GENERALIZATIONS OF DEC’S INEQUALITY FOR TWO
ISOTONIC FUNCTIONALS

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1. Introduction

In [2], Daykin, Elizer and Carlitz obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality, which, in the version from [3, p. 87], can be stated as:

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} \varphi (a_i, b_i) \sum_{i=1}^{n} \psi (a_i, b_i) \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,
\]

where and \( a_i, b_i \in [0, \infty) \) for each \( i \in \{1, ..., n\} \) and \( (\varphi, \psi) \) is a pair of functions defined on \([0, \infty) \times [0, \infty)\) and satisfying the conditions

(i) \( \varphi (a, b) \psi (a, b) = a^2 b^2 \) for any \( a, b \in [0, \infty) \);

(ii) \( \varphi (ka, kb) = k^2 \varphi (a, b) \) for any \( a, b, k \in [0, \infty) \);

(iii) \( \frac{b \varphi(a, 1)}{a \varphi(b, 1)} + \frac{a \varphi(b, 1)}{b \varphi(a, 1)} \leq \frac{a}{b} + \frac{b}{a} \) for any \( a, b \in (0, \infty) \).

As examples of such pairs of functions, which will be called for simplicity \( (DEC) \)-pairs, we can indicate the following functions: \( \varphi (a, b) = a^2 + b^2 \), \( \psi (a, b) = \frac{a^2 b^2}{a^2 + b^2} \) and \( \varphi (a, b) = a^{1+\alpha} b^{1-\alpha}, \psi (a, b) = a^{1-\alpha} b^{1+\alpha} \) with \( \alpha \in [0, 1] \). The first pair generates the famous Milne’s inequality:

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} (a_i^2 + b_i^2) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,
\]

while the second generates the Callebaut’s inequality:

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.
\]

We observe that from the property (iii) we have the inequality

\[
2 \leq \frac{b \varphi(a, 1)}{a \varphi(b, 1)} + \frac{a \varphi(b, 1)}{b \varphi(a, 1)} \leq \frac{a}{b} + \frac{b}{a},
\]

for any \( a, b > 0 \).
If in this inequality we choose \( a = \frac{u}{v} \) and \( b = \frac{z}{w} \), then we get

\[
2 \leq \frac{zv\varphi\left(\frac{u}{v}, 1\right)}{uv\varphi\left(\frac{z}{w}, 1\right)} + \frac{uw\varphi\left(\frac{z}{w}, 1\right)}{zw\varphi\left(\frac{u}{v}, 1\right)} \leq \frac{uw}{vz} + \frac{vz}{uw},
\]

From the property (ii) we have

\[
zv\varphi\left(\frac{u}{v}, 1\right) = \frac{z}{v}\varphi(u, v) \quad \text{and} \quad uw\varphi\left(\frac{z}{w}, 1\right) = \frac{u}{w}\varphi(z, w),
\]

which give from (1.3) that

\[
2 \leq \frac{zw\varphi(u, v)}{uw\varphi(z, w)} + \frac{uw\varphi(z, w)}{zw\varphi(u, v)} \leq \frac{uw}{vz} + \frac{vz}{uw},
\]

for any \( u, v, z, w > 0 \).

Utilising the property (i) we have

\[
\varphi(z, w) = z^2w^2 \psi(z, w) \quad \text{and} \quad \varphi(u, v) = u^2v^2 \psi(u, v),
\]

which, from (1.4), produces the inequality

\[
2 \leq \frac{\varphi(u, v) \psi(z, w)}{zwuv} + \frac{\varphi(z, w) \psi(u, v)}{uwzw} \leq \frac{uw}{vz} + \frac{vz}{uw},
\]

i.e., the inequality

\[
2uvwz \leq \varphi(u, v) \psi(z, w) + \varphi(z, w) \psi(u, v) \leq u^2w^2 + v^2z^2,
\]

for any \( u, v, z, w \geq 0 \).

**Definition 1.** We say that the pair of functions \((\varphi, \psi)\) defined on \([0, \infty) \times [0, \infty)\) and with positive values is of pre-(DEC) type if it satisfies the condition (1.5).

We observe that, as shown above, if \((\varphi, \psi)\) is a (DEC) pair, then it is of pre-(DEC) type.

