GENERALIZATIONS OF DEC'S INEQUALITY FOR TWO ISOTONIC FUNCTIONALS

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Abstract. In this paper we obtain some inequalities for two isotonic functionals that generalize the well known Daykin, Elizer and Carlitz (DEC's) inequality. Applications for several classes of mean are provided as well.

1. Introduction

In [2], Daykin, Elizer and Carlitz obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality, which, in the version from [3, p. 87], can be stated as:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} \varphi(a_i, b_i) \sum_{i=1}^{n} \psi(a_i, b_i) \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{DEC}$$

where and $a_i,\ b_i \in [0,\infty)$ for each $i \in \{1, ..., n\}$ and (φ, ψ) is a pair of functions defined on $[0,\infty) \times [0,\infty)$ and satisfying the conditions

(i) $\varphi(a,b)\psi(a,b) = a^2b^2$ for any $a,\ b \in [0,\infty)$;

(ii) $\varphi(ka, kb) = k^2\varphi(a,b)$ for any $a,\ b,\ k \in [0,\infty)$;

(iii) $\frac{b\varphi(a,1)}{a\varphi(b,1)} + \frac{a\varphi(b,1)}{b\varphi(a,1)} \leq \frac{a}{b} + \frac{b}{a}$ for any $a,\ b \in (0,\infty)$.

As examples of such pairs of functions, which will be called for simplicity *(DEC)-pairs,* we can indicate the following functions: $\varphi(a,b) = a^2+b^2$, $\psi(a,b) = \frac{a^2b^2}{a^2+b^2}$ and $\varphi(a,b) = a^{1+\alpha}b^{1-\alpha}$, $\psi(a,b) = a^{1-\alpha}b^{1+\alpha}$ with $\alpha \in [0,1]$. The first pair generates the famous *Milne's inequality:*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} \left(a_i^2 + b_i^2\right) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{1.1}$$

while the second generates the *Callebaut's inequality:*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{1.2}$$

We observe that from the property (iii) we have the inequality

$$2 \leq \frac{b\varphi(a,1)}{a\varphi(b,1)} + \frac{a\varphi(b,1)}{b\varphi(a,1)} \leq \frac{a}{b} + \frac{b}{a},$$

for any $a,\ b > 0$.

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If in this inequality we choose $a = \frac{u}{v}$ and $b = \frac{z}{w}$, then we get

$$2 \le \frac{zv\varphi\left(\frac{u}{v},1\right)}{uw\varphi\left(\frac{z}{w},1\right)} + \frac{uw\varphi\left(\frac{z}{w},1\right)}{zv\varphi\left(\frac{u}{v},1\right)} \le \frac{uw}{vz} + \frac{vz}{uw}. \tag{1.3}$$

From the property (ii) we have

$$zv\varphi\left(\frac{u}{v},1\right) = \frac{z}{v}\varphi(u,v) \text{ and } uw\varphi\left(\frac{z}{w},1\right) = \frac{u}{w}\varphi(z,w),$$

which give from (1.3) that

$$2 \le \frac{zw\varphi(u,v)}{uv\varphi(z,w)} + \frac{uv\varphi(z,w)}{zw\varphi(u,v)} \le \frac{uw}{vz} + \frac{vz}{uw}, \tag{1.4}$$

for any $u, v, z, w > 0$.

Utilising the property (i) we have

$$\varphi(z,w) = \frac{z^2w^2}{\psi(z,w)} \text{ and } \varphi(u,v) = \frac{u^2v^2}{\psi(u,v)},$$

which, from (1.4), produces the inequality

$$2 \le \frac{\varphi(u,v)\psi(z,w)}{zwuv} + \frac{\varphi(z,w)\psi(u,v)}{uvzw} \le \frac{uw}{vz} + \frac{vz}{uw},$$

i.e., the inequality

$$2uvzw \le \varphi(u,v)\psi(z,w) + \varphi(z,w)\psi(u,v) \le u^2w^2 + v^2z^2, \tag{1.5}$$

for any u, v, z, $w \ge 0$.

