New inequalities for quotients of circular and hyperbolic functions

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Abstract. This paper deals with new inequalities involving the quotients $\frac{\sin x}{\sinh x}$, $\frac{\cos x}{\cosh x}$ and $\frac{\tan x}{\tanh x}$. The proofs are based on l'Hôpital's rule of monotonicity, series expansions using Bernoulli numbers, and some analytical techniques. Some of the obtained inequalities have resemblance with Adamović-Mitrinović, Wilker and Cusa-Huygens type inequalities.

1 Introduction

We begin with the following two results recently established by C. Chesneau and Y. J. Bagul [8] for the quotients of circular and hyperbolic functions. For similar results for the products of these functions we refer to [9] and references therein.

Theorem 1. *[8, Proposition 2]: For $x \in (0, \alpha)$ where $\alpha \in (0, \pi/2)$, we have*

$$e^{-\beta x^2} \leqslant \frac{\cos x}{\cosh x}, \tag{1.1}$$

with $\beta = \ln(\cosh \alpha / \cos \alpha)/\alpha^2$.

Theorem 2. *[8, Proposition 4]: For $x \in (0, \pi/2)$, we have*

$$e^{-\gamma x^2} < \frac{\sin x}{\sinh x}, \tag{1.2}$$

with $\gamma = 4\ln(\sinh \pi/2)/\pi^2 \approx 0.337794$.

The inequalities (1.1) and (1.2) are generalized in [17]. We can obtain similar types of exponential bounds for both the quotients $\frac{\cos x}{\cosh x}$ and $\frac{\sin x}{\sinh x}$ by using exponential bounds of $\frac{\sin x}{x}$, $\frac{x}{\sinh x}$, $\cos x$ and $\cosh x$ given in [3, 4, 10] after slight rearrangement of terms as follows:

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Theorem 3. *For $x \in [0, \alpha]$ where $\alpha \in (0, \pi/2)$, we have*

$$e^{-(A+1/2)x^2} \leqslant \frac{\cos x}{\cosh x} \leqslant e^{-(B+1/2)x^2}, \tag{1.3}$$

with $A = -\ln(\cos\alpha)/\alpha^2$ and $B = 4\ln[\cosh(\pi/2)]/\pi^2 \approx 0.372844$.

Theorem 4. *For $x \in (0, \pi/2)$, we have*

$$e^{-(C+1/6)x^2} < \frac{\sin x}{\sinh x} < e^{-(D+1/6)x^2}, \tag{1.4}$$

with $C = -4\ln(2/\pi)/\pi^2 \approx 0.183019$ and $D = 4\ln[2\sinh(\pi/2)/\pi]/\pi^2 \approx 0.154774$.

Motivated by these results, the main purpose of this paper is to establish improved upper bounds for $\frac{\cos x}{\cosh x}$ and $\frac{\sin x}{\sinh x}$ and to obtain some other interesting inequalities involving these functions. Inequalities involving $\frac{\tan x}{\tanh x}$ will also be investigated.

2 Preliminaries

The following series expansions can be found in [15, 1.411]:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}; \quad |x| < \pi, \tag{2.1}$$

$$\coth x = \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^{n-1} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}; \quad |x| < \pi, \tag{2.2}$$

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}; \quad |x| < \pi, \tag{2.3}$$

and

$$\operatorname{cosech} x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)}{(2n)!}B_{2n}x^{2n-1}; \quad |x| < \pi, \tag{2.4}$$

where B_{2n} are the even-indexed Bernoulli numbers, see [14, p. 231]. From expansion (2.1), we obtain

$$\frac{\tanh x}{\tan x} = \frac{\tanh x}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\tanh x; \quad |x| < \pi, \tag{2.5}$$

and

$$\left(\frac{\sinh x}{\sin x}\right)^2 = -(\cot x)' \sinh^2 x$$
$$= \left(\frac{\sinh x}{x}\right)^2 + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}\sinh^2 x; \quad |x| < \pi. \tag{2.6}$$

Similarly, from (2.3), (2.4) we respectively have

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)}{(2n)!}|B_{2n}|x^{2n}; \quad |x| < \pi, \tag{2.7}$$

and

$$\frac{x}{\sinh x} = 1 - \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)}{(2n)!}B_{2n}x^{2n}; \quad |x| < \pi. \tag{2.8}$$

The following l'Hôpital's rule of monotonicity [1] has widespread applications and is proved to be an important tool in the field of analytic inequalities.

