GENERALIZATIONS OF MILNE INEQUALITY FOR TWO ISOTONIC FUNCTIONALS

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Abstract. In this paper we obtain some inequalities for two isotonic functionals that generalize the well known Milne's inequality. Applications for integrals and n-tuples of real numbers are provided as well.

1. Introduction

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalized if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0$, $k \in E$).

The following inequality is known in the literature as Milne inequality [3] for the Lebesgue measurable positive functions f, g, p on E

$$\int_E pf d\mu \int_E pg d\mu \geq \int_E p(f+g) d\mu \int_E p\left(\frac{fg}{f+g}\right) d\mu, \tag{1.1}$$

provided that all the Lebesgue integrals exist and are finite.

The discrete version is as follows

$$\sum_{k=1}^n p_k a_k \sum_{k=1}^n p_k b_k \geq \sum_{k=1}^n p_k (a_k + b_k) \sum_{k=1}^n p_k \left(\frac{a_k b_k}{a_k + b_k}\right), \tag{1.2}$$

where a_k, $b_k > 0$, $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$.

For related results, see [1], [2], [4], [5] and [6].

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In this paper we obtain some inequalities for two isotonic functionals that generalize the well known Milne's inequality. Applications for integrals and n-tuples of real numbers are provided as well.

2. Main Results

We start with the following identity of interest:

Lemma 1. *Let f, $g : E \to (0, \infty)$, then for all s, $r \in E$ we have the identity*

$$\begin{aligned} & f(r)g(s)+f(s)g(r) \\ & -\frac{f(r)g(r)}{f(r)+g(r)}(f(s)+g(s))-\frac{f(s)g(s)}{f(s)+g(s)}(f(r)+g(r)) \\ & =\frac{(f(s)g(r)-f(r)g(s))^2}{(f(s)+g(s))(f(r)+g(r))} \\ & =\frac{f^2(s)}{f(s)+g(s)}\frac{g^2(r)}{f(r)+g(r)}+\frac{f^2(r)}{f(r)+g(r)}\frac{g^2(s)}{f(s)+g(s)} \\ & -2\frac{f(s)g(s)}{f(s)+g(s)}\frac{f(r)g(r)}{f(r)+g(r)}. \end{aligned} \tag{2.1}$$

Proof. Observe that

$$\begin{aligned} I & := (f(r)g(s)+f(s)g(r))(f(r)+g(r))(f(s)+g(s)) \\ & = (f(r)g(s)+f(s)g(r))(f(r)f(s)+f(r)g(s)+g(r)f(s)+g(r)g(s)) \\ & = f(s)f^2(r)g(s)+f^2(r)g^2(s)+f(r)f(s)g(r)g(s)+f(r)g^2(s)g(r) \\ & + f^2(s)f(r)g(r)+f(r)f(s)g(r)g(s)+f^2(s)g^2(r)+f(s)g(s)g^2(r) \end{aligned}$$

and

$$\begin{aligned} J & := f(r)g(r)(f(s)+g(s))^2+f(s)g(s)(f(r)+g(r))^2 \\ & = f(r)g(r)\left(f^2(s)+2f(s)g(s)+g^2(s)\right) \\ & + f(s)g(s)\left(f^2(r)+2f(r)g(r)+g^2(r)\right) \\ & = f^2(s)f(r)g(r)+2f(s)g(s)f(r)g(r)+f(r)g^2(s)g(r) \\ & + f(s)f^2(r)g(s)+2f(s)g(s)f(r)g(r)+f(s)g(s)g^2(r) \end{aligned}$$

for all s, $r \in E$.

Using the above equalities, we deduce that

$$\begin{aligned} I-J & = f^2(s)g^2(r)+f^2(r)g^2(s)-2f(s)g(s)f(r)g(r) \\ & = (f(s)g(r)-f(r)g(s))^2 \end{aligned}$$

for all s, $r \in E$.

This proves the first identity in (2.1). The second identity is obvious by expanding the binomial. □

Theorem 1. *Let* $f,\, g : E \to (0,\infty)$ *be such that* $f,\ g,\ \frac{fg}{f+g},\ \frac{f^2}{f+g}\frac{g^2}{f+g} \in L$, *then for all isotonic linear functionals* A, B *defined on* L,

$$\begin{aligned}
&A(f)\,B(g)+A(g)\,B(f) \\
&-A(f+g)\,B\left(\frac{fg}{f+g}\right)-A\left(\frac{fg}{f+g}\right)B(f+g) \\
&=A\left(\frac{f^2}{f+g}\right)B\left(\frac{g^2}{f+g}\right)+A\left(\frac{g^2}{f+g}\right)B\left(\frac{f^2}{f+g}\right) \\
&-2A\left(\frac{fg}{f+g}\right)B\left(\frac{fg}{f+g}\right) \\
&\geq 0.
\end{aligned} \tag{2.2}$$