Definition 1. *We say that the pair of functions (φ, ψ) defined on $[0,\infty) \times [0,\infty)$ and with positive values is of pre-(DEC) type if it satisfies the condition (1.5).*

We observe that, as shown above, if (φ, ψ) is a *(DEC) pair*, then it is of pre-*(DEC) type.*

In this paper we obtain some inequalities for two isotonic functionals that generalize the well known DEC's inequality. Applications for integrals and n-tuples of real numbers are provided as well.

2. Main Results

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) f, $g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \ge 0$, then $A(f) \ge 0$.
The mapping A is said to be *normalized if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0$, $k \in E$).

We have the following refinement of Schwarz inequality for isotonic functionals:

Theorem 1. *Let (φ, ψ) be a pre-(DEC) pair. Assume that f, $g : E \to \mathbb{R}$ are such that f^2, g^2, fg, $\varphi(f,g)$, $\psi(f,g) \in L$, then for all isotonic linear functionals A, B defined on L,*

$$2A(fg)B(fg) \leq A[\varphi(f,g)]B[\psi(f,g)] + A[\psi(f,g)]B[\varphi(f,g)] \leq A(f^2)B(g^2) + A(g^2)B(f^2). \tag{2.1}$$

In particular, for $B = A$, we obtain the refinement of Schwarz's inequality

$$A^2(fg) \leq A[\varphi(f,g)]A[\psi(f,g)] \leq A(f^2)A(g^2). \tag{2.2}$$

Proof. From the inequalities (1.5) we get for $u = f(t)$, $v = g(t)$, $z = f(s)$ and $w = g(s)$ with t, $s \in E$, then

$$\begin{aligned} &2f(t)g(t)f(s)g(s) \\ &\leq \varphi(f(t),g(t))\psi(f(s),g(s)) + \varphi(f(s),g(s))\psi(f(t),g(t)) \\ &\leq f^2(t)g^2(s) + g^2(t)f^2(s), \end{aligned} \tag{2.3}$$

for all t, $s \in E$.

Fix $s \in E$, then the inequality (2.3) can be written in the order of L as

$$2f(s)g(s)fg \leq \psi(f(s),g(s))\varphi(f,g) + \varphi(f(s),g(s))\psi(f,g) \leq g^2(s)f^2 + f^2(s)g^2. \tag{2.4}$$

If we apply the functional A to this inequality we deduce

$$\begin{aligned} &2f(s)g(s)A(fg) \\ &\leq \psi(f(s),g(s))A[\varphi(f,g)] + \varphi(f(s),g(s))A[\psi(f,g)] \\ &\leq g^2(s)A(f^2) + f^2(s)A(g^2) \end{aligned}$$

for all $s \in E$.

This inequality can be written in the order of L as

$$\begin{aligned} 2A(fg)fg &\leq A[\varphi(f,g)]\psi(f,g) + A[\psi(f,g)]\varphi(f,g) \\ &\leq A(f^2)g^2 + A(g^2)f^2 \end{aligned}$$

and if we apply to this inequality the functional B, then we get (2.1). □

Corollary 1. *Assume that f, $g : E \to \mathbb{R}$ are such that f^2, g^2, fg, f^2+g^2, $\frac{f^2g^2}{f^2+g^2} \in L$, then for all isotonic linear functionals A, B defined on L,*

$$\begin{aligned} &2A(fg)B(fg) \\ &\leq A(f^2+g^2)B\left(\frac{f^2g^2}{f^2+g^2}\right) + A\left(\frac{f^2g^2}{f^2+g^2}\right)B(f^2+g^2) \\ &\leq A(f^2)B(g^2) + A(g^2)B(f^2). \end{aligned} \tag{2.5}$$

In particular, for $B = A$, we obtain the Milne refinement of Schwarz's inequality

$$A^2(fg) \leq A(f^2+g^2)A\left(\frac{f^2g^2}{f^2+g^2}\right) \leq A(f^2)A(g^2). \tag{2.6}$$

The proof follows by Theorem 1 for the *(DEC) pair* $\varphi(a,b) = a^2 + b^2$, $\psi(a,b) = \frac{a^2b^2}{a^2+b^2}$, $a,\ b > 0$.

