IMPROVEMENTS OF JENSEN-MERCER TYPE DISCRETE INEQUALITIES FOR CONVEX FUNCTIONS ON FINITE INTERVALS

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Abstract. In this paper we obtain some new refinements of Mercer's discrete inequality for univariate functions defined on finite intervals and provide some examples for particular functions of interest.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

We also recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}
& n \min_{i\in\{1,\ldots,n\}} \{p_i\} \left[\frac{1}{n}\sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \\
& \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \\
& \leq n \max_{i\in\{1,\ldots,n\}} \{p_i\} \left[\frac{1}{n}\sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right],
\end{aligned} \tag{1.2}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\ldots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\ldots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

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In 2003, A. McD. Mercer [19] obtained the following inequality for convex functions of a real variable $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$ and the finite sequences $x_k\in[m,M]$, and $p_k\geq 0$, $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$,

$$f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\leq f(m)+f(M)-\sum_{k=1}^{n}p_kf(x_k) \tag{1.3}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [17], [20], [26], in relation with majorization theory [24], for convex functions of selfadjoint operators in Hilbert spaces [16], [18], [21], [22] and for operator convex functions in Hilbert spaces [23] and [26].

In this paper we obtain some new refinements of Mercer's discrete inequality for univariate functions defined on finite intervals and provide some examples for particular functions of interest.

2. Main Results

The following refinement of Mercer's inequality holds:

Theorem 1. *Let $f:[m,M]\to\mathbb{R}$ be a convex function, $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^{n}p_k=1$. Then*

$$\begin{aligned} &f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\ &\leq \min_{i\in\{1,...,n\}}\{p_i\}\left[\sum_{k=1}^{n}f(m+M-x_k)-nf\left(m+M-\frac{1}{n}\sum_{k=1}^{n}x_k\right)\right]\\ &+f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\ &\leq\sum_{k=1}^{n}p_kf(m+M-x_k)\leq f(m)+f(M)-\sum_{k=1}^{n}p_kf(x_k). \end{aligned} \tag{2.1}$$

Proof. By the convexity of f we have

$$f(m+M-x_k)+f(x_k)\leq f(m)+f(M)$$

for $k\in\{1,...,n\}$.

Indeed, since $x_k\in[m,M]$ then $x_k=(1-t_k)m+t_kM$ with some $t_k\in[0,1]$, for $k\in\{1,...,n\}$ and

$$\begin{aligned} &f(m+M-x_k)+f(x_k)\\ &=f(m+M-(1-t_k)m-t_kM)+f((1-t_k)m+t_kM)\\ &=f(t_km+(1-t_k)M)+f((1-t_k)m+t_kM)\\ &\leq t_kf(m)+(1-t_k)f(M)+(1-t_k)f(m)+t_kf(M)\\ &=f(m)+f(M), \end{aligned}$$

for $k\in\{1,...,n\}$.

If we multiply this inequality by $p_k \geq 0$ and summing over k from 1 to n, then we get

$$\sum_{k=1}^{n} p_k f(m+M-x_k) + \sum_{k=1}^{n} p_k f(x_k) \leq f(m) + f(M),$$

namely

$$\sum_{k=1}^{n} p_k f(m+M-x_k) \leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k).$$

This implies that

$$\begin{aligned} \sum_{k=1}^{n} p_k f(m+M-x_k) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\ + f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\ \leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k). \end{aligned} \tag{2.2}$$

If we apply the first inequality in (1.2) for the convex function $\Phi(t) = f(m+M-t)$, $t \in [m, M]$ then we have

$$\begin{aligned} 0 \leq \min_{i\in\{1,\dots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(m+M-x_k) - nf\left(m+M-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right] \\ \leq \sum_{k=1}^{n} p_k f(m+M-x_k) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \end{aligned} \tag{2.3}$$

By making use of (2.2) and (2.3) we get the desired result (2.1). □

We also have:

Corollary 1. *With the assumptions of Theorem 1, we have the following refinement of Mercer's inequality:*

$$\begin{aligned} 0 \leq \sum_{k=1}^{n} p_k f(m+M-x_k) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\ - \min_{i\in\{1,\dots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(m+M-x_k) - nf\left(m+M-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right] \\ \leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right). \end{aligned} \tag{2.4}$$

The proof follows by Theorem 1 observing that the difference between the extreme terms is greater than the difference between the internal ones.

