 \left|\int_t^s p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt =: B_0(f,p,x,y,s).\end{aligned}$$

Moreover, we have the bounds

$$(3.11)\quad B_0(f,p,x,y,s) \le \begin{cases} \left(\sup_{t\in[0,1]}\left|\int_t^s p(\tau)\,d\tau\right|\right)\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^1 \left|\int_t^s p(\tau)\,d\tau\right|^p dt\right)^{1/p}\left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^1 \left|\int_t^s p(\tau)\,d\tau\right| dt\right)\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|.\end{cases}$$

If we take $s=1/2$ in Corollary 2, then we obtain

$$\begin{aligned}(3.12)\quad &\left\|\left(\int_{1/2}^1 p(\tau)\,d\tau\right) f(y) + \left(\int_0^{1/2} p(\tau)\,d\tau\right) f(x) \right.\\ &\left. - \int_0^1 p(t) f((1-t)x+ty)\,dt\right\| \\ &\le \int_0^1 \left|\int_t^{1/2} p(\tau)\,d\tau\right| \|Df((1-t)x+ty)(y-x)\|\,dt =: B_0(f,p,x,y,).\end{aligned}$$

Moreover, we have the bounds

$$(3.13)\quad B_0(f,p,x,y) \le \begin{cases} \left(\sup_{t\in[0,1]}\left|\int_t^{1/2} p(\tau)\,d\tau\right|\right)\int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^1 \left|\int_t^{1/2} p(\tau)\,d\tau\right|^p dt\right)^{1/p}\left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^1 \left|\int_t^{1/2} p(\tau)\,d\tau\right| dt\right)\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|.\end{cases}$$

We also have the mixed inequalities:

Corollary 4. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}(3.14)\quad & \left\| \frac{1}{2}\left\{ \left(\int_s^1 p(\tau)\,d\tau \right) f(y) + \left(\int_0^s p(\tau)\,d\tau \right) f(x) \right\} \right. \\ & \left. + \frac{1}{2}\left(\int_0^1 p(\tau)\,d\tau \right) f((1-s)x+sy) - \int_0^1 p(t) f((1-t)x+ty)\,dt \right\| \\ & \le \frac{1}{2}\int_0^s \left| \int_0^t p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau \right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ & + \frac{1}{2}\int_s^1 \left| \int_t^1 p(\tau)\,d\tau - \int_s^t p(\tau)\,d\tau \right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ & =: B_{1/2}(f,p,x,y,s).\end{aligned}$$

Moreover, we have the bounds

$$(3.15)\qquad B_{1/2}(f,p,x,y,s)$$

$$\le \frac{1}{2} \times \begin{cases} \max\left\{ \sup_{t\in[0,s]} \left| \int_0^t p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau \right|, \right. \\ \left. \sup_{t\in[s,1]} \left| \int_t^1 p(\tau)\,d\tau - \int_s^t p(\tau)\,d\tau \right| \right\} \\ \times \int_0^1 \|Df((1-t)x+ty)(y-x)\|\,dt \\ \\ \left(\int_0^s \left| \int_0^t p(\tau)\,d\tau - \int_t^s p(\tau)\,d\tau \right|^p dt + \int_s^1 \left| \int_t^1 p(\tau)\,d\tau - \int_s^t p(\tau)\,d\tau \right|^p dt \right)^{1/p} \\ \times \left(\int_0^1 \|Df((1-t)x+ty)(y-x)\|^q\,dt \right)^{1/q}, p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^s \left| \int_t^1 p(\tau)\,d\tau - \int_s^t p(\tau)\,d\tau \right| dt + \int_s^1 \left| \int_t^1 p(\tau)\,d\tau - \int_s^t p(\tau)\,d\tau \right| dt \right) \\ \times \sup_{t\in[0,1]} \|Df((1-t)x+ty)(y-x)\|. \end{cases}$$