Corollary 2. *Assume that* $f,\ g : E \to \mathbb{R}$ *are such that* $f^2,\ g^2,\ fg,\ f^{1+\alpha}g^{1-\alpha}$, $f^{1-\alpha}g^{1+\alpha} \in L$, *with* $\alpha \in [0,1]$, *then for all isotonic linear functionals* $A,\ B$ *defined on* L,

$$\begin{aligned} &2A(fg)B(fg) \\ &\leq A\left(f^{1+\alpha}g^{1-\alpha}\right)B\left(f^{1-\alpha}g^{1+\alpha}\right) + A\left(f^{1-\alpha}g^{1+\alpha}\right)B\left(f^{1+\alpha}g^{1-\alpha}\right) \\ &\leq A\left(f^2\right)B\left(g^2\right) + A\left(g^2\right)B\left(f^2\right). \end{aligned} \tag{2.7}$$

In particular, for $B = A$, *we obtain the Callebaut refinement of Schwarz's inequality*

$$A^2(fg) \leq A\left(f^{1+\alpha}g^{1-\alpha}\right)A\left(f^{1-\alpha}g^{1+\alpha}\right) \leq A\left(f^2\right)A\left(g^2\right). \tag{2.8}$$

The proof follows by Theorem 1 for the *(DEC) pair* $\varphi(a,b) = a^{1+\alpha}b^{1-\alpha}$, $\psi(a,b) = a^{1-\alpha}b^{1+\alpha}$ with $\alpha \in [0,1]$ and $a,\ b > 0$.

A function $M : [0,\infty)\times[0,\infty) \to [0,\infty)$ is caled *a mean* if it satisfy the properties:

(1) $M(x,x) = x$ for all $x \geq 0$,
(2) $M(\lambda x, \lambda y) = \lambda M(x,y)$ for $\lambda > 0$ and $x,\ y \in [0,\infty)$ (positive homogenity),
(3) If $x_1 < x_2$, then $M(x_1,y) \leq M(x_2,y)$ and if $y_1 < y_2$, then $M(x,y_1) \leq M(x,y_2)$ (monotonicity on both arguments),
(4) $M(x,y) = M(y,x)$ for all $x,\ y \geq 0$.

We call $M : [0,\infty)\times[0,\infty) \to [0,\infty)$ a *pre-mean* if it satisfies the conditions 2 and 3.

Lemma 1. *If* $M : [0,\infty)\times[0,\infty) \to [0,\infty)$ *is a pre-mean on* $[0,\infty)$, *then the pair of functions* $\varphi(u,v) = M^2(u,v)$ *and* $\psi(u,v) = \frac{u^2v^2}{M^2(u,v)}$ *is of pre-(DEC) type.*

Proof. Assume that $\frac{w}{z} < \frac{v}{u}$ for positive $u,\ v,\ w,\ z$. Then $\frac{z}{w} > \frac{u}{v}$ and by the monotonicity of M we get

$$M\left(1,\frac{w}{z}\right) \leq M\left(1,\frac{v}{u}\right) \text{ and } M\left(\frac{z}{w},1\right) \geq M\left(\frac{u}{v},1\right).$$

Since M is nonnegative, hence

$$\left(M^2\left(1,\frac{w}{z}\right) - M^2\left(1,\frac{v}{u}\right)\right)\left(M^2\left(\frac{z}{w},1\right) - M^2\left(\frac{u}{v},1\right)\right) \leq 0.$$

Taking into acccount that M is positive homogeneous, then by multiplying with $u^2,\ v^2,\ w^2,\ z^2$ and rearranging, we derive (see also [4])

$$0 \geq \left(z^2M^2(u,v) - u^2M^2(z,w)\right)\left(w^2M^2(u,v) - v^2M^2(z,w)\right).$$

This inequality also holds if we assume that that $\frac{w}{z} > \frac{v}{u}$ for positive $u,\ v,\ w,\ z$.