Lemma 1. *([l'Hôpital's rule of monotonicity] [1]): Let f, g be two real valued functions which are continuous on $[a, b]$ and differentiable on (a, b), where $-\infty < a < b < \infty$ and $g'(x) \neq 0$, for $\forall x \in (a, b)$. Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)}$$

and

$$B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Then

I. $A(x)$ and $B(x)$ are increasing on (a, b) if f'/g' is increasing on (a, b) and

II. $A(x)$ and $B(x)$ are decreasing on (a, b) if f'/g' is decreasing on (a, b).

The strictness of the monotonicity of $A(x)$ and $B(x)$ depends on the strictness of monotonicity of f'/g'.

3 Main results

We now state and prove the first main result of the paper.

Proposition 1. *If $x \in [0, \alpha]$ where $\alpha \in (0, \pi/2)$ then*

$$e^{-ax^2} \leqslant \frac{\cos x}{\cosh x} \leqslant e^{-x^2}, \tag{3.1}$$

with $a = \ln(\cosh \alpha / \cos \alpha)/\alpha^2$.

Proof. We have to show that

$$1 < f(x) < a \quad (0 < x < \pi/2),$$

where

$$f(x) = \frac{\ln(\cosh x / \cos x)}{x^2}.$$

Let

$$g_1(x) = \ln(\cosh x / \cos x), \ g_2(x) = \tanh x + \tan x,$$

and

$$h_1(x) = x^2, \ h_2(x) = 2x.$$

Then

$$g_i(0+) = h_i(0+) = 0 \ (i = 1, 2), \ \frac{g_1'(x)}{h_1'(x)} = \frac{g_2(x)}{h_2(x)},$$

and

$$\frac{g_2'(x)}{h_2'(x)} = \frac{P(x)}{2}$$

with $P(x) = \operatorname{sech}^2 x + \sec^2 x$. It has derivative

$$P'(x) = 2\left(\tan x \sec^2 x - \tanh x \operatorname{sech}^2 x\right).$$

Now $\tan x > \tanh x$ and $\sec^2 x > \operatorname{sech}^2 x$ in $(0, \pi/2)$ imply $P'(x) > 0$ which in turn implies that $P(x)$ is increasing in $(0, \pi/2)$. Applying Lemma 1, gives that $f(x)$ is increasing in the same interval. Since $f(0+) = 1$ by l'Hôpital's rule and $f(\alpha-) = \frac{\ln(\cosh \alpha / \cos \alpha)}{\alpha^2}$ we obtain (3.1). □

It should be noted that the lower bound in (3.1) is nothing but lower bound in (1.1) and upper bound in (3.1) is sharper than the corresponding upper bound in (1.3). The right inequality of (3.1) is in fact true in $(0, \pi/2)$.

In what follows, similar bounds for $\frac{\sin x}{\sinh x}$ as in (3.1) are proposed.

Proposition 2. *If $x \in (0, \pi/2)$ then*

$$e^{-bx^2} < \frac{\sin x}{\sinh x} < e^{-x^2/3}, \tag{3.2}$$

with $b = \frac{4 \ln(\sinh \pi/2)}{\pi^2} \approx 0.337794$.

The following lemma is important as it leads to prove Proposition 2 and gives very sharp bounds for $\frac{x}{\tan x}$ in $(0, \pi/2)$.

Lemma 2. $\lambda(x)=\frac{\coth x-\cot x}{x}$ *is positive increasing in* $(0,\pi)$*. In particular we have the following inequalities:*

$$\frac{x}{\tanh x}-cx^2<\frac{x}{\tan x}<\frac{x}{\tanh x}-\frac{2}{3}x^2;\quad x\in(0,\pi/2), \tag{3.3}$$

and

$$\frac{x}{\tan x}<\frac{x}{\tanh x}-\frac{2}{3}x^2;\quad x\in(0,\pi) \tag{3.4}$$

where $c=\frac{2\coth(\pi/2)}{\pi}=0.694126\cdots$.