In this paper we obtain some inequalities for two isotonic functionals that generalize the well known DEC’s inequality. Applications for integrals and \(n\)-tuples of real numbers are provided as well.

2. **Main Results**

Let \( L \) be a linear class of real-valued functions \( g : E \to \mathbb{R} \) having the properties

(L1) \( f, g \in L \) imply \( (\alpha f + \beta g) \in L \) for all \( \alpha, \beta \in \mathbb{R} \);

(L2) \( 1 \in L \), i.e., if \( f_0(t) = 1, t \in E \) then \( f_0 \in L \).

An isotonic linear functional \( A : L \to \mathbb{R} \) is a functional satisfying

(A1) \( A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \) for all \( f, g \in L \) and \( \alpha, \beta \in \mathbb{R} \).

(A2) If \( f \in L \) and \( f \geq 0 \), then \( A(f) \geq 0 \).

The mapping \( A \) is said to be normalized if

(A3) \( A(1) = 1 \).

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. We note that common examples of such isotonic linear functionals \( A \) are given by

\[
A(g) = \int_E gd\mu \quad \text{or} \quad A(g) = \sum_{k \in E} p_k g_k,
\]
where $\mu$ is a positive measure on $E$ in the first case and $E$ is a subset of the natural numbers \( \mathbb{N} \), in the second \( (p_k \geq 0, k \in E) \).

We have the following refinement of Schwarz inequality for isotonic functionals:

**Theorem 1.** Let \((\varphi, \psi)\) be a pre-(DEC) pair. Assume that \(f, g : E \to \mathbb{R}\) are such that \(f^2, g^2, f g, \varphi(f, g), \psi(f, g) \in L\), then for all isotonic linear functionals \(A, B\) defined on \(L\),

\[
(2.1) \quad 2A(fg)B(fg) \leq A[\varphi(f, g)]B[\psi(f, g)] + A[\psi(f, g)]B[\varphi(f, g)]
\]

\[
\leq A(f^2)B(g^2) + A(g^2)B(f^2).
\]

In particular, for \(B = A\), we obtain the refinement of Schwarz’s inequality

\[
(2.2) \quad A^2(fg) \leq A[\varphi(f, g)]A[\psi(f, g)] \leq A(f^2)A(g^2).
\]

**Proof.** From the inequalities (1.5) we get for \(A\)

\[
(2.3) \quad 2f(t)g(t)f(s)g(s)
\]

\[
\leq \varphi(f(t), g(t))\psi(f(s), g(s)) + \varphi(f(s), g(s))\psi(f(t), g(t))
\]

\[
\leq f^2(t)g^2(s) + g^2(t)f^2(s),
\]

for all \(t, s \in E\).

Fix \(s \in E\), then the inequality (2.3) can be written in the order of \(L\) as

\[
(2.4) \quad 2f(s)g(s)fg \leq \psi(f(s), g(s))\varphi(f, g) + \varphi(f(s), g(s))\psi(f, g)
\]

\[
\leq g^2(s)f^2 + f^2(s)g^2.
\]

If we apply the functional \(A\) to this inequality we deduce

\[
2f(s)g(s)A(fg)
\]

\[
\leq \psi(f(s), g(s))A[\varphi(f, g)] + \varphi(f(s), g(s))A[\psi(f, g)]
\]

\[
\leq g^2(s)A(f^2) + f^2(s)A(g^2)
\]

for all \(s \in E\).

This inequality can be written in the order of \(L\) as

\[
2A(fg)fg \leq A[\varphi(f, g)]\psi(f, g) + A[\psi(f, g)]\varphi(f, g)
\]

\[
\leq A(f^2)g^2 + A(g^2)f^2
\]

and if we apply to this inequality the functional \(B\), then we get (2.1). \(\square\)

**Corollary 1.** Assume that \(f, g : E \to \mathbb{R}\) are such that \(f^2, g^2, f g, f^2 + g^2, \frac{f^2g^2}{f^2 + g^2} \in L\), then for all isotonic linear functionals \(A, B\) defined on \(L\),

\[
(2.5) \quad 2A(fg)B(fg)
\]

\[
\leq A(f^2 + g^2)B\left(\frac{f^2g^2}{f^2 + g^2}\right) + A\left(\frac{f^2g^2}{f^2 + g^2}\right)B(f^2 + g^2)
\]

\[
\leq A(f^2)B(g^2) + A(g^2)B(f^2).
\]