In particular, for $B=A$, *we derive*

$$\begin{aligned}
&A(f)\,A(g)-A(f+g)\,A\left(\frac{fg}{f+g}\right) \\
&=A\left(\frac{f^2}{f+g}\right)A\left(\frac{g^2}{f+g}\right)-A^2\left(\frac{fg}{f+g}\right)\geq 0.
\end{aligned} \tag{2.3}$$

Proof. Fix $r\in E$, then by (2.1) we get

$$\begin{aligned}
&g(r)\,f+f(r)\,g \\
&-\frac{f(r)\,g(r)}{f(r)+g(r)}(f+g)-(f(r)+g(r))\frac{fg}{f+g} \\
&=\frac{g^2(r)}{f(r)+g(r)}\frac{f^2}{f+g}+\frac{f^2(r)}{f(r)+g(r)}\frac{g^2}{f+g}-2\frac{f(r)\,g(r)}{f(r)+g(r)}\frac{fg}{f+g}\geq 0
\end{aligned} \tag{2.4}$$

in the order of L.

If we apply to this equality the functional A, then we get

$$\begin{aligned}
&g(r)\,A(f)+f(r)\,A(g) \\
&-\frac{f(r)\,g(r)}{f(r)+g(r)}(A(f)+A(g))-(f(r)+g(r))\,A\left(\frac{fg}{f+g}\right) \\
&=\frac{g^2(r)}{f(r)+g(r)}A\left(\frac{f^2}{f+g}\right)+\frac{f^2(r)}{f(r)+g(r)}A\left(\frac{g^2}{f+g}\right) \\
&-2\frac{f(r)\,g(r)}{f(r)+g(r)}A\left(\frac{fg}{f+g}\right)\geq 0
\end{aligned}$$

for all $r\in E$.

This inequality can be written in the order of L as

$$\begin{aligned}
&A(f)\,g+A(g)\,f-(A(f)+A(g))\frac{fg}{f+g}-A\left(\frac{fg}{f+g}\right)(f+g) \\
&=A\left(\frac{f^2}{f+g}\right)\frac{g^2}{f+gr}+A\left(\frac{g^2}{f+g}\right)\frac{f^2}{f+g} \\
&-2A\left(\frac{fg}{f+g}\right)\frac{fg}{f+g}\geq 0.
\end{aligned}$$

If we apply to this equation the functional B we derive the desired result (2.2). □

Corollary 1. *With the assumptions of Theorem 1 we have the Milne inequality for positive functionals* [5]

$$\frac{A(f)A(g)}{A(f)+A(g)} \geq A\left(\frac{fg}{f+g}\right). \tag{2.5}$$

We also have:

Corollary 2. *Assume that* $h,\ \ell : E \to \mathbb{R}$ *are such that* $h^2,\ \ell^2,\ \frac{h^2\ell^2}{h^2+\ell^2},\ h\ell \in L$, *then*

$$A\left(h^2\right)A\left(\ell^2\right) \geq A\left(\frac{h^2\ell^2}{h^2+\ell^2}\right)A\left(h^2+\ell^2\right) \geq A^2(h\ell). \tag{2.6}$$

The first inequality in (2.6) follows by (2.5) for $f = h^2$, $g = \ell^2$. The second inequality follows by Schwarz inequality.

We have the following refinement and reverse of Milne inequality:

Theorem 2. *Let* $f,\ g : E \to (0,\infty)$ *be such that* $f,\ g,\ fg,\ \frac{fg}{f+g},\ f^2,\ g^2 \in L$ *and there exists* $0 < m < M$ *such that*

$$m \leq f+g \leq M \tag{2.7}$$

on L, *then for all isotonic linear functionals* A, B *defined on* L,

$$\begin{aligned} &\frac{1}{M^2}\left(A\left(f^2\right)B\left(g^2\right) - 2A(fg)B(gf) + A\left(g^2\right)B\left(f^2\right)\right) \\ &\leq A(f)B(g) + A(g)B(f) \\ &- A(f+g)B\left(\frac{fg}{f+g}\right) - A\left(\frac{fg}{f+g}\right)B(f+g) \\ &\leq \frac{1}{m^2}\left(A\left(f^2\right)B\left(g^2\right) - 2A(fg)B(gf) + A\left(g^2\right)B\left(f^2\right)\right). \end{aligned} \tag{2.8}$$

In particular,

$$\begin{aligned} \frac{1}{M^2}\left(A\left(f^2\right)A\left(g^2\right) - A^2(fg)\right) &\leq A(f)A(g) - A(f+g)A\left(\frac{fg}{f+g}\right) \\ &\leq \frac{1}{m^2}\left(A\left(f^2\right)A\left(g^2\right) - A^2(fg)\right). \end{aligned} \tag{2.9}$$