Proof. Utilizing (2.1) and (2.2) we write

$$\begin{aligned}\lambda(x)&=\frac{\coth x-\cot x}{x}\\&=\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|\left[(-1)^{n-1}+1\right]x^{2n-2}\\&=\sum_{n=1}^{\infty}a_nx^{2n-2}\end{aligned}$$

where $a_n\geqslant 0$; $\forall n$. Thus $\lambda(x)=\frac{2}{3}+\frac{4x^4}{945}+\frac{4x^8}{93555}+\cdots$. This clearly shows that $\lambda(x)$ is positive increasing in $(0,\pi)$. With the limits $\lambda(0+)=\frac{2}{3}$ and $\lambda(\pi/2-)=\frac{2\coth(\pi/2)}{\pi}$ we get inequalities (3.3) and (3.4). □

The inequality (3.3) is very sharp and can be studied further independently for its refinement and generalization. Let us now prove Proposition 2.

Proof of Proposition 2. Let

$$f(x)=\frac{\ln(\sinh x/\sin x)}{x^2}=\frac{g(x)}{h(x)},$$

where $g(x)=\ln(\sinh x/\sin x)$ and $h(x)=x^2$ with $g(0+)=0$ and $h(0)=0$. Differentiation gives

$$\frac{g'(x)}{h'(x)}=\frac{1}{2}\frac{\coth x-\cot x}{x}=\frac{1}{2}\lambda(x)$$

which is increasing in $(0,\pi/2)$ by Lemma 2. So

$$f(0+)<f(x)<f(\pi/2-).$$

With the limits $f(0+)=\lim_{x\to 0+}\frac{1}{2}\lambda(x)=\frac{1}{2}\frac{2}{3}=\frac{1}{3}$ by Lemma 2 and $f(\pi/2-)=\frac{4\ln(\sinh(\pi/2))}{\pi^2}\approx 0.337794$ we finish the proof. □

Here, too, it should be noted that the lower bound in (3.2) is nothing but lower bound in (1.2) and upper bound in (3.2) is sharper than the corresponding upper bound in (1.4). Moreover, the constants obtained in Propositions 1 and 2 are optimal.

Remark 1. *An immediate consequence of Propositions 1 and 2 is the following inequality:*

$$\frac{\cos x}{\cosh x} < \frac{\sin x}{\sinh x}; \quad x \in (0, \pi/2) \tag{3.5}$$

which can also be obtained from the obvious relation $\tanh x < \tan x$. *Similarly from Propositions 1 and 2, we can have the inequality*

$$\frac{\cos x}{\cosh x} < \left(\frac{\sin x}{\sinh x}\right)^{1/b}; \quad x \in (0, \pi/2) \tag{3.6}$$

where $1/b \approx 2.960383$.

Now we ask the natural question: what can be the best possible exponent of $\frac{\sin x}{\sinh x}$ in the above inequality (2.4)? can we expect it to be 3? The affirmative answer can be seen in the following theorem.

Theorem 5. *If* $x \in (0, \pi/2)$ *then the inequality*

$$\left(\frac{\tanh x}{\tan x}\right)^{1/2} < \frac{\sin x}{\sinh x} \tag{3.7}$$

holds true with the best possible constant $1/2$. *Equivalently, we have*

$$\frac{\cos x}{\cosh x} < \left(\frac{\sin x}{\sinh x}\right)^{3}; \quad x \in (0, \pi/2), \tag{3.8}$$

with the best possible constant 3.

Before entering the proof of Theorem 5, we prove two lemmas as follows.

Lemma 3. $\xi(x) = \cos x \cosh x$ *is strictly positive decreasing in* $(0, \pi/2)$.

Proof. The proof is easy and straightforward since,

$$\xi'(x) = \cos x \sinh x - \sin x \cosh x < 0$$

by (3.5). □

Lemma 4. $\tau(x) = \frac{\sin^2 x + \sinh^2 x}{\sin x \sinh x} = \frac{\sin x}{\sinh x} + \frac{\sinh x}{\sin x}$ *is strictly increasing in* $(0, \pi/2)$.