In particular, for \(B = A\), we obtain the Milne refinement of Schwarz’s inequality

\[
(2.6) \quad A^2(fg) \leq A(f^2 + g^2)A\left(\frac{f^2g^2}{f^2 + g^2}\right) \leq A(f^2)A(g^2).
\]
The proof follows by Theorem 1 for the (DEC) pair \( \varphi(a, b) = a^2 + b^2, \psi(a, b) = \frac{a^2b^2}{a^2+b^2}, a, b > 0 \).

**Corollary 2.** Assume that \( f, g : E \to \mathbb{R} \) are such that \( f^2, g^2, fg, f^{1+\alpha}g^{1-\alpha}, f^{1-\alpha}g^{1+\alpha} \in L \), with \( \alpha \in [0, 1] \), then for all isotonic linear functionals \( A, B \) defined on \( L \),

\[
(2.7) \quad 2A(fg)B(fg) \\
\leq A(f^{1+\alpha}g^{1-\alpha})B(f^{1-\alpha}g^{1+\alpha}) + A(f^{1-\alpha}g^{1+\alpha})B(f^{1+\alpha}g^{1-\alpha}) \\
\leq A(f^2)B(g^2) + A(g^2)B(f^2).
\]

In particular, for \( B = A \), we obtain the Callebaut refinement of Schwarz’s inequality

\[
(2.8) \quad A^2(fg) \leq A(f^{1+\alpha}g^{1-\alpha})A(f^{1-\alpha}g^{1+\alpha}) \leq A(f^2)A(g^2).
\]

The proof follows by Theorem 1 for the (DEC) pair \( \varphi(a, b) = a^{1+\alpha}b^{1-\alpha}, \psi(a, b) = a^{1-\alpha}b^{1+\alpha} \) with \( \alpha \in [0, 1] \) and \( a, b > 0 \).

A function \( M : [0, \infty) \times [0, \infty) \to [0, \infty) \) is called a mean if it satisfies the properties:

1. \( M(x, x) = x \) for all \( x \geq 0 \),
2. \( M(\lambda x, \lambda y) = \lambda M(x, y) \) for \( \lambda > 0 \) and \( x, y \in [0, \infty) \) (positive homogeneity),
3. If \( x_1 < x_2 \), then \( M(x_1, y) \leq M(x_2, y) \) and if \( y_1 < y_2 \), then \( M(x, y_1) \leq M(x, y_2) \) (monotonicity on both arguments),
4. \( M(x, y) = M(y, x) \) for all \( x, y \geq 0 \).

We call \( M : [0, \infty) \times [0, \infty) \to [0, \infty) \) a pre-mean if it satisfies the conditions 2 and 3.

**Lemma 1.** If \( M : [0, \infty) \times [0, \infty) \to [0, \infty) \) is a pre-mean on \([0, \infty)\), then the pair of functions \( \varphi(u, v) = M^2(u, v) \) and \( \psi(u, v) = \frac{u^2v^2}{M^2(u, v)} \) is of pre-(DEC) type.

**Proof.** Assume that \( \frac{w}{z} < \frac{u}{v} \) for positive \( u, v, w, z \). Then \( \frac{z}{w} > \frac{v}{u} \) and by the monotonicity of \( M \) we get

\[
M\left(1, \frac{w}{z}\right) \leq M\left(1, \frac{u}{v}\right) \quad \text{and} \quad M\left(\frac{z}{w}, 1\right) \geq M\left(\frac{u}{v}, 1\right).
\]

Since \( M \) is nonnegative, hence

\[
\left(M^2\left(1, \frac{w}{z}\right) - M^2\left(1, \frac{u}{v}\right)\right) \left(M^2\left(\frac{z}{w}, 1\right) - M^2\left(\frac{u}{v}, 1\right)\right) \leq 0.
\]

Taking into account that \( M \) is positive homogeneous, then by multiplying with \( u^2, v^2, w^2, z^2 \) and rearranging, we derive (see also [4])

\[
0 \geq (z^2M^2(u, v) - u^2M^2(z, w))(w^2M^2(u, v) - v^2M^2(z, w)).
\]

This inequality also holds if we assume that that \( \frac{w}{z} > \frac{u}{v} \) for positive \( u, v, w, z \).