Proof. From the condition (2.8) and the identity (2.1) we have the inequality

$$\begin{aligned} &f(r)g(s) + f(s)g(r) \\ &- \frac{f(r)g(r)}{f(r)+g(r)}(f(s)+g(s)) - \frac{f(s)g(s)}{f(s)+g(s)}(f(r)+g(r)) \\ &\geq \frac{1}{M^2}(f(s)g(r) - f(r)g(s))^2 \\ &= \frac{1}{M^2}\left(f^2(s)g^2(r) - 2f(s)g(r)f(r)g(s) + f^2(r)g^2(s)\right) \end{aligned} \tag{2.10}$$

for all $s,\ r \in E$.

Fix $r \in E$, then we have in the order of L that

$$\begin{aligned} &f(r)g + g(r)f - \frac{f(r)g(r)}{f(r)+g(r)}(f+g) - (f(r)+g(r))\frac{fg}{f+g} \\ &\geq \frac{1}{M^2}\left(g^2(r)f^2 - 2g(r)f(r)fg + f^2(r)g^2\right). \end{aligned} \tag{2.11}$$

If we take the functional A in (2.11), then

$$\begin{aligned}(2.12)\qquad & f(r)A(g)+g(r)A(f)\\ & -\frac{f(r)g(r)}{f(r)+g(r)}(A(f)+A(g))-(f(r)+g(r))A\left(\frac{fg}{f+g}\right)\\ & \geq \frac{1}{M^2}\left(g^2(r)A\left(f^2\right)-2g(r)f(r)A(fg)+f^2(r)A\left(g^2\right)\right)\end{aligned}$$

for all $r\in E$.

This can be written in the order of L as follows

$$\begin{aligned}(2.13)\qquad & A(g)f+A(f)g-(A(f)+A(g))\frac{fg}{f+g}-A\left(\frac{fg}{f+g}\right)(f+g)\\ & \geq \frac{1}{M^2}\left(A\left(f^2\right)g^2-2A(fg)gf+A\left(g^2\right)f^2\right).\end{aligned}$$

If we take the functional B in (2.13), then we get the first inequality in (2.8).

The second inequality follows in a similar way and we omit the details. □

Corollary 3. *Assume that* $h,\ \ell : E\to\mathbb{R}$ *are such that* $h^2,\ \ell^2,\ \frac{h^2\ell^2}{h^2+\ell^2},\ h^2\ell^2,\ h^4,\ \ell^4\in L$, *and there exists the constants* $0<k<K$ *such that*

$$k\leq h^2+\ell^2\leq K,$$

then

$$\begin{aligned}(2.14)\qquad & \frac{1}{K^2}\left(A\left(h^4\right)A\left(\ell^4\right)-A^2\left(h^2\ell^2\right)\right)\\ & \leq A\left(h^2\right)A\left(\ell^2\right)-A\left(\frac{h^2\ell^2}{h^2+\ell^2}\right)A\left(h^2+\ell^2\right)\\ & \leq \frac{1}{k^2}\left(A\left(h^4\right)A\left(\ell^4\right)-A^2\left(h^2\ell^2\right)\right).\end{aligned}$$

From a different perspective, we also have:

Theorem 3. *Let* $f,\ g : E\to(0,\infty)$ *be such that* $f,\ g,\ \frac{g}{f},\ \frac{fg}{f+g},\ \frac{g^2}{f}\in L$ *and there exists* $0<p<P$ *such that*

$$(2.15)\qquad p\leq\frac{g}{f}\leq P,$$

on L, *then for all isotonic linear functionals* A, B *defined on* L,

$$\begin{aligned}(2.16)\qquad & \frac{1}{(1+P)^2}\left(A(f)B\left(\frac{g^2}{f}\right)-2A\left(\frac{g}{f}\right)B\left(\frac{g}{f}\right)+A\left(\frac{g^2}{f}\right)B(f)\right)\\ & \leq A(f)B(g)+A(g)B(f)\\ & -A(f+g)B\left(\frac{fg}{f+g}\right)-A\left(\frac{fg}{f+g}\right)B(f+g)\\ & \leq \frac{1}{(1+p)^2}\left(A(f)B\left(\frac{g^2}{f}\right)-2A\left(\frac{g}{f}\right)B\left(\frac{g}{f}\right)+A\left(\frac{g^2}{f}\right)B(f)\right).\end{aligned}$$

In particular,

$$\frac{1}{(1+P)^2}\left(A(f)\,A\left(\frac{g^2}{f}\right)-A^2\left(\frac{g}{f}\right)\right) \le A(f)\,A(g)-A(f+g)\,A\left(\frac{fg}{f+g}\right) \le \frac{1}{(1+p)^2}\left(A(f)\,A\left(\frac{g^2}{f}\right)-A^2\left(\frac{g}{f}\right)\right). \tag{2.17}$$

Proof. We have

$$\frac{(f(s)\,g(r)-f(r)\,g(s))^2}{(f(s)+g(s))(f(r)+g(r))} = \frac{f^2(s)\,f^2(r)\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2}{(f(s)+g(s))(f(r)+g(r))} = \frac{f(s)\,f(r)\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2}{\left(1+\frac{g(s)}{f(s)}\right)\left(1+\frac{g(r)}{f(r)}\right)}$$

for all $s, r \in E$.