Proof.

$$\begin{aligned}(\sin x \sinh x)^2\, \tau'(x) &= \sin x \sinh^2 x \cosh x + \sin^2 x \sinh x \cos x \\ &\quad - \sin^3 x \cosh x - \sinh^3 x \cos x \\ &= \sinh^2 x\, (\sin x \cosh x - \sinh x \cos x) \\ &\quad - \sin^2 x\, (\sin x \cosh x - \sinh x \cos x) \\ &= (\sin x \cosh x - \sinh x \cos x)\left(\sinh^2 x - \sin^2 x\right)\end{aligned}$$

which is positive by (3.5) and the fact that $\sinh x > \sin x$. This proves our lemma. □

We are now in a position to prove Theorem 5.

Proof of Theorem 5. Suppose

$$f(x) = \frac{\ln(\sin x/\sinh x)}{\ln(\tanh x/\tan x)} = \frac{g(x)}{h(x)},$$

where $g(x) = \ln(\sin x/\sinh x)$ and $h(x) = \ln(\tanh x/\tan x)$ with $g(0+) = 0 = h(0+)$. Then

$$\begin{aligned}\frac{g'(x)}{h'(x)} &= \frac{\sin x\cosh x - \sinh x\cos x}{\sinh x\cosh x - \sin x\cos x}(\cos x\cosh x)\\ &= q(x)(\cos x\cosh x).\end{aligned}$$

And

$$q(x) = \frac{\sin x\cosh x - \sinh x\cos x}{\sinh x\cosh x - \sin x\cos x} = \frac{q_1(x)}{q_2(x)},$$

where $q_1(x) = \sin x\cosh x - \sinh x\cos x$, $q_2(x) = \sinh x\cosh x - \sin x\cos x$ with $q_1(0) = q_2(0) = 0$. By differentiation

$$\frac{q_1'(x)}{q_2'(x)} = \frac{\sin x\sinh x}{\sin^2 x + \sinh^2 x} = \frac{1}{\tau(x)}$$

which is strictly decreasing by Lemma 4. By Lemma 1, $q(x)$ is strictly decreasing in $(0, \pi/2)$ and it is obvious that $q(x)$ is positive. By Lemma 3, $\cos x\cosh x$ is positive decreasing. Consequently $\frac{g'(x)}{h'(x)}$ is strictly decreasing in $(0, \pi/2)$ and so is $f(x)$ by Lemma 1 again. Hence

$$f(x) < f(0+); \quad 0 < x < \pi/2.$$

Lastly $f(0+) = \lim_{x\to 0+} q(x)\lim_{x\to 0+}(\cos x\cosh x) = \lim_{x\to 0+}\frac{1}{\tau(x)} = \frac{1}{2}$ completes the proof. $\square$

Note: The inequality (3.8) has close resemblance with Mitrinović-Adamović inequality, see e.g., [1, 19, 23, 29].

In the following corollary, we present an inequality for ratio functions which is exactly similar to the one known as Wilker's inequality [12, 20, 22, 25, 28, 30].

Corollary 1. *For $x \in (0, \pi/2)$, we have*

$$\left(\frac{\sin x}{\sinh x}\right)^2 + \frac{\tan x}{\tanh x} > 2. \tag{3.9}$$

Proof. For $x \in (0, \pi/2)$, the inequality (3.7) can be written as

$$\left(\frac{\sin x}{\sinh x}\right)^2 > \frac{\tanh x}{\tan x}.$$

This implies

$$\left(\frac{\sin x}{\sinh x}\right)^2+\frac{\tan x}{\tanh x}>\frac{\tanh x}{\tan x}+\frac{\tan x}{\tanh x}>2,$$

as $u+\frac{1}{u}>2$ for any $u>0$. □

In Proposition 3, we establish another upper bound for $\frac{\sin x}{\sinh x}$.