If we choose in Corollary 4, $s = 1/2$, then we get

$$\begin{aligned}(3.16)\quad & \left\| \frac{1}{2}\left\{ \left(\int_{1/2}^1 p(\tau)\,d\tau \right) f(y) + \left(\int_0^{1/2} p(\tau)\,d\tau \right) f(x) \right\} \right. \\ & \left. + \frac{1}{2}\left(\int_0^1 p(\tau)\,d\tau \right) f\left(\frac{x+y}{2} \right) - \int_0^1 p(t) f((1-t)x+ty)\,dt \right\| \\ & \le \frac{1}{2}\int_0^{1/2} \left| \int_0^t p(\tau)\,d\tau - \int_t^{1/2} p(\tau)\,d\tau \right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ & + \frac{1}{2}\int_{1/2}^1 \left| \int_t^1 p(\tau)\,d\tau - \int_{1/2}^t p(\tau)\,d\tau \right| \|Df((1-t)x+ty)(y-x)\|\,dt \\ & =: B_{1/2}(f,p,x,y).\end{aligned}$$

Moreover, we have the bounds

$$(3.17)\quad B_{1/2}(f,p,x,y)$$
$$\leq \frac{1}{2}\times\begin{cases} \max\left\{\sup_{t\in[0,1/2]}\left|\int_0^t p(\tau)\,d\tau-\int_t^{1/2}p(\tau)\,d\tau\right|,\right.\\ \left.\sup_{t\in[1/2,1]}\left|\int_t^1 p(\tau)\,d\tau-\int_{1/2}^t p(\tau)\,d\tau\right|\right\}\\ \times\int_0^1\|Df((1-t)x+ty)(y-x)\|\,dt\\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau-\int_t^{1/2}p(\tau)\,d\tau\right|^p dt+\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau-\int_{1/2}^t p(\tau)\,d\tau\right|^p dt\right)^{1/p}\\ \times\left(\int_0^1\|Df((1-t)x+ty)(y-x)\|^q\,dt\right)^{1/q},p,q>1,\frac{1}{p}+\frac{1}{q}=1\\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau-\int_t^{1/2}p(\tau)\,d\tau\right|dt+\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau-\int_{1/2}^t p(\tau)\,d\tau\right|dt\right)\\ \times\sup_{t\in[0,1]}\|Df((1-t)x+ty)(y-x)\|.\end{cases}$$

Remark 1. *Since the Fréchet derivative satisfies the condition*

$$\|Df(a)(b)\|\leq\|Df(a)\|\,\|b\|$$

for $a\in C$ and $b\in E$, then for all x, $y\in C$ we also have the chain of inequalities

$$(3.18)\quad \left\|(1-\alpha)\left\{\left(\int_s^1 p(\tau)\,d\tau\right)f(y)+\left(\int_0^s p(\tau)\,d\tau\right)f(x)\right\}\right.$$
$$\left.+\alpha\left(\int_0^1 p(\tau)\,d\tau\right)f((1-s)x+sy)-\int_0^1 p(t)\,f((1-t)x+ty)\,dt\right\|$$
$$\leq\|y-x\|\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|\|Df((1-t)x+ty)\|\,dt$$
$$+\|y-x\|\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\|Df((1-t)x+ty)\|\,dt$$
$$=:\tilde{B}(f,p,x,y,\alpha,s).$$

Moreover, we have the bounds

$$(3.19)\quad \tilde{B}(f,p,x,y,\alpha,s)\leq\|y-x\|$$
$$\times\begin{cases} \max\left\{\sup_{t\in[0,s]}\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|,\right.\\ \left.\sup_{t\in[s,1]}\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|\right\}\\ \times\int_0^1\|Df((1-t)x+ty)\|\,dt\\ \\ \left(\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|^p dt+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p}\\ \times\left(\int_0^1\|Df((1-t)x+ty)\|^q\,dt\right)^{1/q},p,q>1,\frac{1}{p}+\frac{1}{q}=1\\ \\ \left(\int_0^s\left|\alpha\int_0^s p(\tau)\,d\tau-\int_t^s p(\tau)\,d\tau\right|dt+\int_s^1\left|\int_s^t p(\tau)\,d\tau-\alpha\int_s^1 p(\tau)\,d\tau\right|dt\right)\\ \times\sup_{t\in[0,1]}\|Df((1-t)x+ty)\|.\end{cases}$$

The interested reader may state the corresponding particular cases for $\alpha=1$, $\alpha=0$, $\alpha=1/2$ and $s=1/2$, however the details are omitted.