Since

$$0 < p \le \frac{g(s)}{f(s)} \le P,$$

hence

$$(1+p)^2 \le \left(1+\frac{g(s)}{f(s)}\right)\left(1+\frac{g(r)}{f(r)}\right) \le (1+P)^2,$$

which shows that

$$\frac{f(s)\,f(r)}{(1+P)^2}\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2 \le \frac{f(s)\,f(r)\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2}{\left(1+\frac{g(s)}{f(s)}\right)\left(1+\frac{g(r)}{f(r)}\right)} \le \frac{f(s)\,f(r)}{(1+p)^2}\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2$$

for all $s, r \in E$.

Therefore, we get

$$\frac{1}{(1+P)^2}\left(f(s)\,\frac{g^2(r)}{f(r)}-2\frac{g(s)}{f(s)}\frac{g(r)}{f(r)}+f(r)\,\frac{g^2(s)}{f(s)}\right) \le \frac{(f(s)\,g(r)-f(r)\,g(s))^2}{(f(s)+g(s))(f(r)+g(r))} \le \frac{1}{(1+p)^2}\left(f(s)\,\frac{g^2(r)}{f(r)}-2\frac{g(s)}{f(s)}\frac{g(r)}{f(r)}+f(r)\,\frac{g^2(s)}{f(s)}\right)$$

for all $s, r \in E$. □

Corollary 4. *Assume that $h, \ell : E \to \mathbb{R}$ are such that h^2, ℓ^2, $\frac{h^2\ell^2}{h^2+\ell^2}$, $\frac{\ell^2}{h^2}$, $\frac{\ell^4}{h^2} \in L$, and there exists the constants $0 < q < Q$ such that*

$$q \le \frac{\ell^2}{h^2} \le Q, \tag{2.18}$$

then

$$\frac{1}{(1+Q)^2}\left(A\left(h^2\right)A\left(\frac{\ell^4}{h^2}\right)-A^2\left(\frac{\ell^2}{h^2}\right)\right) \le A\left(h^2\right)A\left(\ell^2\right)-A\left(\frac{h^2\ell^2}{h^2+\ell^2}\right)A\left(h^2+\ell^2\right) \le \frac{1}{(1+q)^2}\left(A\left(h^2\right)A\left(\frac{\ell^4}{h^2}\right)-A^2\left(\frac{\ell^2}{h^2}\right)\right). \tag{2.19}$$

We also have the following upper bound:

Theorem 4. *Let $f,\ g: E\to(0,\infty)$ be such that $f,\ g,\ \frac{fg}{f+g}\in L$ and there exists $0<p<P$ such that the condition (2.15) holds on L, then for all isotonic linear functionals $A,\ B$ defined on L,*

$$0 \le A(f)B(g)+A(g)B(f) - A(f+g)B\left(\frac{fg}{f+g}\right)-A\left(\frac{fg}{f+g}\right)B(f+g) \le \frac{(P-p)^2}{(1+p)^2}A(f)B(f). \tag{2.20}$$

In particular,

$$0\le 1-\frac{A(f+g)}{A(f)A(g)}A\left(\frac{fg}{f+g}\right)\le\frac{(P-p)^2}{(1+p)^2}\frac{A(f)}{A(g)}. \tag{2.21}$$

Proof. As above, we have

$$\frac{f(s)f(r)\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2}{\left(1+\frac{g(s)}{f(s)}\right)\left(1+\frac{g(r)}{f(r)}\right)} \le \frac{f(s)f(r)}{(1+p)^2}\left(\frac{g(r)}{f(r)}-\frac{g(s)}{f(s)}\right)^2 \le \frac{(P-p)^2}{(1+p)^2}f(s)f(r)$$

therefore

$$\frac{(f(s)g(r)-f(r)g(s))^2}{(f(s)+g(s))(f(r)+g(r))}\le\frac{(P-p)^2}{(1+p)^2}f(s)f(r)$$

for all $s,\ r\in E$.

