SOME GENERALIZATIONS OF JENSEN-MERCER TYPE INEQUALITY FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. In this paper we extend Mercer's discrete inequality for univariate functions to the case of convex functions on convex subsets of linear spaces and provide some natural applications for norms.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\ldots,p_n)$ is a probability sequence and $\mathbf{x}=(x_1,\ldots,x_n)\in C^n$, then

$$f\left(\sum_{i=1}^{n}p_i x_i\right)\le\sum_{i=1}^{n}p_i f\left(x_i\right), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [18] obtained the following inequality for convex functions of a real variable $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$ and the finite sequences $x_k\in[m,M]$, and $p_k\ge 0$, $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$,

$$f\left(m+M-\sum_{k=1}^{n}p_k x_k\right)\le f\left(m\right)+f\left(M\right)-\sum_{k=1}^{n}p_k f\left(x_k\right) \tag{1.2}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [16], [19], [25], in relation with majorization theory [23], for convex functions of selfadjoint operators in Hilbert spaces [15], [17], [20], [21] and for operator convex functions in Hilbert spaces [22] and [25].

In the recent paper [11], we obtained the following generalization of Jensen-Mercer inequality for convex functions on convex subsets of linear spaces:

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Theorem 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C and x, $y \in C$. If $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k f(x + y - x_k) \tag{1.3}$$
$$\leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x)$$
$$\leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k).$$

Since the distance between the extreme terms is greater than the distance between the internal ones, we can state the following corollary as well:

Corollary 1. *With the assumptions of Theorem 1 we have*

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x) - \sum_{k=1}^{n} p_k f(x + y - x_k) \tag{1.4}$$
$$\leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right).$$

Motivated by the above results, in this paper we obtain some generalizations of (1.3) for convex functions on convex subsets of linear spaces and provide some natural applications for normed spaces.

2. Main Results

We have the following Mercer's type inequalities for functions defined on convex subsets C in a linear space X.

Theorem 2. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C, x, $y \in C$ and $c \in X$ with $c + x$, $c + y \in C$. If $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(c + x + y - \sum_{k=1}^{n} p_k x_k\right) \tag{2.1}$$
$$\leq \sum_{k=1}^{n} p_k f(c + x + y - x_k)$$
$$\leq \left(\sum_{k=1}^{n} p_k t_k\right) f(c + x) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(c + y)$$
$$\leq f(c + x) + f(c + y) - \sum_{k=1}^{n} p_k f(c + x_k).$$

Proof. For x, $y \in C$ and $t \in [0,1]$, put $x_t := (1 - t) x + ty$. Since $c + x$, $c + y \in C$, hence by the convexity of C we have

$$(1 - t)(c + x) + t(c + y) = c + x_t.$$

By the convexity of f on C we then have

$$
\begin{aligned}
f(c+x_t) &= f[(1-t)(c+x)+t(c+y)] \\
&\leq (1-t) f(c+x)+t f(c+y) \\
&= f(c+x)+f(c+y) \\
&\quad -[t f(c+x)+(1-t) f(c+y)] \\
&\leq f(c+x)+f(c+y) \\
&\quad - f[t(c+x)+(1-t)(c+y)] \\
&= f(c+x)+f(c+y)-f[c+tx+(1-t) y] \\
&= f(c+x)+f(c+y) \\
&\quad - f[c+x+y-(1-t) x-t y] \\
&= f(c+x)+f(c+y)-f(c+x+y-x_t).
\end{aligned} \tag{2.2}
$$

Since $t_k \in [0,1]$, then by taking $t=t_k$ in (2.2) we get

$$
\begin{aligned}
f(c+x_k) &\leq (1-t_k) f(c+x)+t_k f(c+y) \\
&\leq f(c+x)+f(c+y)-f(c+x+y-x_k)
\end{aligned} \tag{2.3}
$$

for all $k \in \{1,...,n\}$.