4. Some Examples for Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [13].

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-s)x + sy\|)\, ds. \tag{4.1}$$

We also have:

Lemma 4. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|Df((1-t)x + ty)(y - x)\| \leq \|y - x\| f_a'(\|(1-t)x + ty\|) \tag{4.2}$$

for all $t \in [0, 1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (4.1) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds$$

namely

$$\begin{aligned} \left\| \frac{f(u + \varepsilon v) - f(u)}{\varepsilon} \right\| &\leq \|v\| \int_0^1 f_a'(\|(1-s)u + s(u + \varepsilon v)\|)\, ds, \ \varepsilon \neq 0 \\ &= \|v\| \int_0^1 f_a'(\|u + s\varepsilon v\|)\, ds \end{aligned}$$

and by taking $\varepsilon \to 0$ we get, by the property of integral, that

$$\|Df(u)(v)\| \leq \|v\| f_a'(\|u\|). \tag{4.3}$$

Now, if we take in (4.3) $u = (1-t)x + ty$ and $v = y - x$, then we get (4.2). $\square$

We have the following result:

Theorem 3. *Let* $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ *be a function defined by power series with complex coefficients and convergent on the open disk* $D(0,R)\subset\mathbb{C}$, $R>0$. *Also, let* $p:[0,1]\to\mathbb{C}$ *be a Lebesgue integrable function on* $[0,1]$. *For any* $x, y\in\mathcal{B}$ *with* $\|x\|, \|y\|<R$ *we have*

$$\begin{aligned}
&\left\|\left(\int_0^1 p(\tau)\,d\tau\right) f((1-s)x+sy)-\int_0^1 p(t)\, f((1-t)x+ty)\,dt\right\| \\
&\le \|y-x\|\int_0^s\left|\int_0^t p(\tau)\,d\tau\right| f_a'(\|(1-t)x+ty\|)\,dt \\
&+\|y-x\|\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right| f_a'(\|(1-t)x+ty\|)\,dt \\
&=: C_1(f,p,x,y,s).
\end{aligned} \tag{4.4}$$

Moreover, we have the bounds

$$\begin{aligned}
&C_1(f,p,x,y,s) \\
&\le \|y-x\|\begin{cases}
\max\left\{\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau\right|, \sup_{t\in[s,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\} \\
\times\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \\[2ex]
\left(\int_0^s\left|\int_0^t p(\tau)\,d\tau\right|^p dt+\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\
\times\left(\int_0^1\left[f_a'(\|(1-t)x+ty\|)\right]^q dt\right)^{1/q}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\left(\int_0^s\left|\int_0^t p(\tau)\,d\tau\right| dt+\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right| dt\right) \\
\times\sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|).
\end{cases}
\end{aligned} \tag{4.5}$$

The proof follows by Theorem 2 and Lemma 4.

We have the following midpoint inequality:

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
&\left\|\left(\int_0^1 p(\tau)\,d\tau\right) f\left(\frac{x+y}{2}\right)-\int_0^1 p(t)\, f((1-t)x+ty)\,dt\right\| \\
&\le \|y-x\|\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right| f_a'(\|(1-t)x+ty\|)\,dt \\
&+\|y-x\|\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right| f_a'(\|(1-t)x+ty\|)\,dt \\
&=: C_1(f,p,x,y).
\end{aligned} \tag{4.6}$$

Moreover, we have the bounds

$$C_1(f,p,x,y) \tag{4.7}$$
$$\leq \|y-x\| \begin{cases} \max\left\{\sup_{t\in[0,1/2]}\left|\int_0^t p(\tau)\,d\tau\right|, \sup_{t\in[1/2,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\} \\ \times \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right|^p dt + \int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p} \\ \times\left(\int_0^1 \left[f_a'(\|(1-t)x+ty\|)\right]^q dt\right)^{1/q}, \\ p,\ q>1, \frac{1}{p}+\frac{1}{q}=1 \\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right| dt + \int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right| dt\right) \\ \times \sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|). \end{cases}$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{4.8}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \tag{4.9}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{4.10}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1).$$

