SOME NEW INEQUALITIES FOR CONVEX FUNCTIONS ON LINEAR SPACES RELATED TO JENSEN'S RESULT

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Abstract. In this paper we obtain some new discrete inequalities for convex functions on linear spaces related to Jensen's result and provide some natural applications for norms.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [18] obtained the following inequality for convex functions of a real variable $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ and the finite sequences $x_k \in [m, M]$, and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$,

$$f\left(m + M - \sum_{k=1}^{n} p_k x_k\right) \leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) \tag{1.2}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [16], [19], [25], in relation with majorization theory [23], for convex functions of selfadjoint operators in Hilbert spaces [15], [17], [20], [21] and for operator convex functions in Hilbert spaces [22] and [25].

In the recent paper [11], we obtained the following generalization of Jensen-Mercer inequality for convex functions on convex subsets of linear spaces:

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Theorem 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C and $x,\ y \in C$. If $x_k := (1 - t_k)\, x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0,\ k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k f\left(x + y - x_k\right) \tag{1.3}$$
$$\leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x)$$
$$\leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k).$$

Since the distance between the extreme terms is greater than the distance between the internal ones, we can state the following corollary as well:

Corollary 1. *With the assumptions of Theorem 1 we have*

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x) - \sum_{k=1}^{n} p_k f(x + y - x_k) \tag{1.4}$$
$$\leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right).$$

The inequalities in (1.3) and (1.4) are valid for sequences $\{x_k\}_{k\in\{1,...,n\}}$ in the interval $[x, y] := \{(1 - s)\, x + sy,\ s \in [0, 1]\}$. In order to remove this restriction and let the sequence $\{x_k\}_{k\in\{1,...,n\}}$ be in the convex set C we should modify the terms in the inequality as follows.

2. Main Results

We have the following main result where the involved sequence belongs to the convex subset C with no restriction.

Theorem 2. *Let $f : C \subset X \to \mathbb{R}$ be a convex function defined on the convex subset C in the linear space X. If $x_k \in C$ for $k \in \{1, ..., n\}$ and $a,\ b \in X$ with $a - x_k$, $b - x_k \in C$ for $k \in \{1, ..., n\}$, then for all $\lambda \in [0, 1]$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$,*

$$f\left((1 - \lambda)\, a + \lambda b - \sum_{k=1}^{n} p_k x_k\right) \tag{2.1}$$
$$\leq \sum_{k=1}^{n} p_k f\left((1 - \lambda)\, a + \lambda b - x_k\right)$$
$$\leq (1 - \lambda) \sum_{k=1}^{n} p_k f(a - x_k) + \lambda \sum_{k=1}^{n} p_k f(b - x_k)$$
$$\leq \sum_{k=1}^{n} p_k f(a - x_k) + \sum_{k=1}^{n} p_k f(b - x_k) - \sum_{k=1}^{n} p_k f\left(\lambda a + (1 - \lambda)\, b - x_k\right)$$

and

$$0 \leq (1-\lambda)\sum_{k=1}^{n} p_k f(a-x_k) + \lambda \sum_{k=1}^{n} p_k f(b-x_k) - \sum_{k=1}^{n} p_k f((1-\lambda)a+\lambda b - x_k) \leq \sum_{k=1}^{n} p_k f(a-x_k) + \sum_{k=1}^{n} p_k f(b-x_k) - \sum_{k=1}^{n} p_k f(\lambda a + (1-\lambda) b - x_k) - f\left((1-\lambda)a+\lambda b - \sum_{k=1}^{n} p_k x_k\right). \tag{2.2}$$

Proof. By using Jensen's inequality, we have

$$\begin{aligned} f\left((1-\lambda)a+\lambda b - \sum_{k=1}^{n} p_k x_k\right) &= f\left(\sum_{k=1}^{n} p_k [(1-\lambda)a+\lambda b - x_k]\right) \\ &= f\left(\sum_{k=1}^{n} p_k [\lambda (a-x_k) + (1-\lambda)(b-x_k)]\right) \\ &\leq \sum_{k=1}^{n} p_k f(\lambda (a-x_k) + (1-\lambda)(b-x_k)) \\ &= \sum_{k=1}^{n} p_k f((1-\lambda)a+\lambda b - x_k) \end{aligned}$$

which proves the first inequality in (2.1).