Proposition 3. *If* $x\in(0,\pi)$ *then*

$$\frac{\sin x}{\sinh x}<\sqrt{\frac{x+\sin x\cos x}{x+\sinh x\cosh x}}=\sqrt{\frac{2x+\sin 2x}{2x+\sinh 2x}}. \tag{3.10}$$

Proof. By Lemma 2, $\lambda'(x)>0$ in $(0,\pi)$. It means that

$$x\left(\operatorname{cosec}^2 x-\operatorname{cosech}^2 x\right)-(\coth x-\cot x)>0,$$

which is equivalent to

$$x\operatorname{cosec}^2 x+\cot x>\coth x+x\operatorname{cosech}^2 x$$

or

$$\frac{x+\sin x\cos x}{\sin^2 x}>\frac{x+\sinh x\cosh x}{\sinh^2 x}.$$

This gives desired inequality. □

Some computations and Graphing calculator at www.symbolab.com suggest that the upper bound of $\frac{\sin x}{\sinh x}$ in (3.10) is sharper than the corresponding upper bound in (1.4) except for a little portion as $x\to\pi/2$. We present the following graphical comparison in support of our claim.

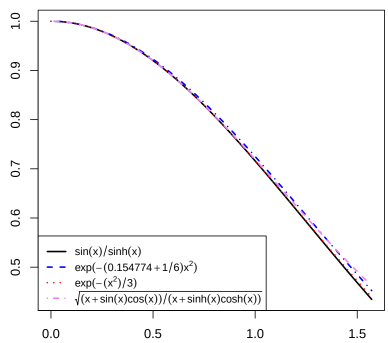

Figure 1: Upper bounds of $\frac{\sin x}{\sinh x}$ in (1.4), (3.2) and (3.10) for $x\in(0,\pi/2)$.

Inspired by Corollary 1, we prove

Theorem 6. *For $x \in (0, \pi)$, the inequality*

$$\left(\frac{\sinh x}{\sin x}\right)^2 + \frac{\tanh x}{\tan x} > 2 \tag{3.11}$$

holds true.

Proof. Adding (2.5) and (2.6), and using the well-known inequality (see e.g., [28])

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2; \quad x > 0,$$

we get

$$\left(\frac{\sinh x}{\sin x}\right)^2 + \frac{\tanh x}{\tan x} > 2 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| f_n(x) x^{2n-1} \sinh x,$$

for $x \in (0, \pi)$, where $f_n(x) = \left((2n-1)\frac{\sinh x}{x} - \frac{1}{\cosh x}\right)$. Since, $(2n-1)\frac{\sinh x}{x} > \frac{1}{\cosh x}$ for all $x > 0$ and $n \geqslant 1$, our assertion is proved. $\square$

Let us find exponential bounds for $\frac{\tanh x}{\tan x}$.

Proposition 4. *For $x \in (0, \alpha]$ where $\alpha \in (0, \pi/2)$, it is true that*

$$e^{-cx^2} < \frac{\tanh x}{\tan x} < e^{-\frac{2}{3}x^2}, \tag{3.12}$$

with the best possible constants $c = \frac{\ln(\tan \alpha / \tanh \alpha)}{\alpha^2}$ and $-\frac{2}{3}$.

Proof. We want to prove that

$$-\frac{2}{3} < f(x) < a; \quad x \in (0, \alpha],$$

where

$$f(x) = \frac{\ln(\tan x / \tanh x)}{x^2}.$$

Let $g(x) = \ln(\tan x / \tanh x)$ and $h(x) = x^2$. We can see that $g(0+) = 0 = h(0)$. After differentiating we get

$$\begin{aligned} \frac{g'(x)}{h'(x)} &= \frac{\tanh x \sec^2 x - \tan x \operatorname{sech}^2 x}{2x \tan x \tanh x} \\ &= \frac{\sinh x \cosh x - \sin x \cos x}{2x \sin x \cos x \sinh x \cosh x} \\ &= \frac{1}{2x^2}\left(\frac{2x}{\sin 2x} - \frac{2x}{\sinh 2x}\right) \end{aligned}$$