Since $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$, then by multiplying (2.3) with $p_k \geq 0$ and summing over k from 1 to n, we get

$$
\begin{aligned}
& \sum_{k=1}^{n} p_k f(c+x_k) \\
& \leq \sum_{k=1}^{n} p_k [(1-t_k) f(c+x)+t_k f(c+y)] \\
& = \left[\left(1-\sum_{k=1}^{n} p_k t_k\right) f(c+x)+\left(\sum_{k=1}^{n} p_k t_k\right) f(c+y)\right] \\
& \leq f(c+x)+f(c+y)-\sum_{k=1}^{n} p_k f(c+x+y-x_k).
\end{aligned} \tag{2.4}
$$

By Jensen's inequality we also have

$$
\sum_{k=1}^{n} p_k f(c+x+y-x_k) \geq f\left(c+x+y-\sum_{k=1}^{n} p_k x_k\right). \tag{2.5}
$$

Therefore, by (2.4) and (2.5) we get the following inequality of interest

$$\begin{aligned}
&\sum_{k=1}^{n} p_k f(c+x_k) \\
&\leq \left[\left(1-\sum_{k=1}^{n} p_k t_k\right) f(c+x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(c+y) \right] \\
&\leq f(c+x) + f(c+y) - \sum_{k=1}^{n} p_k f(c+x+y-x_k) \\
&\leq f(c+x) + f(c+y) - f\left(c+x+y-\sum_{k=1}^{n} p_k x_k\right),
\end{aligned} \tag{2.6}$$

which is clearly equivalent to (2.1). □

Remark 1. *Since the distance between the extreme terms is greater than the distance between the internal ones, hence we can state that*

$$\begin{aligned}
0 &\leq \left(\sum_{k=1}^{n} p_k t_k\right) f(c+x) + \left(1-\sum_{k=1}^{n} p_k t_k\right) f(c+y) \\
&\quad - \sum_{k=1}^{n} p_k f(c+x+y-x_k) \\
&\leq f(c+x) + f(c+y) - \sum_{k=1}^{n} p_k f(c+x_k) - f\left(c+x+y-\sum_{k=1}^{n} p_k x_k\right),
\end{aligned} \tag{2.7}$$

provided that $f : C \subset X \to \mathbb{R}$ is a convex function on the convex subset C, $x, y \in C$ and $c \in X$ with $c+x, c+y \in C$ while $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$.

Corollary 2. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C and $x, y \in C$. If $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$, then* [11]

$$\begin{aligned}
f\left(x+y-\sum_{k=1}^{n} p_k x_k\right) &\leq \sum_{k=1}^{n} p_k f(x+y-x_k) \\
&\leq \left(\sum_{k=1}^{n} p_k t_k\right) f(x) + \left(1-\sum_{k=1}^{n} p_k t_k\right) f(y) \\
&\leq f(x) + f(y) - \sum_{k=1}^{n} p_k f(x_k).
\end{aligned} \tag{2.8}$$

The proof follows by (2.1) on choosing $c = 0$.

Corollary 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C, $x, y \in C$ and $2y - x \in C$. If $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$,*

$k \in \{1, ..., n\}$ *with* $\sum_{k=1}^{n} p_k = 1$, *then*

$$f\left(2y - \sum_{k=1}^{n} p_k x_k\right) \le \sum_{k=1}^{n} p_k f(2y - x_k) \le \left(\sum_{k=1}^{n} p_k t_k\right) f(y) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(2y - x) \le f(y) + f(2y - x) - \sum_{k=1}^{n} p_k f(y - x + x_k). \tag{2.9}$$

Follows by (2.1) on choosing $c = y - x$.

Corollary 4. *Let* $f : C \subset X \to \mathbb{R}$ *be a convex function on the convex subset* C, $x, y \in C$ *with* $\frac{x-y}{2}, \frac{y-x}{2} \in C$. *If* $x_k := (1 - t_k)x + t_k y$ *with* $t_k \in [0, 1]$ *and* $p_k \ge 0$, $k \in \{1, ..., n\}$ *with* $\sum_{k=1}^{n} p_k = 1$, *then*

$$f\left(\frac{x+y}{2} - \sum_{k=1}^{n} p_k x_k\right) \le \sum_{k=1}^{n} p_k f\left(\frac{x+y}{2} - x_k\right) \le \left(\sum_{k=1}^{n} p_k t_k\right) f\left(\frac{x-y}{2}\right) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f\left(\frac{y-x}{2}\right) \le f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - \sum_{k=1}^{n} p_k f\left(x_k - \frac{x+y}{2}\right). \tag{2.10}$$

The proof is obvious by (2.1) on choosing $c = -\frac{x+y}{2}$.

We say that a convex subset C in X is *even*, if $-C = C$. The function $f : C \to \mathbb{R}$ on the even set C is also called *even*, if $f(-x) = f(x)$ for all $x \in C$.