If we consider the exponential function $f(x) = \exp x$, then for $x,\ y \in \mathcal{B}$

$$\begin{aligned} \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt &= \int_0^1 \exp(\|(1-t)x+ty\|)\,dt \\ &\le \int_0^1 \exp((1-t)\|x\| + t\|y\|)\,dt \\ &= \int_0^1 \exp(\|x\| + t(\|y\|-\|x\|))\,dt \\ &= \begin{cases} \frac{\exp\|y\|-\exp\|x\|}{\|y\|-\|x\|} \text{ if } \|y\| \neq \|x\| \\ \\ \exp\|x\| \text{ if } \|y\| = \|x\|. \end{cases} \\ &=: E_1(x,y). \end{aligned} \tag{4.11}$$

Also

$$\begin{aligned} \int_0^1 [f_a'(\|(1-t)x+ty\|)]^q\,dt &= \int_0^1 \exp[q(\|(1-t)x+ty\|)]\,dt \\ &\le \int_0^1 \exp[q((1-t)\|x\| + t\|y\|)]\,dt \\ &= \begin{cases} \frac{\exp(q\|y\|)-\exp(q\|x\|)}{q(\|y\|-\|x\|)} \text{ if } \|y\| \neq \|x\| \\ \\ \exp(q\|x\|) \text{ if } \|y\| = \|x\| \end{cases} \\ &=: E_q(x,y),\ q>1. \end{aligned} \tag{4.12}$$

Moreover, for $x,\ y \in \mathcal{B}$

$$\begin{aligned} \sup_{t\in[0,1]} f_a'(\|(1-t)x+ty\|) &\le \sup_{t\in[0,1]} \exp((1-t)\|x\| + t\|y\|) \\ &= \max\{\|x\|,\|y\|\}. \end{aligned} \tag{4.13}$$

By making use of Theorem 3 and (4.11)-(4.13), we have for $p:[0,1]\to\mathbb{C}$, a Lebesgue integrable function on $[0,1]$ and any $x,\ y\in\mathcal{B}$ and $s\in[0,1]$, that

$$\begin{aligned}(4.14)\quad &\left\|\left(\int_0^1 p(\tau)\,d\tau\right)\exp((1-s)x+sy)-\int_0^1 p(t)\exp((1-t)x+ty)\,dt\right\|\\ &\le\|y-x\|\int_0^s\left|\int_0^t p(\tau)\,d\tau\right|\exp(\|(1-t)x+ty\|)\,dt\\ &+\|y-x\|\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right|\exp(\|(1-t)x+ty\|)\,dt\\ &=:D_1(f,p,x,y,s).\end{aligned}$$

Moreover, we have the bounds

$$(4.15)\quad D_1(f,p,x,y,s)$$
$$\le\|y-x\|\begin{cases}\max\left\{\sup_{t\in[0,s]}\left|\int_0^t p(\tau)\,d\tau\right|,\sup_{t\in[s,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\}E_1(x,y)\\ \\ \left(\int_0^s\left|\int_0^t p(\tau)\,d\tau\right|^p dt+\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p}[E_q(x,y)]^{1/q},\\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \left(\int_0^s\left|\int_0^t p(\tau)\,d\tau\right|dt+\int_s^1\left|\int_t^1 p(\tau)\,d\tau\right|dt\right)\max\{\|x\|,\|y\|\}\end{cases}$$

for $x,\ y\in\mathcal{B}$ and $s\in[0,1]$.

In addition, we have the particular inequalities

$$\begin{aligned}(4.16)\quad &\left\|\left(\int_0^1 p(\tau)\,d\tau\right)\exp\left(\frac{x+y}{2}\right)-\int_0^1 p(t)\exp((1-t)x+ty)\,dt\right\|\\ &\le\|y-x\|\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right|\exp(\|(1-t)x+ty\|)\,dt\\ &+\|y-x\|\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right|\exp(\|(1-t)x+ty\|)\,dt\\ &=:D_1(f,p,x,y)\end{aligned}$$