Therefore, for any positive numbers \( u, v, w, z \) we have

\[
0 \geq (z^2M^2(u, v) - u^2M^2(z, w))(w^2M^2(u, v) - v^2M^2(z, w)) = z^2w^2M^4(u, v) + u^2v^2M^4(z, w) - u^2w^2M^2(z, w)M^2(u, v) - z^2v^2M^2(u, v)M^2(z, w),
\]

which gives

\[
u^2w^2M^2(z, w)M^2(u, v) + z^2v^2M^2(u, v)M^2(z, w) \geq z^2w^2M^4(u, v) + u^2v^2M^4(z, w) \]
and dividing by $M^2 (z, w) M^2 (u, v)$ we get
\[
2u^2 w^2 + 2v^2 w^2 \geq \frac{z^2 w^2 M^4 (u, v) + u^2 v^2 M^4 (z, w)}{M^2 (z, w) M^2 (u, v)}
\]
\[
= \frac{z^2 w^2 M^2 (u, v)}{M^2 (z, w)} + \frac{u^2 v^2 M^2 (z, w)}{M^2 (u, v)}
\]
\[
= M^2 (u, v) \frac{z^2 w^2}{M^2 (z, w)} + \frac{u^2 v^2}{M^2 (u, v)} M^2 (z, w)
\]
\[
= \phi (u, v) \psi (z, w) + \phi (z, w) \psi (u, v),
\]
where
\[
\phi (u, v) = M^2 (u, v) \quad \text{and} \quad \psi (u, v) = \frac{u^2 v^2}{M^2 (u, v)}. 
\]
We also have
\[
M^2 (u, v) \frac{z^2 w^2}{M^2 (z, w)} + \frac{u^2 v^2}{M^2 (u, v)} M^2 (z, w)
\geq 2M (u, v) \frac{zw}{M (z, w)} \frac{uv}{M (u, v)} M (z, w) = 2zwuv,
\]
which shows that
\[
2uwz \leq \phi (u, v) \psi (z, w) + \phi (z, w) \psi (u, v) \leq u^2 w^2 + v^2 z^2,
\]
namely $(\phi, \psi)$ is of pre-(DEC) type. 

**Corollary 3.** If $M : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a pre-mean on $[0, \infty)$ and $f^2, g^2, fg, M^2 (f, g)$, $\frac{f^2 g^2}{M^2 (f, g)} \in L$, then for all isotonic linear functionals $A, B$ defined on $L$,

\[
(2.9) \quad 2A (fg) B (fg)
\leq A \left[ M^2 (f, g) \right] B \left( \frac{f^2 g^2}{M^2 (f, g)} \right) + A \left( \frac{f^2 g^2}{M^2 (f, g)} \right) B \left[ M^2 (f, g) \right]
\leq A (f^2) B (g^2) + A (g^2) B (f^2).
\]

In particular, for $B = A$, we obtain the refinement of Schwarz’s inequality

\[
(2.10) \quad A^2 (fg) \leq A \left[ M^2 (f, g) \right] A \left( \frac{f^2 g^2}{M^2 (f, g)} \right) \leq A (f^2) A (g^2).
\]

**Remark 1.** The univariate integral version of (2.10) was obtained by S. M. Sitnik, see for instance [5]. The isotonic functional version of (2.10) was obtained by L. Nikolova and S. Varošanec in [4].