By the convexity of f we have

$$\begin{aligned} & f((1-\lambda)a+\lambda b - x_k) \\ &= f((1-\lambda)(a-x_k)+\lambda(b-x_k)) \\ &\leq (1-\lambda) f(a-x_k) + \lambda f(b-x_k) \\ &= f(a-x_k) + f(b-x_k) - [\lambda f(a-x_k) + (1-\lambda) f(b-x_k)] \end{aligned}$$

for $k \in \{1, ..., n\}$.

If we multiply this inequality by $p_k \geq 0$ and sum over k from 1 to n, then

$$\begin{aligned} & \sum_{k=1}^{n} p_k f((1-\lambda)a+\lambda b - x_k) \\ &\leq (1-\lambda)\sum_{k=1}^{n} p_k f(a-x_k) + \lambda \sum_{k=1}^{n} p_k f(b-x_k) \\ &\leq \sum_{k=1}^{n} p_k f(a-x_k) + \sum_{k=1}^{n} p_k f(b-x_k) \\ &- \sum_{k=1}^{n} p_k [\lambda f(a-x_k) + (1-\lambda) f(b-x_k)], \end{aligned}$$

which proves the second inequality in (2.1).

By the convexity of f we also have

$$\begin{aligned}\lambda f(a-x_k)+(1-\lambda)f(b-x_k) &\geq f[\lambda(a-x_k)+(1-\lambda)(b-x_k)]\\ &= f(\lambda a+(1-\lambda)b-x_k)\end{aligned}$$

for $k\in\{1,...,n\}$.

If we multiply this inequality by $p_k\geq 0$ and sum over k from 1 to n, then we get

$$\sum_{k=1}^{n}p_k[\lambda f(a-x_k)+(1-\lambda)f(b-x_k)]\geq\sum_{k=1}^{n}p_k f(\lambda a+(1-\lambda)b-x_k),$$

which implies that

$$\begin{aligned}&\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)-\sum_{k=1}^{n}p_k[\lambda f(a-x_k)+(1-\lambda)f(b-x_k)]\\ &\leq\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)-\sum_{k=1}^{n}p_k f(\lambda a+(1-\lambda)b-x_k),\end{aligned}$$

which proves the last part of (2.1).

The inequality (2.2) follows by the fact that the distance between the extreme terms is greater than the distance between the internal ones. □

Corollary 2. *Let $f:C\subset X\to\mathbb{R}$ be a convex function defined on the convex subset C in the linear space X. If $x_k\in C$ for $k\in\{1,...,n\}$ and $a,\ b\in X$ with $a-x_k,\ b-x_k\in C$ for $k\in\{1,...,n\}$, then for all $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$,*

$$\begin{aligned}f\left(\frac{a+b}{2}-\sum_{k=1}^{n}p_k x_k\right)&\\ \leq\sum_{k=1}^{n}p_k f\left(\frac{a+b}{2}-x_k\right)&\\ \leq\frac{1}{2}\left[\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)\right]&\\ \leq\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)-\sum_{k=1}^{n}p_k f\left(\frac{a+b}{2}-x_k\right)&\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}0&\leq\frac{1}{2}\left[\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)\right]-\sum_{k=1}^{n}p_k f\left(\frac{a+b}{2}-x_k\right)\\ &\leq\sum_{k=1}^{n}p_k f(a-x_k)+\sum_{k=1}^{n}p_k f(b-x_k)-\sum_{k=1}^{n}p_k f\left(\frac{a+b}{2}-x_k\right)\\ &\quad-f\left(\frac{a+b}{2}-\sum_{k=1}^{n}p_k x_k\right).\end{aligned}\tag{2.4}$$