By taking the functionals A and B as above, we derive the desired result (2.20). □

We also have:

Corollary 5. *Assume that $h,\ \ell: E\to\mathbb{R}$ are such that $h^2,\ \ell^2,\ \frac{h^2\ell^2}{h^2+\ell^2}\in L$ and there exists the constants $0<q<Q$ such that (2.18) is valid, then*

$$0\le 1-\frac{A\left(h^2+\ell^2\right)}{A(h^2)A(\ell^2)}A\left(\frac{h^2\ell^2}{h^2+\ell^2}\right)\le\frac{(Q-q)^2}{(1+q)^2}\frac{A\left(h^2\right)}{A(\ell^2)}. \tag{2.22}$$

3. Integral Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)| w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below.

We assume that $\int_\Omega w d\mu = 1$ and $\int_\Omega v d\mu = 1$. Then by (2.2) we get

$$\begin{aligned}(3.1)\quad & \int_\Omega vf d\mu \int_\Omega wg d\mu + \int_\Omega vg d\mu \int_\Omega wf d\mu \\ & - \int_\Omega v(f+g)\, d\mu \int_\Omega w\left(\frac{fg}{f+g}\right) d\mu \\ & - \int_\Omega w(f+g)\, d\mu \int_\Omega v\left(\frac{fg}{f+g}\right) d\mu \\ & = \int_\Omega v\left(\frac{f^2}{f+g}\right) d\mu \int_\Omega w\left(\frac{g^2}{f+g}\right) d\mu + \int_\Omega v\left(\frac{g^2}{f+g}\right) d\mu \int_\Omega w\left(\frac{f^2}{f+g}\right) d\mu \\ & - 2\int_\Omega v\left(\frac{fg}{f+g}\right) d\mu \int_\Omega w\left(\frac{fg}{f+g}\right) d\mu \\ & \geq 0,\end{aligned}$$

provided that all the integrals in (3.1) are finite.

In particular,

$$\begin{aligned}(3.2)\qquad & \int_\Omega vf d\mu \int_\Omega vg d\mu - \int_\Omega v(f+g)\, d\mu \int_\Omega v\left(\frac{fg}{f+g}\right) d\mu \\ & = \int_\Omega v\left(\frac{f^2}{f+g}\right) d\mu \int_\Omega v\left(\frac{g^2}{f+g}\right) d\mu - \left(\int_\Omega v\frac{fg}{f+g} d\mu\right)^2 \geq 0,\end{aligned}$$

provided all these integrals exists.

If there exists $0 < m < M$ such that

$$(3.3)\qquad m \leq f + g \leq M \ \mu\text{-almost everywhere on } \Omega,$$

then for all $v,\ w \geq 0$ we have

$$\begin{aligned}(3.4)\qquad & \frac{1}{M^2}\left(\int_\Omega vf^2 d\mu \int_\Omega wg^2 d\mu + \int_\Omega vg^2 d\mu \int_\Omega wf^2 d\mu\right. \\ & \left. -2\int_\Omega vfg d\mu \int_\Omega wgf d\mu\right)\end{aligned}$$

$$\leq \int_\Omega vf d\mu \int_\Omega wg d\mu + \int_\Omega vg d\mu \int_\Omega wf d\mu$$
$$- \int_\Omega v(f+g) d\mu \int_\Omega w \left(\frac{fg}{f+g} \right) d\mu$$
$$- \int_\Omega w(f+g) d\mu \int_\Omega v \left(\frac{fg}{f+g} \right) d\mu$$
$$\leq \frac{1}{m^2} \left(\int_\Omega vf^2 d\mu \int_\Omega wg^2 d\mu + \int_\Omega vg^2 d\mu \int_\Omega wf^2 d\mu \right.$$
$$\left. -2 \int_\Omega vfg d\mu \int_\Omega wgf d\mu \right).$$

In particular, we have

$$\frac{1}{M^2} \left(\int_\Omega vf^2 d\mu \int_\Omega vg^2 d\mu - \left(\int_\Omega vfg d\mu \right)^2 \right) \tag{3.5}$$
$$\leq \int_\Omega vf d\mu \int_\Omega vg d\mu - \int_\Omega v(f+g) d\mu \int_\Omega v \left(\frac{fg}{f+g} \right) d\mu$$
$$\leq \frac{1}{m^2} \left(\int_\Omega vf^2 d\mu \int_\Omega vg^2 d\mu - \left(\int_\Omega vfg d\mu \right)^2 \right).$$