**Corollary 4.** If $f, g > 0$ with $f^2, g^2, fg, \min \{ f^2, g^2 \}, \max \{ f^2, g^2 \} \in L$, then for all isotonic linear functionals $A, B$ defined on $L$,

\[
(2.11) \quad 2A (fg) B (fg) \leq A (\min \{ f^2, g^2 \}) B (\max \{ f^2, g^2 \})
\]
\[
+ A (\max \{ f^2, g^2 \}) B (\min \{ f^2, g^2 \})
\leq A (f^2) B (g^2) + A (g^2) B (f^2).
\]

In particular, for $B = A$, we obtain the refinement of Schwarz’s inequality

\[
(2.12) \quad A^2 (fg) \leq A (\min \{ f^2, g^2 \}) A (\max \{ f^2, g^2 \}) \leq A (f^2) A (g^2).
\]
Proof. We know that for any \( a, b > 0 \) we have \( ab = \min \{ a, b \} \max \{ a, b \} \). By utilising (2.9) for \( M(f, g) = \min \{ f, g \} \) and since \( \max \{ f, g \} = \frac{f_2}{\min(f, g)} \), hence we get (2.11).

\[ \varphi \]

Remark 2. The univariate integral version of (2.12) was obtained by S. M. Sitnik, see for instance [5]. In [1] the authors also provided four different proofs of this case, but none simpler than the one of Corollary 4.

The proof follows by Theorem 1 and Lemma 1.

Theorem 2. Let \((\varphi, \psi)\) be a pre-(DEC) pair. Assume that \( f, g : E \to \mathbb{R} \) are such that \( f^2 g^2, f^{2+2\alpha} g^{2-2\alpha}, f^{2-2\alpha} g^{2+2\alpha}, \varphi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha}), \psi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha}) \in L, \alpha \in [0, 1] \), then for all isotonic linear functionals \( A, B \) defined on \( L \),

\[
2 A(f^2 g^2) B(f^2 g^2) \leq A[\varphi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] B[\psi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] + B[\varphi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] A[\psi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] \leq A(f^{2+2\alpha} g^{2-2\alpha}) B(f^{2-2\alpha} g^{2+2\alpha}) + A(f^{2-2\alpha} g^{2+2\alpha}) B(f^{2+2\alpha} g^{2-2\alpha}),
\]

and, in particular

\[
A^2(f^2 g^2) \leq A[\varphi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] A[\psi(f^{1+\alpha} g^{1-\alpha}, f^{1-\alpha} g^{1+\alpha})] \leq A(f^{2+2\alpha} g^{2-2\alpha}) A(f^{2-2\alpha} g^{2+2\alpha}).
\]

Proof. From (1.5) we get for \( u = f^{1+\alpha}(t) g^{1-\alpha}(t), v = f^{1-\alpha}(t) g^{1+\alpha}(t), z = f^{1+\alpha}(s) g^{1-\alpha}(s) \) and \( w = f^{1-\alpha}(s) g^{1+\alpha}(s) \), where \( t, s \in E \) we get

\[
2 f^{1+\alpha}(t) g^{1-\alpha}(t) f^{1-\alpha}(t) g^{1+\alpha}(t) f^{1+\alpha}(s) g^{1-\alpha}(s) f^{1-\alpha}(s) g^{1+\alpha}(s) \leq \varphi(f^{1+\alpha}(t) g^{1-\alpha}(t), f^{1-\alpha}(t) g^{1+\alpha}(t)) \times \psi(f^{1+\alpha}(s) g^{1-\alpha}(s), f^{1-\alpha}(s) g^{1+\alpha}(s)) + \varphi(f^{1+\alpha}(s) g^{1-\alpha}(s), f^{1-\alpha}(s) g^{1+\alpha}(s)) \times \psi(f^{1+\alpha}(t) g^{1-\alpha}(t), f^{1-\alpha}(t) g^{1+\alpha}(t)) \leq f^{2+2\alpha}(t) g^{2-2\alpha}(t) f^{2-2\alpha}(s) g^{2+2\alpha}(s) + f^{2-2\alpha}(t) g^{2+2\alpha}(t) f^{2+2\alpha}(s) g^{2-2\alpha}(s),
\]

namely

\[
2 f^2(t) g^2(s) f^2(s) g^2(s) \leq \varphi(f^{1+\alpha}(t) g^{1-\alpha}(t), f^{1-\alpha}(t) g^{1+\alpha}(t)) \times \psi(f^{1+\alpha}(s) g^{1-\alpha}(s), f^{1-\alpha}(s) g^{1+\alpha}(s)) + \varphi(f^{1+\alpha}(s) g^{1-\alpha}(s), f^{1-\alpha}(s) g^{1+\alpha}(s)) \times \psi(f^{1+\alpha}(t) g^{1-\alpha}(t), f^{1-\alpha}(t) g^{1+\alpha}(t)) \leq f^{2+2\alpha}(t) g^{2-2\alpha}(t) f^{2-2\alpha}(s) g^{2+2\alpha}(s) + f^{2-2\alpha}(t) g^{2+2\alpha}(t) f^{2+2\alpha}(s) g^{2-2\alpha}(s),
\]

for \( t, s \in E \).