and the upper bounds

$$(4.17)\quad D_1(f,p,x,y)$$
$$\le\|y-x\|\begin{cases}\max\left\{\sup_{t\in[0,1/2]}\left|\int_0^t p(\tau)\,d\tau\right|,\sup_{t\in[1/2,1]}\left|\int_t^1 p(\tau)\,d\tau\right|\right\}E_1(x,y)\\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right|^p dt+\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right|^p dt\right)^{1/p}[E_q(x,y)]^{1/q},\\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \left(\int_0^{1/2}\left|\int_0^t p(\tau)\,d\tau\right|dt+\int_{1/2}^1\left|\int_t^1 p(\tau)\,d\tau\right|dt\right)\max\{\|x\|,\|y\|\}\end{cases}$$

for all $x,\ y\in\mathcal{B}$.

The interested reader may apply the above inequalities for the other functions listed in (4.8)-(4.10). The details are omitted.

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[3] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[4] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[5] L. Ciurdariu, A note concerning several Hermite-Hadamard inequalities for different types of convex functions. *Int. J. Math. Anal.* **6** (2012), no. 33-36, 1623–1639.

[6] R. Coleman, *Calculus on Normed Vector Spaces*, Springer, 2012.

[7] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications. *J. Math. Anal. Appl.* **245** (2000), no. 2, 489–501.

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.[Online http://rgmia.org/monographs/hermite_hadamard.html].

[9] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral and applications. *Korean J. Comput. Appl. Math.* **7** (2000), no. 3, 611–627.

[10] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[11] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[12] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[13] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* No. **29** (2015), 61–83.

[14] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications. *Kodai Math. J.* **39** (2016), no. 1, 227–251.

[15] S. S. Dragomir, Weighted norm inequalities of trapezoid type for Frechet differentiable functions in Banach spaces, *RGMIA Res. Rep. Coll.* **23** (2020), Art. 22, 19 pp. [Online https://rgmia.org/papers/v23/v23a22.pdf].

[16] S. S. Dragomir, Weighted norm inequalities of midpoint type for Frechet differentiable functions in Banach spaces, *RGMIA Res. Rep. Coll.* **23** (2020), Art. 20, 20 pp. [Online https://rgmia.org/papers/v23/v23a20.pdf].

[17] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[18] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[19] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[20] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[21] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[22] Y. Feng and W. Zhao, Refinement of Hermite-Hadamard inequality. *Far East J. Math. Sci.* **68** (2012), no. 2, 245–250.
[23] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.
[24] X. Gao, A note on the Hermite-Hadamard inequality. *J. Math. Inequal.* **4** (2010), no. 4, 587–591.
[25] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, Hermite-Hadamard type and Fejér type inequalities for general weights (I). *J. Inequal. Appl.* **2013**, 2013:170.
[26] U. S. Kırmacı and R. Dikici, On some Hermite-Hadamard type inequalities for twice differentiable mappings and applications. *Tamkang J. Math.* **44** (2013), no. 1, 41–51.
[27] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.
[28] M. Muddassar, M. I. Bhatti and M. Iqbal, Some new s-Hermite-Hadamard type inequalities for differentiable functions and their applications. *Proc. Pakistan Acad. Sci.* **49** (2012), no. 1, 9–17.
[29] M. Matić and J. Pečarić, Note on inequalities of Hadamard's type for Lipschitzian mappings. *Tamkang J. Math.* **32** (2001), no. 2, 127–130.
[30] M. Z. Sarikaya, On new Hermite Hadamard Fejér type integral inequalities. *Stud. Univ. Babeş-Bolyai Math.* **57** (2012), no. 3, 377–386.
[31] S. Wąsowicz and A. Witkowski, On some inequality of Hermite-Hadamard type. *Opuscula Math.* **32** (2012), no. 3, 591–600.
[32] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. *J. Funct. Spaces Appl.* **2012**, Art. ID 980438, 14 pp.
[33] G. Zabandan, A. Bodaghi and A. Kılıçman, The Hermite-Hadamard inequality for r-convex functions. *J. Inequal. Appl.* **2012**, 2012:215, 8 pp.
[34] C.-J. Zhao, W.-S. Cheung and X.-Y. Li, On the Hermite-Hadamard type inequalities. *J. Inequal. Appl.* **2013**, 2013:228.

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