Further, assume that h^2, ℓ^2, $\frac{h^2\ell^2}{h^2+\ell^2}$, $h^2\ell^2$, h^4, ℓ^4 are in $L_w(\Omega, \mu)$ and there exists the constants $0 < k < K$ such that

$$k \leq h^2 + \ell^2 \leq K, \ \mu\text{-almost everywhere on } \Omega, \tag{3.6}$$

then by (2.14) we have

$$\frac{1}{K^2} \left(\int_\Omega wh^4 d\mu \int_\Omega w\ell^4 d\mu - \left(\int_\Omega wh^2\ell^2 d\mu \right)^2 \right) \tag{3.7}$$
$$\leq \int_\Omega wh^2 d\mu \int_\Omega w\ell^2 d\mu - \int_\Omega w \left(\frac{h^2\ell^2}{h^2+\ell^2} \right) d\mu \int_\Omega w \left(h^2 + \ell^2 \right) d\mu$$
$$\leq \frac{1}{k^2} \left(\int_\Omega wh^4 d\mu \int_\Omega w\ell^4 d\mu - \left(\int_\Omega wh^2\ell^2 d\mu \right)^2 \right).$$

Also assume that f, g, $\frac{g}{f}$, $\frac{fg}{f+g}$, $\frac{g^2}{f}$ are in $L_w(\Omega, \mu)$ and there exists $0 < p < P$ such that

$$p \leq \frac{g}{f} \leq P \ \mu\text{-almost everywhere on } \Omega \tag{3.8}$$

then by (2.17) we get

$$\frac{1}{(1+P)^2} \left(\int_\Omega wf d\mu \int_\Omega w \left(\frac{g^2}{f} \right) d\mu - \left(\int_\Omega w \frac{g}{f} d\mu \right)^2 \right) \tag{3.9}$$
$$\leq \int_\Omega wf d\mu \int_\Omega wg d\mu - \int_\Omega w(f+g) d\mu \int_\Omega w \left(\frac{fg}{f+g} \right) d\mu$$
$$\leq \frac{1}{(1+p)^2} \left(\int_\Omega wf d\mu \int_\Omega w \left(\frac{g^2}{f} \right) d\mu - \left(\int_\Omega w \frac{g}{f} d\mu \right)^2 \right)$$

and by (2.21) we get

$$0\leq 1-\frac{\int_{\Omega}w\left(f+g\right)d\mu}{\int_{\Omega}wfd\mu\int_{\Omega}wgd\mu}\int_{\Omega}w\left(\frac{fg}{f+g}\right)d\mu \leq\frac{\left(P-p\right)^{2}}{\left(1+p\right)^{2}}\frac{\int_{\Omega}wfd\mu}{\int_{\Omega}wgd\mu}. \tag{3.10}$$

If there exists the constants $0<q<Q$ such that

$$q\leq\frac{\ell^{2}}{h^{2}}\leq Q \ \mu\text{-almost everywhere on } \Omega, \tag{3.11}$$

then we have

$$\begin{aligned} &\frac{1}{\left(1+Q\right)^{2}}\left(\int_{\Omega}wh^{2}d\mu\int_{\Omega}w\left(\frac{\ell^{4}}{h^{2}}\right)d\mu-\left(\int_{\Omega}w\frac{\ell^{2}}{h^{2}}d\mu\right)^{2}\right)\\ &\leq\int_{\Omega}wh^{2}d\mu\int_{\Omega}w\ell^{2}d\mu-\int_{\Omega}w\left(\frac{h^{2}\ell^{2}}{h^{2}+\ell^{2}}\right)d\mu\int_{\Omega}w\left(h^{2}+\ell^{2}\right)d\mu\\ &\leq\frac{1}{\left(1+q\right)^{2}}\left(\int_{\Omega}wh^{2}d\mu\int_{\Omega}w\left(\frac{\ell^{4}}{h^{2}}\right)d\mu-\left(\int_{\Omega}w\frac{\ell^{2}}{h^{2}}d\mu\right)^{2}\right). \end{aligned} \tag{3.12}$$

From (2.22) we also get

$$0\leq 1-\frac{\int_{\Omega}w\left(h^{2}+\ell^{2}\right)d\mu}{\int_{\Omega}wh^{2}d\mu\int_{\Omega}w\ell^{2}d\mu}\int_{\Omega}w\left(\frac{h^{2}\ell^{2}}{h^{2}+\ell^{2}}\right)d\mu \leq\frac{\left(Q-q\right)^{2}}{\left(1+q\right)^{2}}\frac{\int_{\Omega}wh^{2}d\mu}{\int_{\Omega}w\ell^{2}d\mu}. \tag{3.13}$$

4. Discrete Inequalities

We consider the n-tuples of positive numbers $a=\left(a_{1},...,a_{n}\right)$, $b=\left(b_{1},...,b_{n}\right)$ and the probability distributions $p=\left(p_{1},...,p_{n}\right)$, $q=\left(q_{1},...,q_{n}\right)$ i.e. $p_{i}\geq 0$, $q_{i}\geq 0$ for any $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_{i}=\sum_{i=1}^{n}q_{i}=1.$