Remark 1. *If $x, y \in X$ such that x_k, $2x - x_k$, $2y - x_k \in C$ for $k \in \{1, ..., n\}$, then for all $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$ we get by (2.3) that*

$$
\begin{aligned}
(2.5)\qquad & f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \\
& \leq \sum_{k=1}^{n} p_k f\left(x + y - x_k\right) \\
& \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(2x - x_k\right) + \sum_{k=1}^{n} p_k f\left(2y - x_k\right)\right] \\
& \leq \sum_{k=1}^{n} p_k f\left(2x - x_k\right) + \sum_{k=1}^{n} p_k f\left(2y - x_k\right) - \sum_{k=1}^{n} p_k f\left(x + y - x_k\right),
\end{aligned}
$$

where $f : C \subset X \to \mathbb{R}$ is a convex function defined on the convex subset C.

We also have the inequalities

$$
\begin{aligned}
(2.6)\qquad & 0 \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(2x - x_k\right) + \sum_{k=1}^{n} p_k f\left(2y - x_k\right)\right] - \sum_{k=1}^{n} p_k f\left(x + y - x_k\right) \\
& \leq \sum_{k=1}^{n} p_k f\left(2x - x_k\right) + \sum_{k=1}^{n} p_k f\left(2y - x_k\right) - \sum_{k=1}^{n} p_k f\left(x + y - x_k\right) \\
& - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right).
\end{aligned}
$$

We can improve the inequality (2.3) as follows:

Corollary 3. *With the assumptions of Theorem 2, we have*

$$
\begin{aligned}
(2.7)\qquad & f\left(\frac{a+b}{2} - \sum_{k=1}^{n} p_k x_k\right) \\
& \leq \frac{1}{2}\left[f\left((1-\lambda) a + \lambda b - \sum_{k=1}^{n} p_k x_k\right) + f\left(\lambda a + (1-\lambda) b - \sum_{k=1}^{n} p_k x_k\right)\right] \\
& \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left((1-\lambda) a + \lambda b - x_k\right) + \sum_{k=1}^{n} p_k f\left(\lambda a + (1-\lambda) b - x_k\right)\right] \\
& \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right)\right] \\
& \leq \sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right) \\
& - \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(\lambda a + (1-\lambda) b - x_k\right) + \sum_{k=1}^{n} p_k f\left((1-\lambda) a + \lambda b - x_k\right)\right] \\
& \leq \sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right) - \sum_{k=1}^{n} p_k f\left(\frac{a+b}{2} - x_k\right).
\end{aligned}
$$

Proof. If we write the inequality (2.1) for $1-\lambda$ instead of λ, then we get

$$\begin{aligned} f\left(\lambda a+(1-\lambda) b-\sum_{k=1}^{n} p_k x_k\right) & \\ \leq \sum_{k=1}^{n} p_k f(\lambda a+(1-\lambda) b-x_k) & \\ \leq \lambda \sum_{k=1}^{n} p_k f(a-x_k)+(1-\lambda) \sum_{k=1}^{n} p_k f(b-x_k) & \\ \leq \sum_{k=1}^{n} p_k f(a-x_k)+\sum_{k=1}^{n} p_k f(b-x_k)-\sum_{k=1}^{n} p_k f((1-\lambda) a+\lambda b-x_k) . & \end{aligned} \tag{2.8}$$