Utilization of (2.7) and (2.8) yields

$$\begin{aligned}\frac{g'(x)}{h'(x)} &= \frac{1}{2x^2}\left(\sum_{n=1}^{\infty}\frac{2^{2n+1}(2^{2n-1}-1)}{(2n)!}|B_{2n}|x^{2n} + \sum_{n=1}^{\infty}\frac{2^{2n+1}(2^{2n-1}-1)}{(2n)!}B_{2n}x^{2n}\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n+1}(2^{2n-1}-1)}{(2n)!}\left(|B_{2n}|+B_{2n}\right)x^{2n-2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n+1}(2^{2n-1}-1)}{(2n)!}|B_{2n}|\left(1+(-1)^{n-1}\right)x^{2n-2}\end{aligned}$$

which is strictly increasing in $(0,\alpha]$. By Lemma 1, $f(x)$ is also strictly increasing in $(0,\alpha]$. By the limits $f(0+)=\frac{2}{3}$ and $f(\alpha-)=\frac{\ln(\tan\alpha/\tanh\alpha)}{\alpha^2}$, the proof is now complete. □

The right inequality of (3.12) is, of course, holds for $x\in(0,\pi/2)$ and this inequality with the left inequality of (3.2) provides an alternative simple proof of Theorem 5.

We proceed to obtain a simple Jordan type inequality for $\frac{\sin x}{\sinh x}$. The details of Jordan's inequality can be found in [1, 4, 6, 16, 27] and references therein.

Proposition 5. *For $x\in(0,\pi/2)$ we have*

$$1-\frac{x^2}{3}<\frac{\sin x}{\sinh x}<1. \tag{3.13}$$

Proof. The right inequality is obvious as $\sin x<\sinh x$. For left inequality, let us set

$$T(x)=\sin x-\sinh x+\frac{x^2}{3}\sinh x.$$

Successive differentiation gives

$$T'(x)=\cos x-\cosh x+\frac{x^2}{3}\cosh x+\frac{2x}{3}\sinh x,$$

$$T''(x)=-\sin x-\sinh x+\frac{x^2}{3}\sinh x+\frac{4x}{3}\cosh x+\frac{2}{3}\sinh x$$

and

$$T'''(x)=(\cosh x-\cos x)+2x\sinh x+\frac{x^2}{3}\cosh x>0,$$

implying that $T''(x)$ is increasing on $(0,\pi/2)$ and, $T''(x)>T''(0)=0$ fortiori, $T(x)>0$ gives inequality (3.13). □

Motivated by Cusa-Huygens inequality [2, 5, 13, 20, 21, 24] which is stated as

$$\frac{\sin x}{x}<\frac{2+\cos x}{3};\ x\in(0,\pi/2)$$

we present a very similar inequality in the next theorem.

Theorem 7. *If $x \in (0, \pi/2)$ then the following inequality holds true:*

$$\frac{\sin x}{\sinh x} < \frac{2 + \cos x}{2 + \cosh x}. \tag{3.14}$$

Proof. Suppose that,

$$f(x) = 2(\sinh x - \sin x) - (\sin x \cosh x - \sinh x \cos x).$$

On differentiating continuously four times we get successive derivatives as follows:

$$f'(x) = 2(\cosh x - \cos x) - 2 \sin x \sinh x,$$

$$f''(x) = 2(\sinh x + \sin x) - 2(\cos x \sinh x + \sin x \cosh x),$$

$$f'''(x) = 2(\cosh x + \cos x) - 4 \cos x \cosh x$$

and

$$f^{iv}(x) = 2(\sinh x - \sin x) + 4(\sin x \cosh x - \cos x \sinh x) > 0.$$

Now since $\sinh x > \sin x$ and by (2.3) we get that $f^{iv}(x) > 0$, implying taht $f'''(x)$ is increasing on $(0, \pi/2)$. Hence $f'''(x) > f'''(0) = 0$ fortiori, $f(x) > 0$ gives the desired inequality. □

The inequality (3.14) is very sharp. This claim can be verified from the following figure.

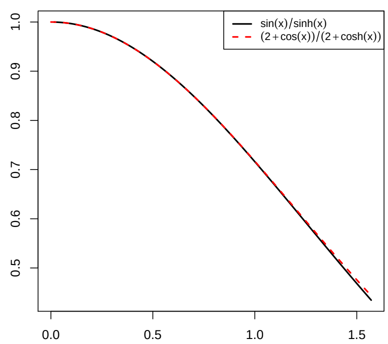

Figure 2: Graphs of functions in (3.14) for $x \in (0, \pi/2)$.