From (3.1) we get for the discrete counting measure that

$$\begin{aligned} &\sum_{k=1}^{n}q_{k}a_{k}\sum_{k=1}^{n}p_{k}b_{k}+\sum_{k=1}^{n}q_{k}b_{k}\sum_{k=1}^{n}p_{k}a_{k}-\sum_{k=1}^{n}q_{k}\left(a_{k}+b_{k}\right)\sum_{k=1}^{n}p_{k}\left(\frac{a_{k}b_{k}}{a_{k}+b_{k}}\right)\\ &-\sum_{k=1}^{n}p_{k}\left(a_{k}+b_{k}\right)\sum_{k=1}^{n}q_{k}\left(\frac{a_{k}b_{k}}{a_{k}+b_{k}}\right)\\ &=\sum_{k=1}^{n}q_{k}\left(\frac{a_{k}^{2}}{a_{k}+b_{k}}\right)\sum_{k=1}^{n}p_{k}\left(\frac{b_{k}^{2}}{a_{k}+b_{k}}\right)\\ &+\sum_{k=1}^{n}q_{k}\left(\frac{b_{k}^{2}}{a_{k}+b_{k}}\right)\sum_{k=1}^{n}p_{k}\left(\frac{a_{k}^{2}}{x_{k}+b_{k}}\right)\\ &-2\sum_{k=1}^{n}q_{k}\left(\frac{x_{k}b_{k}}{x_{k}+b_{k}}\right)\sum_{k=1}^{n}p_{k}\left(\frac{x_{k}b_{k}}{x_{k}+b_{k}}\right)\\ &\geq 0. \end{aligned} \tag{4.1}$$

In particular,

$$\sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k - \sum_{k=1}^{n} p_k (a_k + b_k) \sum_{k=1}^{n} p_k \left(\frac{a_k b_k}{a_k + b_k} \right) = \sum_{k=1}^{n} p_k \left(\frac{a_k^2}{a_k + b_k} \right) \sum_{k=1}^{n} p_k \left(\frac{b_k^2}{a_k + b_k} \right) - \left(\sum_{k=1}^{n} p_k \left(\frac{a_k b_k}{a_k + b_k} \right) \right)^2 \geq 0. \tag{4.2}$$

If there exists $0 < m < M$ such that

$$m \leq a_k + b_k \leq M, \; k \in \{1, ..., n\}, \tag{4.3}$$

then from (3.4)

$$\begin{aligned} &\frac{1}{M^2} \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 + \sum_{k=1}^{n} q_k b_k^2 \sum_{k=1}^{n} p_k a_k^2 - 2 \sum_{k=1}^{n} q_k a_k b_k \sum_{k=1}^{n} p_k b_k a_k \right) \\ &\leq \sum_{k=1}^{n} q_k a_k \sum_{k=1}^{n} p_k b_k + \sum_{k=1}^{n} q_k b_k \sum_{k=1}^{n} p_k a_k \\ &- \sum_{k=1}^{n} q_k (a_k + b_k) \sum_{k=1}^{n} p_k \left(\frac{a_k b_k}{a_k + b_k} \right) - \sum_{k=1}^{n} p_k (a_k + b_k) \sum_{k=1}^{n} q_k \left(\frac{a_k b_k}{a_k + b_k} \right) \\ &\leq \frac{1}{m^2} \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 + \sum_{k=1}^{n} q_k b_k^2 \sum_{k=1}^{n} p_k a_k^2 \right. \\ &\left. -2 \sum_{k=1}^{n} q_k a_k b_k \sum_{k=1}^{n} p_k b_k a_k \right). \end{aligned} \tag{4.4}$$

In particular, we have

$$\begin{aligned} &\frac{1}{M^2} \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} q_k b_k^2 - \left(\sum_{k=1}^{n} q_k a_k b_k \right)^2 \right) \\ &\leq \sum_{k=1}^{n} q_k a_k \sum_{k=1}^{n} q_k b_k - \left(\sum_{k=1}^{n} q_k a_k + \sum_{k=1}^{n} q_k b_k \right) \sum_{k=1}^{n} q_k \left(\frac{a_k b_k}{a_k + b_k} \right) \\ &\leq \frac{1}{m^2} \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} q_k b_k^2 - \left(\sum_{k=1}^{n} q_k a_k b_k \right)^2 \right). \end{aligned} \tag{4.5}$$

If there exists the constants $0 < \kappa < K$ such that

$$\kappa \leq x_k^2 + y_k^2 \leq K, \; k \in \{1, ..., n\}, \tag{4.6}$$

then by (3.7) we have

$$\frac{1}{K^2}\left(\sum_{k=1}^{n} p_k x_k^4 \sum_{k=1}^{n} p_k y_k^4 - \left(\sum_{k=1}^{n} p_k x_k^2 y_k^2\right)^2\right) \tag{4.7}$$
$$\leq \sum_{k=1}^{n} p_k x_k^2 \sum_{k=1}^{n} p_k y_k^2 - \sum_{k=1}^{n} p_k \left(\frac{x_k^2 y_k^2}{x_k^2 + y_k^2}\right) \sum_{k=1}^{n} p_k \left(x_k^2 + y_k^2\right)$$
$$\leq \frac{1}{\kappa^2}\left(\sum_{k=1}^{n} p_k x_k^4 \sum_{k=1}^{n} p_k y_k^4 - \left(\sum_{k=1}^{n} p_k x_k^2 y_k^2\right)^2\right).$$