Therefore, for any positive numbers $u,\ v,\ w,\ z$ we have

$$\begin{aligned} 0 &\geq \left(z^2M^2(u,v) - u^2M^2(z,w)\right)\left(w^2M^2(u,v) - v^2M^2(z,w)\right) \\ &= z^2w^2M^4(u,v) + u^2v^2M^4(z,w) \\ &\quad - u^2w^2M^2(z,w)M^2(u,v) - z^2v^2M^2(u,v)M^2(z,w), \end{aligned}$$

which gives

$$\begin{aligned} &u^2w^2M^2(z,w)M^2(u,v) + z^2v^2M^2(u,v)M^2(z,w) \\ &\geq z^2w^2M^4(u,v) + u^2v^2M^4(z,w) \end{aligned}$$

and dividing by $M^2(z,w)\,M^2(u,v)$ we get

$$\begin{aligned}u^2w^2+z^2v^2 &\geq \frac{z^2w^2M^4(u,v)+u^2v^2M^4(z,w)}{M^2(z,w)\,M^2(u,v)}\\ &= \frac{z^2w^2M^2(u,v)}{M^2(z,w)}+\frac{u^2v^2M^2(z,w)}{M^2(u,v)}\\ &= M^2(u,v)\,\frac{z^2w^2}{M^2(z,w)}+\frac{u^2v^2}{M^2(u,v)}M^2(z,w)\\ &= \varphi(u,v)\,\psi(z,w)+\varphi(z,w)\,\psi(u,v)\,,\end{aligned}$$

where

$$\varphi(u,v)=M^2(u,v)\ \text{ and }\psi(u,v)=\frac{u^2v^2}{M^2(u,v)}.$$

We also have

$$\begin{aligned}&M^2(u,v)\,\frac{z^2w^2}{M^2(z,w)}+\frac{u^2v^2}{M^2(u,v)}M^2(z,w)\\ &\geq 2M(u,v)\,\frac{zw}{M(z,w)}\,\frac{uv}{M(u,v)}M(z,w)=2zwuv,\end{aligned}$$

which shows that

$$2uvzw\leq\varphi(u,v)\,\psi(z,w)+\varphi(z,w)\,\psi(u,v)\leq u^2w^2+v^2z^2,$$

namely (φ,ψ) is of pre-*(DEC) type.* □

Corollary 3. *If* $M:[0,\infty)\times[0,\infty)\to[0,\infty)$ *is a pre-mean on* $[0,\infty)$ *and* f^2, g^2, fg, $M^2(f,g)\,,\ \frac{f^2g^2}{M^2(f,g)}\in L$, *then for all isotonic linear functionals* A, B *defined on* L,

$$\begin{aligned}&2A(fg)\,B(fg)\\ &\leq A\left[M^2(f,g)\right]B\left(\frac{f^2g^2}{M^2(f,g)}\right)+A\left(\frac{f^2g^2}{M^2(f,g)}\right)B\left[M^2(f,g)\right]\\ &\leq A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right).\end{aligned}\tag{2.9}$$

In particular, for $B=A$, *we obtain the refinement of Schwarz's inequality*

$$A^2(fg)\leq A\left[M^2(f,g)\right]A\left(\frac{f^2g^2}{M^2(f,g)}\right)\leq A\left(f^2\right)A\left(g^2\right).\tag{2.10}$$

Remark 1. *The univariate integral version of (2.10) was obtained by S. M. Sitnik, see for instance* [5]. *The isotonic functional version of (2.10) was obtained by L. Nikolova and S. Varošanec in* [4].

Corollary 4. *If* $f,\ g>0$ *with* f^2, g^2, fg, $\min\left\{f^2,g^2\right\}$, $\max\left\{f^2,g^2\right\}\in L$, *then for all isotonic linear functionals* A, B *defined on* L,

$$\begin{aligned}2A(fg)\,B(fg)&\leq A\left(\min\left\{f^2,g^2\right\}\right)B\left(\max\left\{f^2,g^2\right\}\right)\\ &\quad+A\left(\max\left\{f^2,g^2\right\}\right)B\left(\min\left\{f^2,g^2\right\}\right)\\ &\leq A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right).\end{aligned}\tag{2.11}$$

In particular, for $B=A$, *we obtain the refinement of Schwarz's inequality*

$$A^2(fg)\leq A\left(\min\left\{f^2,g^2\right\}\right)A\left(\max\left\{f^2,g^2\right\}\right)\leq A\left(f^2\right)A\left(g^2\right).\tag{2.12}$$

Proof. We know that for any $a,\ b>0$ we have $ab=\min\{a,b\}\max\{a,b\}$. By utilising (2.9) for $M(f,g)=\min\{f,g\}$ and since $\max\{f,g\}=\frac{fg}{\min\{f,g\}}$, hence we get (2.11). □

Remark 2. *The univariate integral version of (2.12) was obtained by S. M. Sitnik, see for instance* [5]. *In* [1] *the authors also provided four different proofs of this case, but none simpler than the one of Corollary 4.*

The proof follows by Theorem 1 and Lemma 1.