If we add (2.1) with (2.8), then we obtain

$$\begin{aligned} & f\left((1-\lambda) a+\lambda b-\sum_{k=1}^{n} p_k x_k\right)+f\left(\lambda a+(1-\lambda) b-\sum_{k=1}^{n} p_k x_k\right) \\ & \leq \sum_{k=1}^{n} p_k f((1-\lambda) a+\lambda b-x_k)+\sum_{k=1}^{n} p_k f(\lambda a+(1-\lambda) b-x_k) \\ & \leq \sum_{k=1}^{n} p_k f(a-x_k)+\sum_{k=1}^{n} p_k f(b-x_k) \\ & \leq 2\left[\sum_{k=1}^{n} p_k f(a-x_k)+\sum_{k=1}^{n} p_k f(b-x_k)\right] \\ & -\sum_{k=1}^{n} p_k f(\lambda a+(1-\lambda) b-x_k)-\sum_{k=1}^{n} p_k f((1-\lambda) a+\lambda b-x_k) \end{aligned}$$

namely

$$\begin{aligned} & \frac{1}{2}\left[f\left((1-\lambda) a+\lambda b-\sum_{k=1}^{n} p_k x_k\right)+f\left(\lambda a+(1-\lambda) b-\sum_{k=1}^{n} p_k x_k\right)\right] \\ & \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f((1-\lambda) a+\lambda b-x_k)+\sum_{k=1}^{n} p_k f(\lambda a+(1-\lambda) b-x_k)\right] \\ & \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f(a-x_k)+\sum_{k=1}^{n} p_k f(b-x_k)\right] \\ & \leq \sum_{k=1}^{n} p_k f(a-x_k)+\sum_{k=1}^{n} p_k f(b-x_k) \\ & -\frac{1}{2}\left[\sum_{k=1}^{n} p_k f(\lambda a+(1-\lambda) b-x_k)+\sum_{k=1}^{n} p_k f((1-\lambda) a+\lambda b-x_k)\right], \end{aligned}$$

which proves the second, third and fourth inequalities in (2.7).

By the convexity of f,

$$\frac{1}{2}\left[f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_kx_k\right)+f\left(\lambda a+(1-\lambda)b-\sum_{k=1}^{n}p_kx_k\right)\right]$$
$$\geq f\left(\frac{a+b}{2}-\sum_{k=1}^{n}p_kx_k\right),$$

which proves the first inequality in (2.7).

Moreover,

$$\frac{1}{2}\left[f\left(\lambda a+(1-\lambda)b-x_k\right)+f\left((1-\lambda)a+\lambda b-x_k\right)\right]\geq f\left(\frac{a+b}{2}-x_k\right),$$

which implies by multiplying $p_k\geq 0$ and summing over k from 1 to n that

$$\frac{1}{2}\left[\sum_{k=1}^{n}p_kf\left(\lambda a+(1-\lambda)b-x_k\right)+\sum_{k=1}^{n}p_kf\left((1-\lambda)a+\lambda b-x_k\right)\right]$$
$$\geq\sum_{k=1}^{n}p_kf\left(\frac{a+b}{2}-x_k\right),$$

which implies that

$$\sum_{k=1}^{n}p_kf\left(a-x_k\right)+\sum_{k=1}^{n}p_kf\left(b-x_k\right)$$
$$-\frac{1}{2}\left[\sum_{k=1}^{n}p_kf\left(\lambda a+(1-\lambda)b-x_k\right)+\sum_{k=1}^{n}p_kf\left((1-\lambda)a+\lambda b-x_k\right)\right]$$
$$\leq\sum_{k=1}^{n}p_kf\left(a-x_k\right)+\sum_{k=1}^{n}p_kf\left(b-x_k\right)-\sum_{k=1}^{n}p_kf\left(\frac{a+b}{2}-x_k\right),$$

and the last part of (2.7) is proved. □

Remark 2. *Let $f:C\subset X\to\mathbb{R}$ be a convex function defined on the convex subset C in the linear space X. Now, if we take*

$$y_k=\lambda a+(1-\lambda)b-x_k$$

namely

$$x_k=\lambda a+(1-\lambda)b-y_k$$

then, by (2.3), we get for $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$ that