J. Pečarić and the author obtained in 1989 the following refinement of Jensen inequality (see [25]):

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leq\sum_{i_1,\dots,i_{k+1}=1}^{n}p_{i_1}\dots p_{i_{k+1}}f\left(\frac{x_{i_1}+\dots+x_{i_{k+1}}}{k+1}\right)\leq\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right)\leq\dots\leq\sum_{i=1}^{n}p_if\left(x_i\right), \tag{2.5}$$

for $k\geq1$ and $\mathbf{p}$, $\mathbf{x}$ as above.

If $q_1,\dots,q_k\geq0$ with $\sum_{j=1}^{k}q_j=1$, then the following refinement obtained in 1994 by the author [5] also holds:

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leq\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right)\leq\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(q_1x_{i_1}+\dots+q_kx_{i_k}\right)\leq\sum_{i=1}^{n}p_if\left(x_i\right), \tag{2.6}$$

where $1\leq k\leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.

Remark 1. *From the inequality (2.5), by replacing x_i with $m+M-x_i$, $i\in\{1,...,n\}$ we get*

$$f\left(m+M-\sum_{i=1}^{n}p_ix_i\right)\leq\sum_{i_1,\dots,i_{k+1}=1}^{n}p_{i_1}\dots p_{i_{k+1}}f\left(m+M-\frac{x_{i_1}+\dots+x_{i_{k+1}}}{k+1}\right)\leq\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(m+M-\frac{x_{i_1}+\dots+x_{i_k}}{k}\right)\leq\dots\leq\sum_{i=1}^{n}p_if\left(m+M-x_i\right),$$

and by (2.4) we get the sequence of refinements for Mercer's inequality

$$
\begin{aligned}
(2.7)\qquad 0 &\leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} f\left(m+M-\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \\
&\quad - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(m+M-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\quad - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\
&\quad \ldots \\
&\leq \sum_{k=1}^{n} p_k f\left(m+M-x_k\right) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\
&\quad - \min_{i\in\{1,\ldots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f\left(m+M-x_k\right) - nf\left(m+M-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right] \\
&\leq f(m)+f(M)-\sum_{k=1}^{n} p_k f(x_k) - f\left(m+M-\sum_{k=1}^{n} p_k x_k\right),
\end{aligned}
$$

where $f:[m,M]\to\mathbb{R}$ is a convex function, $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,\ldots,n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

A similar result may be stated by employing inequality (2.6).

In [11] we obtained among others the following two point Taylor's type representation:

Lemma 1. *Let $f: I\to\mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n\geq 1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct x, a, $b\in\mathring{I}$ we have*

$$
\begin{aligned}
(2.8)\qquad f(x) &= \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+\frac{(b-x)(x-a)}{b-a} \\
&\quad \times\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\} \\
&\quad + L_n(x,a,b),
\end{aligned}
$$

where

$$
\begin{aligned}
(2.9)\qquad & L_n(x,a,b) \\
&:= \frac{(b-x)(x-a)}{n!\,(b-a)}\left[(x-a)^n\int_0^1 f^{(n+1)}\left((1-s)a+sx\right)(1-s)^n\,ds\right. \\
&\quad \left. +(-1)^{n+1}(b-x)^n\int_0^1 f^{(n+1)}\left((1-s)x+sb\right)s^n ds\right]
\end{aligned}
$$

The case $n=0$, namely when the function f is locally absolutely continuous on $\mathring{I}$ with the derivative f' existing almost everywhere in $\mathring{I}$ is important and produces the following simple identity for each distinct $x,\ a,\ b\in \mathring{I}$

$$f(x)=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+L(x,a,b), \tag{2.10}$$

where

$$\begin{aligned} L(x,a,b) & \\ :=& \frac{(b-x)(x-a)}{b-a}\left[\int_0^1 f'((1-s)a+sx)\,ds-\int_0^1 f'((1-s)x+sb)\,ds\right]. \end{aligned} \tag{2.11}$$

We have the following refinement of Mercer's inequality in the case that the second derivative is bounded below by a positive constant.