Corollary 5. *Let* $f : C \subset X \to \mathbb{R}$ *be an even convex function on the even convex subset* C *and* $x, y \in C$. *If* $x_k := (1 - t_k)x + t_k y$ *with* $t_k \in [0, 1]$ *and* $p_k \ge 0$, $k \in \{1, ..., n\}$ *with* $\sum_{k=1}^{n} p_k = 1$, *then*

$$f\left(\sum_{k=1}^{n} p_k x_k\right) \le \sum_{k=1}^{n} p_k f(x_k) \le \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(y) \le f(x) + f(y) - \sum_{k=1}^{n} p_k f(x_k - x - y) \tag{2.11}$$

and

$$0 \le \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(y) - \sum_{k=1}^{n} p_k f(x_k) \le f(x) + f(y) - \sum_{k=1}^{n} p_k f(x_k - x - y) - f\left(\sum_{k=1}^{n} p_k x_k\right) \tag{2.12}$$

We take $c = -x - y$ in (2.1) and use the fact that f is even.

The following result providing a dual inequality also holds:

Theorem 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C, $x, y \in C$ and $c \in X$ with $c - C \subset C$. If $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0, 1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(c - \sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k f(c - x_k) \tag{2.13}$$
$$\leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(c - x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(c - y)$$
$$\leq f(c - x) + f(c - y) - \sum_{k=1}^{n} p_k f(c + x + y - x_k)$$

and

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(c - x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(c - y) - \sum_{k=1}^{n} p_k f(c - x_k) \tag{2.14}$$
$$\leq f(c - x) + f(c - y) - \sum_{k=1}^{n} p_k f(c + x + y - x_k) - f\left(c - \sum_{k=1}^{n} p_k x_k\right)$$

Proof. By Jensen's discrete inequality for f, we have

$$f\left(c - \sum_{k=1}^{n} p_k x_k\right) = f\left(\sum_{k=1}^{n} p_k (c - x_k)\right) \leq \sum_{k=1}^{n} p_k f(c - x_k)$$

which proves the first inequality in (2.13).

Let $t \in [0, 1]$ and $x, y \in C$, then by the convexity of f we have

$$f(c - (1 - t) x - ty) = f((1 - t)(c - x) + t(c - y)) \tag{2.15}$$
$$\leq (1 - t) f(c - x) + tf(c - y).$$

If we take $t = t_k$ in (2.15), then we get

$$f(c - x_k) = f(c - (1 - t_k) x - t_k y) \leq (1 - t_k) f(c - x) + t_k f(c - y). \tag{2.16}$$

If we multiply this inequality by $p_k \geq 0$ and sum over k from 1to n, we get

$$\sum_{k=1}^{n} p_k f(c - x_k) \leq \sum_{k=1}^{n} p_k [(1 - t_k) f(c - x) + t_k f(c - y)]$$
$$= \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(c - x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(c - y).$$

By the convexity of f we also have

$$t_k f(c - x) + (1 - t_k) f(c - y) \geq f[t_k (c - x) + (1 - t_k)(c - y)] \tag{2.17}$$

for $k \in \{1, ..., n\}$. Now, observe that

$$t_k (c - x) + (1 - t_k)(c - y)$$
$$= t_k (c) + (1 - t_k)(c) - t_k x - (1 - t_k) y = c - t_k x - (1 - t_k) y$$
$$= c + x + y - ((1 - t_k) x + t_k y) = c + x + y - x_k$$

for $k \in \{1, ..., n\}$, which by (2.17) gives that

$$t_k f(c - x) + (1 - t_k) f(c - y) \geq f(c + x + y - x_k) \tag{2.18}$$

for $k \in \{1, ..., n\}$.