4 Applications

In this section, we see some important consequences of our main results. We first offer a very simple proof of Wu and Srivastava's inequality [26, Lemma 3].

Lemma 5. *([26]) For $x \in (0, \pi/2)$, it is true that*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2.$$

Proof. Inequality (3.10) can be written as

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}; \quad x \in (0, \pi/2). \tag{4.1}$$

C.-P. Chen and J. Sándor [7, Theorem 1.2(iii)] established the inequality

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2; \quad x \in (0, \pi/2). \tag{4.2}$$

Required inequality follows from inequalities (4.1) and (4.2). □

To obtain bounds of $\frac{\sin x}{\sinh x}$ in terms of cosine and hyperbolic cosine functions we continue with

Corollary 2. $\rho(x) = \coth x(\coth x - \cot x)$ *is strictly increasing in* $(0, \pi)$.

Proof.

$$\begin{aligned}\rho(x) &= \frac{x}{\tanh x}\frac{\coth x - \cot x}{x} \\ &= \kappa(x)\lambda(x)\end{aligned}$$

which is strictly positive increasing since $\kappa(x)$ is obviously positive increasing and $\lambda(x)$ is also positive increasing by Lemma 2. □

Corollary 3. $\Psi(x) = \cot x(\coth x - \cot x)$ *is strictly decreasing in* $(0, \pi/2)$.

Proof. By (3.5), $\Psi(x)$ is positive in $(0, \pi/2)$. After differentiating $\Psi(x)$ we get

$$\Psi'(x) = -\cot x \operatorname{cosech}^2 x - \coth x \operatorname{cosec}^2 x + 2\cot x \operatorname{cosec}^2 x.$$

By Corollary 2

$$\rho'(x) > 0 \text{ in } (0, \pi/2).$$

i.e.
$$-2\coth x \operatorname{cosech}^2 x + \cot x \operatorname{cosech}^2 x + \coth x \operatorname{cosec}^2 x > 0.$$
Equivalently,
$$-\coth x \operatorname{cosec}^2 x < (\cot x - 2\coth x) \operatorname{cosech}^2 x.$$
Hence
$$\begin{aligned}\Psi'(x) &< (\cot x - 2\coth x)\operatorname{cosech}^2 x - \cot x \operatorname{cosech}^2 x \\ &+ 2\cot x \operatorname{cosech}^2 x + 2\cot x \operatorname{cosec}^2 x \\ &= -2\coth x \operatorname{cosech}^2 x + 2\cot x \operatorname{cosec}^2 x \\ &= 2\frac{(\sinh^3 x \cos x - \sin^3 x \cosh x)}{\sin^3 x \sinh^3 x} < 0\end{aligned}$$
by 3.8. Thus our claim is proved. $\square$

Note. We can obtain new bounds for $\frac{x}{\tan x}$ with the help of Corollaries 2 and 3; but the new bounds are not as sharp as those obtained in Lemma 2. So we do not present them here.

Proposition 6. *For $x \in (0, \pi/2)$, one has*
$$\cos^{2/3} x < \frac{\sin x}{\sinh x}. \tag{4.3}$$

Proof. Let
$$F(x) = \frac{\ln(\sin x/\sinh x)}{\ln(\cos x)} = \frac{F_1(x)}{F_2(x)},$$
where $F_1(x) = \ln(\sin x/\sinh x)$ and $F_2(x) = \ln(\cos(x))$ with $F_1(0+) = 0 = F_2(0)$. By differentiation we have
$$\frac{F_1'(x)}{F_2'(x)} = \cot x(\coth x - \cot x) = \Psi(x)$$
which is strictly decreasing in $(0, \pi/2)$ by Corollary 3. Therefore $F(x)$ is also strictly decreasing in $(0, \pi/2)$ by Lemma 1. So we can write
$$F(x) < F(0+); \quad x > 0,$$
and $F(0+) = \lim_{x\to 0+} \Psi(x) = \lim_{x\to 0+} \frac{x}{\tan x}\lambda(x) = \frac{2}{3}$ gives (4.3). $\square$

Proposition 7. *For $x \in (0, \pi/2)$ we have*
$$\left(\frac{1}{\cosh x}\right)^h < \frac{\sin x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{2/3} \tag{4.4}$$
with best possible constants $h = \frac{\ln(\sinh \pi/2)}{\ln(\cosh \pi/2)} \approx 0.905994$ and $2/3$.