Theorem 2. *Let $f:[m,M]\to\mathbb{R}$ be a convex function, $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^n p_k=1$. If f is a twice differentiable function on $[m,M]$ and such that*

$$f''(t)\geq\gamma \text{ for all } t\in(m,M) \tag{2.12}$$

for some $\gamma>0$, then

$$\begin{aligned} & f\left(m+M-\sum_{k=1}^n p_k x_k\right) \\ & \leq \sum_{k=1}^n p_k f(m+M-x_k) \\ & \leq \frac{\sum_{k=1}^n p_k x_k-m}{M-m}f(m)+\frac{M-\sum_{k=1}^n p_k x_k}{M-m}f(M) \\ & \leq f(m)+f(M)-\sum_{k=1}^n p_k f(x_k)-\frac{1}{2}\gamma\sum_{k=1}^n p_k(M-x_k)(x_k-m) \\ & \leq f(m)+f(M)-\sum_{k=1}^n p_k f(x_k). \end{aligned} \tag{2.13}$$

Proof. The first inequality in (2.13) follows by Jensen's result.

Since f is convex on $[m,M]$, then

$$f(s)=f\left(\frac{M-s}{M-m}m+\frac{s-m}{M-m}M\right)\leq\frac{M-s}{M-m}f(m)+\frac{s-m}{M-m}f(M) \tag{2.14}$$

for all $s\in[m,M]$.

For all $t\in[m,M]$, $s=m+M-t\in[m,M]$ and by (2.14) we get

$$\begin{aligned} f(m+M-t) &\leq \frac{M-m-M+t}{M-m}f(m)+\frac{m+M-t-m}{M-m}f(M) \\ &= \frac{t-m}{M-m}f(m)+\frac{M-t}{M-m}f(M). \end{aligned}$$

By taking $t=x_k$ in this inequality, we derive

$$f(m+M-x_k)\leq\frac{x_k-m}{M-m}f(m)+\frac{M-x_k}{M-m}f(M),$$

which implies, by multiplying with $p_k \geq 0$ and summing over k, that

$$\begin{aligned}
&\sum_{k=1}^{n} p_k f(m+M-x_k) \\
&\leq \frac{\sum_{k=1}^{n} p_k x_k - m}{M-m} f(m) + \frac{M-\sum_{k=1}^{n} p_k x_k}{M-m} f(M) \\
&= f(m) + f(M) - \left[\frac{M-\sum_{k=1}^{n} p_k x_k}{M-m} f(m) + \frac{\sum_{k=1}^{n} p_k x_k - m}{M-m} f(M)\right] \\
&= f(m) + f(M) \\
&- \left[\frac{M-\sum_{k=1}^{n} p_k x_k}{M-m} f(m) + \frac{\sum_{k=1}^{n} p_k x_k - m}{M-m} f(M) - \sum_{k=1}^{n} p_k f(x_k)\right] \\
&- \sum_{k=1}^{n} p_k f(x_k).
\end{aligned}$$

From (2.1) and (2.2) we obtain

$$\begin{aligned}
&\frac{1}{M-m}\left[(M-t) f(m) + (t-m) f(M)\right] - f(t) \\
&= \frac{(M-t)(t-m)}{M-m} \\
&\times \left[\int_0^1 f'((1-s)t + sM)\,ds - \int_0^1 f'((1-s)m + st)\,ds\right] \\
&= \frac{(M-t)(t-m)}{M-m} \\
&\times \left[\int_0^1 f'(st + (1-s)M)\,ds - \int_0^1 f'((1-s)m + st)\,ds\right].
\end{aligned} \tag{2.15}$$

Since f is twice differentiable and satisfies the condition (2.12), hence

$$\begin{aligned}
&\int_0^1 f'(st + (1-s)M)\,ds - \int_0^1 f'((1-s)m + st)\,ds \\
&= \int_0^1 \left(\int_{(1-s)m+st}^{st+(1-s)M} f''(u)\,du\right) ds \geq \int_0^1 \left(\int_{(1-s)m+st}^{st+(1-s)M} k du\right) ds \\
&= \gamma(M-m) \int_0^1 (1-s)\,ds = \frac{1}{2} v (M-m).
\end{aligned}$$

Therefore

$$\frac{1}{M-m}\left[(M-t) f(m) + (t-m) f(M)\right] - f(t) \geq \frac{1}{2}\gamma(M-t)(t-m) \tag{2.16}$$

for $t \in [m, M]$.