Theorem 2. *Let (φ,ψ) be a pre-(DEC) pair. Assume that $f,\ g:E\to\mathbb{R}$ are such that f^2g^2, $f^{2+2\alpha}g^{2-2\alpha}$, $f^{2-2\alpha}g^{2+2\alpha}$, $\varphi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)$, $\psi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\in L$, $\alpha\in[0,1]$, then for all isotonic linear functionals A, B defined on L,*

$$\begin{aligned}(2.13)\quad & 2A\left(f^2g^2\right)B\left(f^2g^2\right)\\ &\leq A\left[\varphi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]B\left[\psi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]\\ &\quad+B\left[\varphi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]A\left[\psi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]\\ &\leq A\left(f^{2+2\alpha}g^{2-2\alpha}\right)B\left(f^{2-2\alpha}g^{2+2\alpha}\right)+A\left(f^{2-2\alpha}g^{2+2\alpha}\right)B\left(f^{2+2\alpha}g^{2-2\alpha}\right),\end{aligned}$$

and, in particular

$$\begin{aligned}(2.14)\quad A^2\left(f^2g^2\right) &\leq A\left[\varphi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]A\left[\psi\left(f^{1+\alpha}g^{1-\alpha},f^{1-\alpha}g^{1+\alpha}\right)\right]\\ &\leq A\left(f^{2+2\alpha}g^{2-2\alpha}\right)A\left(f^{2-2\alpha}g^{2+2\alpha}\right).\end{aligned}$$

Proof. From (1.5) we get for $u=f^{1+\alpha}(t)\,g^{1-\alpha}(t)$, $v=f^{1-\alpha}(t)\,g^{1+\alpha}(t)$, $z=f^{1+\alpha}(s)\,g^{1-\alpha}(s)$ and $w=f^{1-\alpha}(s)\,g^{1+\alpha}(s)$, where $t,\ s\in E$ we get

$$\begin{aligned}(2.15)\quad & 2f^{1+\alpha}(t)\,g^{1-\alpha}(t)\,f^{1-\alpha}(t)\,g^{1+\alpha}(t)\,f^{1+\alpha}(s)\,g^{1-\alpha}(s)\,f^{1-\alpha}(s)\,g^{1+\alpha}(s)\\ &\leq\varphi\left(f^{1+\alpha}(t)\,g^{1-\alpha}(t),f^{1-\alpha}(t)\,g^{1+\alpha}(t)\right)\\ &\quad\times\psi\left(f^{1+\alpha}(s)\,g^{1-\alpha}(s),f^{1-\alpha}(s)\,g^{1+\alpha}(s)\right)\\ &\quad+\varphi\left(f^{1+\alpha}(s)\,g^{1-\alpha}(s),f^{1-\alpha}(s)\,g^{1+\alpha}(s)\right)\\ &\quad\times\psi\left(f^{1+\alpha}(t)\,g^{1-\alpha}(t),f^{1-\alpha}(t)\,g^{1+\alpha}(t)\right)\\ &\leq f^{2+2\alpha}(t)\,g^{2-2\alpha}(t)\,f^{2-2\alpha}(s)\,g^{2+2\alpha}(s)\\ &\quad+f^{2-2\alpha}(t)\,g^{2+2\alpha}(t)\,f^{2+2\alpha}(s)\,g^{2-2\alpha}(s),\end{aligned}$$

namely

$$\begin{aligned}(2.16)\quad & 2f^2(t)\,g^2(t)\,f^2(s)\,g^2(s)\\ &\leq\varphi\left(f^{1+\alpha}(t)\,g^{1-\alpha}(t),f^{1-\alpha}(t)\,g^{1+\alpha}(t)\right)\\ &\quad\times\psi\left(f^{1+\alpha}(s)\,g^{1-\alpha}(s),f^{1-\alpha}(s)\,g^{1+\alpha}(s)\right)\\ &\quad+\varphi\left(f^{1+\alpha}(s)\,g^{1-\alpha}(s),f^{1-\alpha}(s)\,g^{1+\alpha}(s)\right)\\ &\quad\times\psi\left(f^{1+\alpha}(t)\,g^{1-\alpha}(t),f^{1-\alpha}(t)\,g^{1+\alpha}(t)\right)\\ &\leq f^{2+2\alpha}(t)\,g^{2-2\alpha}(t)\,f^{2-2\alpha}(s)\,g^{2+2\alpha}(s)\\ &\quad+f^{2-2\alpha}(t)\,g^{2+2\alpha}(t)\,f^{2+2\alpha}(s)\,g^{2-2\alpha}(s),\end{aligned}$$

for $t,\ s\in E$.