Proof. Suppose

$$G(x) = \frac{\ln(\sinh x/\sin x)}{\ln(\cosh x)}.$$

We want

$$\frac{2}{3} < G(x) < h; \quad x \in (0, \pi/2).$$

Let $G_1(x) = \ln(\sinh x/\sin x)$ and $G_2(x) = \ln(\cosh x)$. Clearly $G_1(0+) = 0 = G_2(0)$. Differentiation gives

$$\frac{G_1'(x)}{G_2'(x)} = \coth x(\coth x - \cot x) = \rho(x)$$

which is strictly increasing in $(0, \pi/2)$ Corollary 2 and so is $G(x)$ by Lemma 1. Lastly, the limits $G(0+) = \lim_{x\to 0+} G(x) = \lim_{x\to 0+} \rho(x) = \lim_{x\to 0+} \frac{x}{\tanh x}\lambda(x) = \frac{2}{3}$ and $G(\pi/2-) = \frac{\ln(\sinh \pi/2}{\ln(\cosh \pi/2)} \approx 0.905994$ give desired result. □

Remark 2. *Combining (4.3) and (4.4), the following inequality can be written:*

$$\cos^2 x < \left(\frac{\sin x}{\sinh x}\right)^3 < \frac{1}{\cosh^2 x}; \quad x \in (0, \pi/2). \tag{4.5}$$

We conclude this section by noticing that our obtained results give interested inequalities connecting sinc and hyperbolic sinc functions as well as inequalities connecting cosine and hyperbolic cosine functions. For instance, the inequalities (3.1), (3.2), (3.8), (3.12), (3.13), and (3.14) can be written respectively as follows:

$$e^{-ax^2}\cosh x \leqslant \cos x \leqslant e^{-x^2}\cosh x; \quad x \in [0, \alpha], \tag{4.6}$$

where $\alpha \in (0, \pi/2)$ and $a = \ln(\cosh\alpha/\cos\alpha)/\alpha^2$,

$$\left(\frac{\sinh x}{x}\right)e^{-bx^2} < \frac{\sin x}{x} < \left(\frac{\sinh x}{x}\right)e^{-x^2/3}; \quad x \in (0, \pi/2), \tag{4.7}$$

where $b \approx 0.337794$,

$$\left(\frac{\sinh x}{x}\right)^3 \cos x < \left(\frac{\sin x}{x}\right)^3 \cosh x; \quad x \in (0, \pi/2), \tag{4.8}$$

$$e^{-cx^2}\tan x < \tanh x < e^{-\frac{2}{3}x^2}\tan x; \quad x \in (0, \alpha], \tag{4.9}$$

where $\alpha \in (0, \pi/2)$ and $c = \frac{\ln(\tan\alpha/\tanh\alpha)}{\alpha^2}$.

$$\left(1-\frac{x^2}{3}\right)\frac{\sinh x}{x} < \frac{\sin x}{x} < \frac{\sinh x}{x}; \quad x \in (0, \pi/2), \tag{4.10}$$

and

$$\frac{\sin x}{x}\left(\frac{2+\cosh x}{3}\right) < \frac{\sinh x}{x}\left(\frac{2+\cos x}{3}\right); \quad x \in (0, \pi/2). \tag{4.11}$$

5 Conclusion

We obtained sharp exponential bounds for $\frac{\cos x}{\cosh x}$, $\frac{\sin x}{\sinh x}$, and $\frac{\tan x}{\tanh x}$ and established some other inequalities involving these functions. The obtained inequalities are similar to Jordan, Mitrinović-Adamović, Wilker and Cusa-Huygens type for these functions. In an attempt to obtain our main results, we also established very sharp bounds for $\frac{x}{\tan x}$.

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