$$f\left((2\lambda-1)(b-a)+\sum_{k=1}^{n}p_ky_k\right) \tag{2.9}$$
$$\leq\sum_{k=1}^{n}p_kf\left((2\lambda-1)(b-a)+y_k\right)$$
$$\leq(1-\lambda)\sum_{k=1}^{n}p_kf\left((1-\lambda)(a-b)+y_k\right)+\lambda\sum_{k=1}^{n}p_kf\left(\lambda(b-a)+y_k\right)$$
$$\leq\sum_{k=1}^{n}p_kf\left((1-\lambda)(a-b)+y_k\right)+\sum_{k=1}^{n}p_kf\left(\lambda(b-a)+y_k\right)-\sum_{k=1}^{n}p_kf\left(y_k\right)$$

provided that $y_k \in C$ *and*

$$a - x_k = (1-\lambda)(a-b) + y_k,\ b - x_k = \lambda(b-a) + y_k \in C$$

for $k \in \{1, ..., n\}$ *and* $\lambda \in [0,1]$.

For $\lambda = \frac{1}{2}$, *we derive the inequality*

$$\begin{aligned} (2.10) \quad & f\left(\sum_{k=1}^{n} p_k y_k\right) \\ & \leq \sum_{k=1}^{n} p_k f(y_k) \\ & \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(\frac{1}{2}(a-b) + y_k\right) + \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(b-a) + y_k\right)\right] \\ & \leq \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(a-b) + y_k\right) + \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(b-a) + y_k\right) - \sum_{k=1}^{n} p_k f(y_k) \end{aligned}$$

provided that $y_k \in C$ *and* $\frac{1}{2}(a-b) + y_k,\ \frac{1}{2}(b-a) + y_k \in C$ *for* $k \in \{1, ..., n\}$.

From (2.10) we also get

$$\begin{aligned} (2.11) \quad 0 & \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(\frac{1}{2}(a-b) + y_k\right) + \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(b-a) + y_k\right)\right] \\ & - \sum_{k=1}^{n} p_k f(y_k) \\ & \leq \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(a-b) + y_k\right) + \sum_{k=1}^{n} p_k f\left(\frac{1}{2}(b-a) + y_k\right) \\ & - \sum_{k=1}^{n} p_k f(y_k) - f\left(\sum_{k=1}^{n} p_k y_k\right) \end{aligned}$$

provided that $y_k \in C$ *and* $\frac{1}{2}(a-b) + y_k,\ \frac{1}{2}(b-a) + y_k \in C$ *for* $k \in \{1, ..., n\}$.

We also have:

Theorem 3. *Let* $f : C \subset X \to \mathbb{R}$ *be a convex function defined on the convex subset* C *in the linear space* X. *If* $x_k \in C$ *for* $k \in \{1, ..., n\}$ *and* $a, b \in X$ *with* $a - x_k$, $b - x_k \in C$ *for* $k \in \{1, ..., n\}$, *then for all* $\lambda \in [0,1]$ *and* $p_k \geq 0$ *for* $k \in \{1, ..., n\}$

with $\sum_{k=1}^{n} p_k = 1$,

$$
\begin{aligned}
(2.12)\qquad & 0 \le (1-\lambda)\sum_{k=1}^{n} p_k f(a-x_k) + \lambda \sum_{k=1}^{n} p_k f(b-x_k) \\
& - \sum_{k=1}^{n} p_k f((1-\lambda)a + \lambda b - x_k) \\
& \le (1-\lambda)\sum_{k=1}^{n} p_k f(a-x_k) + \lambda \sum_{k=1}^{n} p_k f(b-x_k) \\
& - f\left((1-\lambda)a + \lambda b - \sum_{k=1}^{n} p_k x_k\right) \\
& - \min_{i\in\{1,\dots,n\}}\{p_i\} \\
& \times \left[\sum_{k=1}^{n} f((1-\lambda)a + \lambda b - x_k) - nf\left((1-\lambda)a + \lambda b - \frac{1}{n}\sum_{k=1}^{n} x_k\right)\right] \\
& \le (1-\lambda)\sum_{k=1}^{n} p_k f(a-x_k) + \lambda \sum_{k=1}^{n} p_k f(b-x_k) \\
& - f\left((1-\lambda)a + \lambda b - \sum_{k=1}^{n} p_k x_k\right).
\end{aligned}
$$