If we apply the functional \( A \) over the variable \( t \) and the functional \( B \) over the variable \( s \), we get (2.13).
Corollary 5. Let \((\varphi, \psi)\) be a pre-(DEC) pair. Assume that \(f, g : E \to \mathbb{R}\) are such that \(fg, f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}, \varphi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right), \psi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right) \in L, \alpha \in [0,1]\), then for all isotonic linear functionals \(A, B\) defined on \(L\),

\[
2A(fg) B(fg) \leq A\left[\varphi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] B\left[\psi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] + B\left[\varphi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] A\left[\psi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] \\
\leq A\left(f^{1+\alpha}g^{1-\alpha}\right) B\left(f^{1-\alpha}g^{1+\alpha}\right) + A\left(f^{1-\alpha}g^{1+\alpha}\right) B\left(f^{1+\alpha}g^{1-\alpha}\right),
\]

and, in particular, the following refinement of Callebaut’s inequality

\[
A^2(fg) \leq A\left[\varphi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] A\left[\psi\left(f^{1+\alpha}g^{1-\alpha}, f^{-\alpha}g^{1+\alpha}\right)\right] \\
\leq A\left(f^{1+\alpha}g^{1-\alpha}\right) A\left(f^{1-\alpha}g^{1+\alpha}\right).
\]

3. Some Examples

Following the terminology of [1], a complementary mean is defined as

\[
M^*(x,y) := \frac{xy}{M(x,y)}, \quad x, \ y > 0.
\]

With this notation and if \(M : [0, \infty) \times [0, \infty) \to [0, \infty)\) is a pre-mean on \([0, \infty)\) and \(f^2, g^2, fg, M^2(f,g), [M^*(f,g)]^2 \in L\), then for all isotonic linear functionals \(A, B\) defined on \(L\), we have by (2.9) that

\[
2A(fg) B(fg) \leq A\left[M^2(f,g)\right] B\left([M^*(f,g)]^2\right) + A\left([M^*(f,g)]^2\right) B\left[M^2(f,g)\right] \\
\leq A\left(f^2\right) B\left(g^2\right) + A\left(g^2\right) B\left(f^2\right).
\]

In particular, for \(B = A\), we obtain the refinement of Schwarz’s inequality

\[
A^2(fg) \leq A\left[M^2(f,g)\right] A\left([M^*(f,g)]^2\right) \leq A\left(f^2\right) A\left(g^2\right).
\]

Consider the power means

\[
M_\alpha(x,y) := \begin{cases} 
\left(\frac{x^\alpha + y^\alpha}{2}\right)^{1/\alpha}, & -\infty < \alpha < \infty, \ \alpha \neq 0, \\
\sqrt{xy}, & \alpha = 0, \\
\min\{x,y\}, & \alpha = -\infty, \\
\max\{x,y\}, & \alpha = \infty.
\end{cases}
\]

For power means we have

\[(M_\alpha)^* = M_{-\alpha} \text{ for all } \alpha \in [-\infty, \infty].\]
From (3.1) we obtain the following generalizations of Milne’s inequality (for \( \alpha = 2 \))

\[
2A (fg) B (fg) \leq A \left[ (f^\alpha + g^\alpha)^{2/\alpha} \right] B \left[ \frac{f^2 g^2}{(f^\alpha + g^\alpha)^{2/\alpha}} \right] + A \left[ \frac{f^2 g^2}{(f^\alpha + g^\alpha)^{2/\alpha}} \right] B \left[ (f^\alpha + g^\alpha)^{2/\alpha} \right] \leq A (f^2) B (g^2) + A (g^2) B (f^2),
\]

for all \( \alpha \in [-\infty, \infty] \), provided \( f^2, g^2, fg, (f^\alpha + g^\alpha)^{2/\alpha}, \frac{f^2 g^2}{(f^\alpha + g^\alpha)^{2/\alpha}} \in L \).