If we take $t = x_k \in [m, M]$, $k \in \{1, ..., n\}$ in (2.16), then we get

$$\frac{1}{M-m}\left[(M-x_k) f(m) + (x_k-m) f(M)\right] - f(x_k) \geq \frac{1}{2}\gamma(M-x_k)(x_k-m)$$

and by multiplying with $p_k \geq 0$, for $k \in \{1, ..., n\}$ and summing over k we get

$$\frac{\left(M - \sum_{k=1}^{n} p_k x_k\right) f(m) + \left(\sum_{k=1}^{n} p_k x_k - m\right) f(M)}{M - m} - \sum_{k=1}^{n} p_k f(x_k)$$
$$\geq \frac{1}{2}\gamma \sum_{k=1}^{n} p_k (M - x_k)(x_k - m),$$

which implies that

$$-\left[\frac{\left(M - \sum_{k=1}^{n} p_k x_k\right) f(m) + \left(\sum_{k=1}^{n} p_k x_k - m\right) f(M)}{M - m} - \sum_{k=1}^{n} p_k f(x_k)\right]$$
$$\leq -\frac{1}{2}\gamma \sum_{k=1}^{n} p_k (M - x_k)(x_k - m).$$

Therefore

$$\begin{aligned}
&f(m) + f(M) \\
&- \left[\frac{M - \sum_{k=1}^{n} p_k x_k}{M - m} f(m) + \frac{\sum_{k=1}^{n} p_k x_k - m}{M - m} f(M) - \sum_{k=1}^{n} p_k f(x_k)\right] \\
&- \sum_{k=1}^{n} p_k f(x_k) \\
&\leq f(m) + f(M) - \frac{1}{2}\gamma \sum_{k=1}^{n} p_k (M - x_k)(x_k - m) - \sum_{k=1}^{n} p_k f(x_k) \\
&\leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k),
\end{aligned}$$

and the last part of (2.13) is obtained. □

Remark 2. *From (2.13) we also have the inequalities*

$$\begin{aligned}
0 &\leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) - \frac{1}{2}\gamma \sum_{k=1}^{n} p_k (M - x_k)(x_k - m) \\
&- \sum_{k=1}^{n} p_k f(m + M - x_k) \\
&\leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) - f\left(m + M - \sum_{k=1}^{n} p_k x_k\right)
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
0 &\leq \frac{M - \sum_{k=1}^{n} p_k x_k}{M - m} f(m) + \frac{\sum_{k=1}^{n} p_k x_k - m}{M - m} f(M) \\
&- \sum_{k=1}^{n} p_k f(x_k) - \frac{1}{2}\gamma \sum_{k=1}^{n} p_k (M - x_k)(x_k - m) \\
&\leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) - f\left(m + M - \sum_{k=1}^{n} p_k x_k\right).
\end{aligned} \tag{2.18}$$

3. Some Examples

We consider the convex function $f : [m, M] \subset (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$, then by (2.1) we get

$$\begin{aligned}
(3.1)\qquad & \ln\left(m + M - \sum_{k=1}^{n} p_k x_k\right) \\
& \geq \min_{i\in\{1,\dots,n\}} \{p_i\} \left[\sum_{k=1}^{n} \ln(m + M - x_k) - n \ln\left(m + M - \frac{1}{n}\sum_{k=1}^{n} x_k\right)\right] \\
& + \ln\left(m + M - \sum_{k=1}^{n} p_k x_k\right) \\
& \geq \sum_{k=1}^{n} p_k \ln(m + M - x_k) \geq \ln(m) + \ln(M) - \sum_{k=1}^{n} p_k \ln(x_k),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(3.2)\qquad & m + M - \sum_{k=1}^{n} p_k x_k \\
& \geq \left[\frac{\prod_{k=1}^{n}(m + M - x_k)}{\left(m + M - \frac{1}{n}\sum_{k=1}^{n} x_k\right)^n}\right]^{\min_{i\in\{1,\dots,n\}}\{p_i\}} \left(m + M - \sum_{k=1}^{n} p_k x_k\right) \\
& \geq \prod_{k=1}^{n}(m + M - x_k)^{p_k} \geq \frac{mM}{\prod_{k=1}^{n} x_k^{p_k}},
\end{aligned}$$

for $x_k \in [m, M]$, $p_k \geq 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

From (3.2) we also get

$$\begin{aligned}
(3.3)\qquad & m + M - \sum_{k=1}^{n} p_k x_k - \frac{mM}{\prod_{k=1}^{n} x_k^{p_k}} \\
& \geq \left[\frac{\prod_{k=1}^{n}(m + M - x_k)}{\left(m + M - \frac{1}{n}\sum_{k=1}^{n} x_k\right)^n}\right]^{\min_{i\in\{1,\dots,n\}}\{p_i\}} \left(m + M - \sum_{k=1}^{n} p_k x_k\right) \\
& - \prod_{k=1}^{n}(m + M - x_k)^{p_k} \\
& \geq 0.
\end{aligned}$$