If we apply the functional A over the variable t and the functional B over the variable s, we get (2.13). □

Corollary 5. *Let (φ,ψ) be a pre-(DEC) pair. Assume that $f,\ g:E\to\mathbb{R}$ are such that $fg,\ f^{1+\alpha}g^{1-\alpha},\ f^{1-\alpha}g^{1+\alpha},\ \varphi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right),\psi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\in L,\ \alpha\in[0,1]$, then for all isotonic linear functionals $A,\ B$ defined on L,*

$$\begin{aligned}(2.17)\qquad & 2A(fg)\,B(fg)\\ &\leq A\left[\varphi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]B\left[\psi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]\\ &+B\left[\varphi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]A\left[\psi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]\\ &\leq A\left(f^{1+\alpha}g^{1-\alpha}\right)B\left(f^{1-\alpha}g^{1+\alpha}\right)+A\left(f^{1+\alpha}g^{1-\alpha}\right)B\left(f^{1-\alpha}g^{1+\alpha}\right),\end{aligned}$$

and, in particular, the following refinement of Callebaut's inequality

$$\begin{aligned}(2.18)\qquad A^2(fg)&\leq A\left[\varphi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]A\left[\psi\left(f^{\frac{1+\alpha}{2}}g^{\frac{1-\alpha}{2}},f^{\frac{1-\alpha}{2}}g^{\frac{1+\alpha}{2}}\right)\right]\\ &\leq A\left(f^{1+\alpha}g^{1-\alpha}\right)A\left(f^{1-\alpha}g^{1+\alpha}\right).\end{aligned}$$

3. Some Examples

Following the terminology of [1], a complementary mean is defined as

$$M^*(x,y):=\frac{xy}{M(x,y)},\ x,\ y>0.$$

With this notation and if $M:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a pre-mean on $[0,\infty)$ and $f^2,\ g^2,\ fg,\ M^2(f,g),\ [M^*(f,g)]^2\in L$, then for all *isotonic linear functionals* $A,\ B$ defined on L, we have by (2.9) that

$$\begin{aligned}(3.1)\qquad & 2A(fg)\,B(fg)\\ &\leq A\left[M^2(f,g)\right]B\left[(M^*(f,g))^2\right]+A\left[(M^*(f,g))^2\right]B\left[M^2(f,g)\right]\\ &\leq A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right).\end{aligned}$$

In particular, for $B=A$, we obtain the refinement of Schwarz's inequality

$$(3.2)\qquad A^2(fg)\leq A\left[M^2(f,g)\right]A\left[(M^*(f,g))^2\right]\leq A\left(f^2\right)A\left(g^2\right).$$

Consider the power means

$$M_\alpha(x,y):=\begin{cases}\left(\frac{x^\alpha+y^\alpha}{2}\right)^{1/\alpha}, & -\infty<\alpha<\infty,\ \alpha\neq 0,\\ \sqrt{xy},\ \alpha=0, & \\ \min\{x,y\},\ \alpha=-\infty, & \\ \max\{x,y\},\ \alpha=\infty. & \end{cases}$$

For power means we have

$$(M_\alpha)^*=M_{-\alpha}\text{ for all }\alpha\in[-\infty,\infty].$$

From (3.1) we obtain the following generalizations of *Milne's inequality* (for $\alpha = 2$)

$$2A(fg)B(fg) \le A\left[(f^\alpha + g^\alpha)^{2/\alpha}\right] B\left[\frac{f^2g^2}{(f^\alpha + g^\alpha)^{2/\alpha}}\right] + A\left[\frac{f^2g^2}{(f^\alpha + g^\alpha)^{2/\alpha}}\right] B\left[(f^\alpha + g^\alpha)^{2/\alpha}\right] \le A(f^2)B(g^2) + A(g^2)B(f^2), \tag{3.3}$$

for all $\alpha \in [-\infty, \infty]$, provided $f^2,\ g^2,\ fg,\ (f^\alpha + g^\alpha)^{2/\alpha},\ \frac{f^2g^2}{(f^\alpha+g^\alpha)^{2/\alpha}} \in L$.