In particular,

\[
A^2 (fg) \leq A \left[ (f^\alpha + g^\alpha)^{2/\alpha} \right] A \left[ \frac{f^2 g^2}{(f^\alpha + g^\alpha)^{2/\alpha}} \right] \leq A (f^2) A (g^2).
\]

for all \( \alpha \in [-\infty, \infty] \), provided \( f^2, g^2, fg, (f^\alpha + g^\alpha)^{2/\alpha}, \frac{f^2 g^2}{(f^\alpha + g^\alpha)^{2/\alpha}} \in L \).

The interested reader may state the corresponding inequalities for \( \alpha = 1, \alpha = 1/2 \)
or \( \alpha = 4 \).

Also, consider Radó means (or \( L_\beta \)-means)

\[
R_\beta (x, y) := \begin{cases} 
\left( \frac{x^{\beta+1} - y^{\beta+1}}{(\beta+1)(x-y)} \right)^{1/\beta}, & -\infty < \beta < \infty, \beta \neq 0, -1, \\
\ln \frac{y-x}{y-x} = L (x, y), \text{ logarithmic mean, } \beta = -1, \\
\frac{1}{e} \left( \frac{y^\beta}{x^\beta} \right)^{1/\beta} = I (x, y), \text{ identric mean, } \beta = 0, \\
\min \{x, y\}, \alpha = -\infty, \\
\max \{x, y\}, \alpha = \infty.
\end{cases}
\]

The complementary mean is

\[
R_\beta^* (x, y) = \frac{1}{R_\beta (x^{-1}, y^{-1})}, \text{ for all } \beta \in [-\infty, \infty].
\]

From (3.1) we get

\[
2A (fg) B (fg) \leq A \left[ R^2_\beta (f, g) \right] B \left[ R^{-2}_\beta (f^{-1}, g^{-1}) \right] + A \left[ R^{-2}_\beta (f^{-1}, g^{-1}) \right] B \left[ R^2_\beta (f, g) \right] \leq A (f^2) B (g^2) + A (g^2) B (f^2)
\]

for all \( \beta \in [-\infty, \infty] \), provided \( f^2, g^2, fg, R^2_\beta (f, g), R^{-2}_\beta (f^{-1}, g^{-1}) \in L \).

In particular,

\[
A^2 (fg) \leq A \left[ R^2_\beta (f, g) \right] A \left[ R^{-2}_\beta (f^{-1}, g^{-1}) \right] \leq A (f^2) A (g^2)
\]

for all \( \beta \in [-\infty, \infty] \), provided \( f^2, g^2, fg, R^2_\beta (f, g), R^{-2}_\beta (f^{-1}, g^{-1}) \in L \).

The particular cases \( \beta = -1 \) and \( \beta = 0 \) are of special interest. Their integral version were stated in [1].
As pointed out in [4], one can also consider Seiffert type means defined by

\[ M(x, y) := \frac{|x - y|}{2f\left(\frac{x-y}{x+y}\right)} \quad \text{with} \quad \lim_{z \to 0} \frac{f(z)}{z} = 1. \]

Witkowski, [8], proved that \( M(x, y) \) is increasing on every argument if and only if the function \( (1 + z)f(z)/z \) increases and the function \( (1 - z)f(z)/z \) decreases on \( z \in (0, 1] \). It is checked that functions \( f(z) = \sin z \), \( f(z) = \tan z \), \( f(z) = \sinh z \), \( f(z) = \tanh z \), \( f(z) = \ln(1 + z) \), \( f(z) = \arcsin z \), \( f(z) = \arctan z \), \( f(z) = \text{arsinh} z \), \( f(z) = \text{artanh} z \) satisfy this condition and so we get examples of means which are homogeneous, increasing on every argument (and even symmetric as Witkowski showed). If \( f(z) = \arcsin z \) this is the mean \( P(x, y) \), and if \( f(z) = \arctan z \) this is the mean \( T(x, y) \), introduced by Seiffert in 1987, [6] and 1995, [7], respectively.

References


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