Also assume that there exists $0 < p < P$ such that

$$p \leq \frac{b_k}{a_k} \leq P,\ \ k \in \{1, ..., n\} \tag{4.8}$$

then by (3.9) we get

$$\frac{1}{(1+P)^2}\left(\sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k \left(\frac{b_k^2}{a_k}\right) - \left(\sum_{k=1}^{n} p_k \frac{b_k}{a_k}\right)^2\right) \tag{4.9}$$
$$\leq \sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k - \sum_{k=1}^{n} p_k \left(a_k + b_k\right) \sum_{k=1}^{n} p_k \left(\frac{a_k b_k}{a_k + b_k}\right)$$
$$\leq \frac{1}{(1+p)^2}\left(\sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k \left(\frac{b_k^2}{a_k}\right) - \left(\sum_{k=1}^{n} p_k \frac{b_k}{a_k}\right)^2\right)$$

and by (3.10) we get

$$0 \leq 1 - \frac{\sum_{k=1}^{n} p_k \left(a_k + b_k\right)}{\sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k} \sum_{k=1}^{n} p_k \left(\frac{a_k b_k}{a_k + b_k}\right) \tag{4.10}$$
$$\leq \frac{(P-p)^2}{(1+p)^2} \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k b_k}.$$

If there exists the constants $0 < q < Q$ such that

$$q \leq \frac{y_k^2}{x_k^2} \leq Q,\ \ k \in \{1, ..., n\} \tag{4.11}$$

then by (3.12),

$$\frac{1}{(1+Q)^2}\left(\sum_{k=1}^{n} p_k x_k^2 \sum_{k=1}^{n} p_k \left(\frac{y_k^4}{x_k^2}\right) - \left(\sum_{k=1}^{n} p_k \frac{y_k^2}{x_k^2}\right)^2\right) \tag{4.12}$$
$$\leq \sum_{k=1}^{n} p_k x_k^2 \sum_{k=1}^{n} p_k y_k^2 - \sum_{k=1}^{n} p_k \left(\frac{x_k^2 y_k^2}{x_k^2 + y_k^2}\right) \sum_{k=1}^{n} p_k \left(x_k^2 + y_k^2\right)$$
$$\leq \frac{1}{(1+q)^2}\left(\sum_{k=1}^{n} p_k x_k^2 \sum_{k=1}^{n} p_k \left(\frac{y_k^4}{x_k^2}\right) - \left(\sum_{k=1}^{n} p_k \frac{y_k^2}{x_k^2}\right)^2\right).$$

From (3.13) we also get

$$0 \le 1 - \frac{\sum_{k=1}^{n} p_k \left(x_k^2 + y_k^2\right)}{\sum_{k=1}^{n} p_k x_k^2 \sum_{k=1}^{n} p_k y_k^2} \sum_{k=1}^{n} p_k \left(\frac{x_k^2 y_k^2}{x_k^2 + y_k^2}\right) \le \frac{(Q-q)^2}{(1+q)^2} \frac{\sum_{k=1}^{n} p_k x_k^2}{\sum_{k=1}^{n} p_k y_k^2}. \tag{4.13}$$

References

[1] S. S. DRAGOMIR, A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities. J. *Inequal. Pure Appl. Math.* **4** (2003), no. 3, Article 63, 142 pp.

[2] D. E. DAYKIN, C. J. ELIEZER AND C. CARLITZ, Problem 5563, *Amer. Math. Monthly*, 75 (1968), p. 198 and 76 (1969), 98–100.

[3] E. A. MILNE, Note on Rosseland's integral for the stellar absorption coefficient, *Monthly Notices of the Royal Astronomical Society*, **85** (1925), 979–984

[4] L. NIKOLOVA, S. VAROŠANEC, Properties of some functionals associated with h-concave functions with applications on inequalities, *J. Inequal. Appl.* **2014**, 2014:30.

[5] L. NIKOLOVA, S. VAROŠANEC, Functionals related to DEC inequality, *J. Math. Ineq.*, **9** (2015), Number 4 1227–1243.

[6] S. M. SITNIK, Generalized Young and Cauchy-Bunyakowsky inequalities with applications: a surwey, *Advances in Modern Analysis,* Mathematical Forum Series, Ed. Yu. F. Korobeinik, A. G. Kursaev, 2009, 221–266.

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