From (2.13) we derive

$$\begin{aligned}(3.4)\qquad & \ln\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\ & \geq \sum_{k=1}^{n}p_k\ln\left(m+M-x_k\right)\\ & \geq \frac{\sum_{k=1}^{n}p_kx_k-m}{M-m}\ln\left(m\right)+\frac{M-\sum_{k=1}^{n}p_kx_k}{M-m}\ln\left(M\right)\\ & \geq \ln\left(m\right)+\ln\left(M\right)-\sum_{k=1}^{n}p_k\ln\left(x_k\right)-\frac{1}{2M^2}\sum_{k=1}^{n}p_k\left(M-x_k\right)\left(x_k-m\right)\\ & \leq f\left(m\right)+f\left(M\right)-\sum_{k=1}^{n}p_k\ln\left(x_k\right)\end{aligned}$$

since $f''(x)=\frac{1}{x^2}\geq\frac{1}{M^2}$ for $x\in[m,M]$.

This inequality is equivalent to

$$\begin{aligned}(3.5)\qquad m+M-\sum_{k=1}^{n}p_kx_k & \geq \prod_{k=1}^{n}\left(m+M-x_k\right)^{p_k}\\ & \geq m^{\frac{\sum_{k=1}^{n}p_kx_k-m}{M-m}}M^{\frac{M-\sum_{k=1}^{n}p_kx_k}{M-m}}\\ & \geq \frac{mM}{\prod_{k=1}^{n}x_k^{p_k}\exp\left(\frac{1}{2M^2}\sum_{k=1}^{n}p_k\left(M-x_k\right)\left(x_k-m\right)\right)}\\ & \geq \frac{mM}{\prod_{k=1}^{n}x_k^{p_k}}\end{aligned}$$

for $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^{n}p_k=1$.

Consider the function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f(x)=x^r$ which is convex for $r\in(-\infty,0)\cup[1,\infty)$ and concave for $r\in(0,1)$. If we use (2.1), then we get for $r\in(-\infty,0)\cup[1,\infty)$

$$\begin{aligned}(3.6)\qquad & \left(m+M-\sum_{k=1}^{n}p_kx_k\right)^r\\ & \leq \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\sum_{k=1}^{n}\left(m+M-x_k\right)^r-n\left(m+M-\frac{1}{n}\sum_{k=1}^{n}x_k\right)^r\right]\\ & +\left(m+M-\sum_{k=1}^{n}p_kx_k\right)^r\\ & \leq \sum_{k=1}^{n}p_k\left(m+M-x_k\right)^r\leq m^r+M^r-\sum_{k=1}^{n}p_kx_k^r\end{aligned}$$

for $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^{n}p_k=1$.

The inequality (3.6) implies that

$$0 \leq \sum_{k=1}^{n} p_k (m + M - x_k)^r - \left(m + M - \sum_{k=1}^{n} p_k x_k\right)^r - \min_{i \in \{1,...,n\}} \{p_i\} \left[\sum_{k=1}^{n} (m + M - x_k)^r - n\left(m + M - \frac{1}{n}\sum_{k=1}^{n} x_k\right)^r\right] \leq m^r + M^r - \sum_{k=1}^{n} p_k x_k^r - \left(m + M - \sum_{k=1}^{n} p_k x_k\right)^r. \tag{3.7}$$

From (2.13) we get

$$\left(m + M - \sum_{k=1}^{n} p_k x_k\right)^r \leq \sum_{k=1}^{n} p_k (m + M - x_k)^r \leq \frac{\sum_{k=1}^{n} p_k x_k - m}{M - m} m^r + \frac{M - \sum_{k=1}^{n} p_k x_k}{M - m} M^r \leq m^r + M^r - \sum_{k=1}^{n} p_k x_k^r - \frac{1}{2}\gamma_r \sum_{k=1}^{n} p_k (M - x_k)(x_k - m) \leq m^r + M^r - \sum_{k=1}^{n} p_k x_k^r, \tag{3.8}$$

where $x_k \in [m, M]$, $p_k \geq 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^{n} p_k = 1$, while

$$\gamma_r := r(r-1) \times \begin{cases} m^{r-2} \text{ if } r \geq 2, \\ M^{r-2} \text{if } r \in (-\infty, 0) \cup [1, 2). \end{cases}$$

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