Proof. Further on, we recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
(2.13)\qquad & n \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \\
& \le \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \\
& \le n \max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right],
\end{aligned}
$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

Using (2.13) we have

$$
\begin{aligned}
(2.14)\qquad & \sum_{k=1}^{n} p_k f((1-\lambda)a + \lambda b - x_k) - f\left((1-\lambda)a + \lambda b - \sum_{k=1}^{n} p_k x_k\right) \\
& \ge \min_{i\in\{1,\dots,n\}}\{p_i\} \\
& \times \left[\sum_{k=1}^{n} f((1-\lambda)a + \lambda b - x_k) - nf\left((1-\lambda)a + \lambda b - \frac{1}{n}\sum_{k=1}^{n} x_k\right)\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-\lambda)\sum_{k=1}^{n}p_k f(a-x_k)+\lambda\sum_{k=1}^{n}p_k f(b-x_k)\\
&-\sum_{k=1}^{n}p_k f((1-\lambda)a+\lambda b-x_k)\\
&=(1-\lambda)\sum_{k=1}^{n}p_k f(a-x_k)+\lambda\sum_{k=1}^{n}p_k f(b-x_k)\\
&-\sum_{k=1}^{n}p_k f((1-\lambda)a+\lambda b-x_k)+f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right)\\
&-f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&=(1-\lambda)\sum_{k=1}^{n}p_k f(a-x_k)+\lambda\sum_{k=1}^{n}p_k f(b-x_k)\\
&-f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right)\\
&-\left(\sum_{k=1}^{n}p_k f((1-\lambda)a+\lambda b-x_k)-f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right)\right)\\
&\leq(1-\lambda)\sum_{k=1}^{n}p_k f(a-x_k)+\lambda\sum_{k=1}^{n}p_k f(b-x_k)\\
&-f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right)\\
&-\min_{i\in\{1,\dots,n\}}\{p_i\}\\
&\times\left[\sum_{k=1}^{n}f((1-\lambda)a+\lambda b-x_k)-nf\left((1-\lambda)a+\lambda b-\frac{1}{n}\sum_{k=1}^{n}x_k\right)\right]\\
&\leq(1-\lambda)\sum_{k=1}^{n}p_k f(a-x_k)+\lambda\sum_{k=1}^{n}p_k f(b-x_k)\\
&-f\left((1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_k x_k\right),
\end{aligned}
$$

which proves the desired inequality (2.12). □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$0 \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right)\right] - \sum_{k=1}^{n} p_k f\left(\frac{a+b}{2} - x_k\right) \tag{2.15}$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right)\right] - f\left(\frac{a+b}{2} - \sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,\ldots,n\}}\{p_i\}$$
$$\times\left[\sum_{k=1}^{n} f\left(\frac{a+b}{2} - x_k\right) - nf\left(\frac{a+b}{2} - \frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(a - x_k\right) + \sum_{k=1}^{n} p_k f\left(b - x_k\right)\right] - f\left(\frac{a+b}{2} - \sum_{k=1}^{n} p_k x_k\right).$$