By multiplying with $p_k \geq 0$ in (2.18) and summing over k from 1 to n, we get

$$\left(\sum_{k=1}^{n} p_k t_k\right) f(c-x) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(c-y) \\ \geq \sum_{k=1}^{n} p_k f(c+x+y-x_k).$$

Therefore

$$f(c-x) + f(c-y) \\ - \left[\left(\sum_{k=1}^{n} p_k t_k\right) f(c-x) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(c-y)\right] \\ \leq f(c-x) + f(c-y) - \sum_{k=1}^{n} p_k f(c+x+y-x_k),$$

which proves the second inequality in (2.13). □

Corollary 6. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C, x, $y \in C$ with $x+y-C \subset C$. If $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(x+y-\sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k f(x+y-x_k) \\ \leq \left(\sum_{k=1}^{n} p_k t_k\right) f(x) + \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) \\ \leq f(x) + f(y) - \sum_{k=1}^{n} p_k f(2x+2y-x_k) \tag{2.19}$$

and

$$0 \leq \left(\sum_{k=1}^{n} p_k t_k\right) f(x) + \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) - \sum_{k=1}^{n} p_k f(x+y-x_k) \\ \leq f(x) + f(y) - \sum_{k=1}^{n} p_k f(2x+2y-x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right). \tag{2.20}$$

The proof follows by (2.13) for $c = x+y$.

Corollary 7. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C, x, $y \in C$ with $\frac{x+y}{2} - C \subset C$. If $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$,*

$k \in \{1, ..., n\}$ *with* $\sum_{k=1}^{n} p_k = 1,$ *then*

$$f\left(\frac{x+y}{2} - \sum_{k=1}^{n} p_k x_k\right) \tag{2.21}$$
$$\leq \sum_{k=1}^{n} p_k f\left(\frac{x+y}{2} - x_k\right)$$
$$\leq \left(\sum_{k=1}^{n} p_k t_k\right) f\left(\frac{x-y}{2}\right) + \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f\left(\frac{y-x}{2}\right)$$
$$\leq f\left(\frac{y-x}{2}\right) + f\left(\frac{x-y}{2}\right) - \sum_{k=1}^{n} p_k f\left[3\left(\frac{x+y}{2}\right) - x_k\right].$$

The proof follows by (2.13) on taking $c = \frac{x+y}{2}$.

J. Pečarić and the author obtained in 1989, the following refinement of Jensen inequality (see [24]):

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1} \ldots p_{i_{k+1}} f\left(\frac{x_{i_1} + \cdots + x_{i_{k+1}}}{k+1}\right) \tag{2.22}$$
$$\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$
$$\leq \cdots \leq \sum_{i=1}^{n} p_i f(x_i),$$

for $k \geq 1$ and $\mathbf{p}$, $\mathbf{x}$ as above.

If $q_1, \ldots, q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following refinement obtained in 1994 by the author [5] also holds:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \tag{2.23}$$
$$\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \ldots p_{i_k} f(q_1 x_{i_1} + \cdots + q_k x_{i_k})$$
$$\leq \sum_{i=1}^{n} p_i f(x_i),$$

where $1 \leq k \leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.

Then by (2.22) and (2.23) we obtain

$$\begin{aligned}
&f\left(c+x+y-\sum_{k=1}^{n}p_kx_k\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(c+x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(c+x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\ldots \\
&\leq \sum_{k=1}^{n}p_kf\left(c+x+y-x_k\right)
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&f\left(c+x+y-\sum_{i=1}^{n}p_ix_i\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(c+x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(c+x+y-(q_1x_{i_1}+\cdots+q_kx_{i_k})\right) \\
&\leq \sum_{i=1}^{n}p_if\left(c+x+y-x_k\right),
\end{aligned} \tag{2.25}$$

with the assumptions that $f : C \subset X \to \mathbb{R}$ is a convex function on the convex subset C, $x, y \in C$ and $c \in X$ with $c+x, c+y \in C$ while $x_k := (1-t_k)x+t_ky$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$. These inequalities provide refinements of the first inequality in (2.1).

3. Norm Inequalities

The function $f : X \to [0,\infty)$, $f(x) = \|x\|^p$, $p \geq 1$, is convex and even on the normed space $(X, \|\cdot\|)$. In the following we assume that $x, y \in X$, $t_k \in [0,1]$, $x_k := (1-t_k)x + t_ky$ and $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$.