In particular,

$$A^2(fg) \le A\left[(f^\alpha + g^\alpha)^{2/\alpha}\right] A\left[\frac{f^2g^2}{(f^\alpha + g^\alpha)^{2/\alpha}}\right] \le A(f^2)A(g^2). \tag{3.4}$$

for all $\alpha \in [-\infty, \infty]$, provided $f^2,\ g^2,\ fg,\ (f^\alpha + g^\alpha)^{2/\alpha},\ \frac{f^2g^2}{(f^\alpha+g^\alpha)^{2/\alpha}} \in L$.

The interested reader may state the corresponding inequalities for $\alpha = 1$, $\alpha = 1/2$ or $\alpha = 4$.

Also, consider *Radó means* (or L_p-*means*)

$$R_\beta(x,y) := \begin{cases} \left(\frac{x^{\beta+1}-y^{\beta+1}}{(\beta+1)(x-y)}\right)^{1/\beta}, & -\infty < \beta < \infty,\ \beta \ne 0, -1, \\ \frac{y-x}{\ln y - \ln x} = L(x,y), & \text{logarithmic mean, } \beta = -1, \\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} = I(x,y), & \text{identric mean, } \beta = 0, \\ \min\{x,y\}, & \alpha = -\infty, \\ \max\{x,y\}, & \alpha = \infty. \end{cases}$$

The complementary mean is

$$R_\beta^*(x,y) = \frac{1}{R_\beta(x^{-1}, y^{-1})}, \text{ for all } \beta \in [-\infty, \infty].$$

From (3.1) we get

$$2A(fg)B(fg) \le A\left[R_\beta^2(f,g)\right] B\left[R_\beta^{-2}(f^{-1}, g^{-1})\right] + A\left[R_\beta^{-2}(f^{-1}, g^{-1})\right] B\left[R_\beta^2(f,g)\right] \le A(f^2)B(g^2) + A(g^2)B(f^2) \tag{3.5}$$

for all $\beta \in [-\infty, \infty]$, provided $f^2,\ g^2,\ fg,\ R_\beta^2(f,g),\ R_\beta^{-2}(f^{-1}, g^{-1}) \in L$.

In particular,

$$A^2(fg) \le A\left[R_\beta^2(f,g)\right] A\left[R_\beta^{-2}(f^{-1}, g^{-1})\right] \le A(f^2)A(g^2) \tag{3.6}$$

for all $\beta \in [-\infty, \infty]$, provided $f^2,\ g^2,\ fg,\ R_\beta^2(f,g),\ R_\beta^{-2}(f^{-1}, g^{-1}) \in L$.

The particular cases $\beta = -1$ and $\beta = 0$ are of special interest. Their integral version were stated in [1].

As pointed out in [4], one can also consider *Seiffert type means* defined by

$$M(x,y) := \frac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} \text{ with } \lim_{z\to 0}\frac{f(z)}{z} = 1.$$

Witkowski, [8], proved that $M(x,y)$ is increasing on every argument if and only if the function $(1+z)f(z)/z$ increases and the function $(1-z)f(z)/z$ decreases on $z \in (0,1]$. It is checked that functions $f(z) = \sin z$, $f(z) = \tan z$, $f(z) = \sinh z$, $f(z) = \tanh z$, $f(z) = \ln(1+z)$, $f(z) = \arcsin z$, $f(z) = \arctan z$, $f(z) = \operatorname{arsinh} z$, $f(z) = \operatorname{artanh} z$ satisfy this condition and so we get examples of means which are homogeneous, increasing on every argument (and even symmetric as Witkowski showed). If $f(z) = \arcsin z$ this is the mean $P(x,y)$, and if $f(z) = \arctan z$ this is the mean $T(x,y)$, introduced by Seiffert in 1987, [6] and 1995, [7], respectively.

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