3. Norm Inequalities

The above inequalities can be stated for norms as follows. Consider $(X, \|\cdot\|)$ to be a normed linear space and consider the function $f(x) = \|x\|^p$, $p \geq 1$. Then by (2.1) and (2.2) we get

$$\left\|(1-\lambda)a + \lambda b - \sum_{k=1}^{n} p_k x_k\right\|^p \tag{3.1}$$
$$\leq \sum_{k=1}^{n} p_k \left\|(1-\lambda)a + \lambda b - x_k\right\|^p$$
$$\leq (1-\lambda)\sum_{k=1}^{n} p_k \left\|a - x_k\right\|^p + \lambda \sum_{k=1}^{n} p_k \left\|b - x_k\right\|^p$$
$$\leq \sum_{k=1}^{n} p_k \left\|a - x_k\right\|^p + \sum_{k=1}^{n} p_k \left\|b - x_k\right\|^p - \sum_{k=1}^{n} p_k \left\|\lambda a + (1-\lambda)b - x_k\right\|^p$$

and

$$0 \leq (1-\lambda)\sum_{k=1}^{n} p_k \left\|a - x_k\right\|^p + \lambda \sum_{k=1}^{n} p_k \left\|b - x_k\right\|^p \tag{3.2}$$
$$- \sum_{k=1}^{n} p_k \left\|(1-\lambda)a + \lambda b - x_k\right\|^p$$
$$\leq \sum_{k=1}^{n} p_k \left\|a - x_k\right\|^p + \sum_{k=1}^{n} p_k \left\|b - x_k\right\|^p - \sum_{k=1}^{n} p_k \left\|\lambda a + (1-\lambda)b - x_k\right\|^p$$
$$- \left\|(1-\lambda)a + \lambda b - \sum_{k=1}^{n} p_k x_k\right\|^p$$

for all $a, b, x_k \in X$, $k \in \{1, ..., n\}$, for all $\lambda \in [0,1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

In particular,

$$\left\|\frac{a+b}{2}-\sum_{k=1}^{n}p_kx_k\right\|^p \tag{3.3}$$
$$\leq \sum_{k=1}^{n}p_k\left\|\frac{a+b}{2}-x_k\right\|^p \leq \frac{1}{2}\left(\sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p\right)$$
$$\leq \sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p-\sum_{k=1}^{n}p_k\left\|\frac{a+b}{2}-x_k\right\|^p$$

and

$$0\leq \frac{1}{2}\left(\sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p\right)-\sum_{k=1}^{n}p_k\left\|\frac{a+b}{2}-x_k\right\|^p \tag{3.4}$$
$$\leq \sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p-\sum_{k=1}^{n}p_k\left\|\frac{a+b}{2}-x_k\right\|^p$$
$$-\left\|\frac{a+b}{2}-\sum_{k=1}^{n}p_kx_k\right\|^p$$

for all a, b, $x_k\in X$, $k\in\{1,...,n\}$ and $p_k\geq 0$, $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$.

From (2.7) we also have

$$\left\|\frac{a+b}{2}-\sum_{k=1}^{n}p_kx_k\right\|^p \tag{3.5}$$
$$\leq \frac{1}{2}\left[\left\|(1-\lambda)a+\lambda b-\sum_{k=1}^{n}p_kx_k\right\|^p+\left\|\lambda a+(1-\lambda)b-\sum_{k=1}^{n}p_kx_k\right\|^p\right]$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{n}p_k\|(1-\lambda)a+\lambda b-x_k\|^p+\sum_{k=1}^{n}p_k\|\lambda a+(1-\lambda)b-x_k\|^p\right]$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p\right]$$
$$\leq \sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p$$
$$-\frac{1}{2}\left[\sum_{k=1}^{n}p_k\|\lambda a+(1-\lambda)b-x_k\|^p+\sum_{k=1}^{n}p_k\|(1-\lambda)a+\lambda b-x_k\|^p\right]$$
$$\leq \sum_{k=1}^{n}p_k\|a-x_k\|^p+\sum_{k=1}^{n}p_k\|b-x_k\|^p-\sum_{k=1}^{n}p_k\left\|\frac{a+b}{2}-x_k\right\|^p.$$

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