By using the inequality (2.1) we have for $c \in X$ that

$$\begin{aligned}
&\left\|c+x+y-\sum_{k=1}^{n}p_kx_k\right\|^p \\
&\leq \sum_{k=1}^{n}p_k\left\|c+x+y-x_k\right\|^p \\
&\leq \left(\sum_{k=1}^{n}p_kt_k\right)\|c+x\|^p+\left(1-\sum_{k=1}^{n}p_kt_k\right)\|c+y\|^p \\
&\leq \|c+x\|^p+\|c+y\|^p-\sum_{k=1}^{n}p_k\|c+x_k\|^p
\end{aligned} \tag{3.1}$$

and

$$0 \leq \left(\sum_{k=1}^{n} p_k t_k\right) \|c+x\|^p + \left(1 - \sum_{k=1}^{n} p_k t_k\right) \|c+y\|^p - \sum_{k=1}^{n} p_k \|c+x+y-x_k\|^p \leq \|c+x\|^p + \|c+y\|^p - \sum_{k=1}^{n} p_k \|c+x_k\|^p - \left\|c+x+y-\sum_{k=1}^{n} p_k x_k\right\|^p \tag{3.2}$$

In particular, we have

$$\left\|x+y-\sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \|x+y-x_k\|^p \leq \left(\sum_{k=1}^{n} p_k t_k\right) \|x\|^p + \left(1 - \sum_{k=1}^{n} p_k t_k\right) \|y\|^p \leq \|x\|^p + \|y\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p . \tag{3.3}$$

$$\left\|2y-\sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \|2y-x_k\|^p \leq \left(\sum_{k=1}^{n} p_k t_k\right) \|y\|^p + \left(1 - \sum_{k=1}^{n} p_k t_k\right) \|2y-x\|^p \leq \|y\|^p + \|2y-x\|^p - \sum_{k=1}^{n} p_k \|y-x+x_k\|^p . \tag{3.4}$$

$$\left\|\frac{x+y}{2}-\sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \left\|\frac{x+y}{2}-x_k\right\|^p \leq \left\|\frac{x-y}{2}\right\|^p \leq 2\left\|\frac{x-y}{2}\right\|^p - \sum_{k=1}^{n} p_k \left\|x_k-\frac{x+y}{2}\right\|^p \tag{3.5}$$

and

$$\left\|\sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \|x_k\|^p \leq \left(1 - \sum_{k=1}^{n} p_k t_k\right) \|x\|^p + \left(\sum_{k=1}^{n} p_k t_k\right) \|y\|^p \leq \|x\|^p + \|y\|^p - \sum_{k=1}^{n} p_k \|x_k-x-y\|^p . \tag{3.6}$$

From (2.13) we get for $c \in X$ that

$$\left\| c - \sum_{k=1}^{n} p_k x_k \right\|^p \leq \sum_{k=1}^{n} p_k \left\| c - x_k \right\|^p \tag{3.7}$$
$$\leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k \right) \right] \left\| c - x \right\|^p + \left(\sum_{k=1}^{n} p_k t_k \right) \left\| c - y \right\|^p$$
$$\leq \left\| c - x \right\|^p + \left\| c - y \right\|^p - \sum_{k=1}^{n} p_k \left\| c + x + y - x_k \right\|^p$$

and

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k \right) \right] \left\| c - x \right\|^p + \left(\sum_{k=1}^{n} p_k t_k \right) \left\| c - y \right\|^p - \sum_{k=1}^{n} p_k \left\| c - x_k \right\|^p \tag{3.8}$$
$$\leq \left\| c - x \right\|^p + \left\| c - y \right\|^p - \sum_{k=1}^{n} p_k \left\| c + x + y - x_k \right\|^p - \left\| c - \sum_{k=1}^{n} p_k x_k \right\|^p .$$

In particular,

$$\left\| x + y - \sum_{k=1}^{n} p_k x_k \right\|^p \leq \sum_{k=1}^{n} p_k \left\| x + y - x_k \right\|^p \tag{3.9}$$
$$\leq \left(\sum_{k=1}^{n} p_k t_k \right) \left\| x \right\|^p + \left[1 - \left(\sum_{k=1}^{n} p_k t_k \right) \right] \left\| y \right\|^p$$
$$\leq \left\| x \right\|^p + \left\| y \right\|^p - \sum_{k=1}^{n} p_k \left\| 2x + 2y - x_k \right\|^p$$

and

$$\left\| \frac{x+y}{2} - \sum_{k=1}^{n} p_k x_k \right\|^p \leq \sum_{k=1}^{n} p_k \left\| \frac{x+y}{2} - x_k \right\|^p \leq \left\| \frac{x-y}{2} \right\|^p \tag{3.10}$$
$$\leq 2 \left\| \frac{x-y}{2} \right\|^p - \sum_{k=1}^{n} p_k \left\| 3 \left(\frac{x+y}{2} \right) - x_k \right\|^p .$$

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