SOME INEQUALITIES RELATED TO JENSEN RESULT FOR CONVEX FUNCTIONS ON FINITE INTERVALS

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Abstract. In this paper we obtain some new discrete inequalities for univariate convex functions and provide some natural applications for logarithm and power functions.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\ldots,p_n)$ is a probability sequence and $\mathbf{x}=(x_1,\ldots,x_n)\in C^n$, then

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leq\sum_{i=1}^{n}p_if\left(x_i\right), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [18] obtained the following inequality for convex functions of a real variable $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$ and the finite sequences $x_k\in[m,M]$, and $p_k\geq 0$, $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$,

$$f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\leq f\left(m\right)+f\left(M\right)-\sum_{k=1}^{n}p_kf\left(x_k\right) \tag{1.2}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [16], [19], [25], in relation with majorization theory [23], for convex functions of selfadjoint operators in Hilbert spaces [15], [17], [20], [21] and for operator convex functions in Hilbert spaces [22] and [25].

In this paper we obtain some new discrete inequalities for univariate convex functions that can be seen as counterparts of Mercer's result and provide some natural applications for logarithm and power functions.

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2. Main Result

First of all, we recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} (2.1)\qquad & n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right] \\ & \le \frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right) \\ & \le n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right], \end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

Theorem 1. *Let $f : [m, M] \to \mathbb{R}$ be a convex function, $x_k \in [m, M]$, $p_k \ge 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^{n} p_k = 1$. Then*

$$\begin{aligned} (2.2)\quad & f\left(m+M-\sum_{k=1}^{n}p_kx_k\right) \\ & \ge f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\min_{i\in\{1,\dots,n\}}\{p_i\}\left(\sum_{k=1}^{n}f(x_k)-nf\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)\right) \\ & \ge f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\left[\sum_{k=1}^{n}p_kf(x_k)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \ge 2f\left(\frac{m+M}{2}\right)-\sum_{k=1}^{n}p_kf(x_k). \end{aligned}$$

Proof. Since $m + M - \sum_{k=1}^{n} p_k x_k$, $\frac{m+M}{2} \in [m, M]$, then by the convexity of f on $[m, M]$ we have

$$\begin{aligned} & \frac{1}{2}\left[f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)+f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \ge f\left(\frac{m+M-\sum_{k=1}^{n}p_kx_k+\sum_{k=1}^{n}p_kx_k}{2}\right)=f\left(\frac{m+M}{2}\right), \end{aligned}$$

namely

$$(2.3)\qquad f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)+f\left(\sum_{k=1}^{n}p_kx_k\right)\ge 2f\left(\frac{m+M}{2}\right).$$

By subtracting in both sides of (2.3) the same quantity $\sum_{k=1}^{n} p_k f(x_k)$ we get

$$\begin{aligned} (2.4)\qquad & f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\left[\sum_{k=1}^{n}p_kf(x_k)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \ge 2f\left(\frac{m+M}{2}\right)-\sum_{k=1}^{n}p_kf(x_k). \end{aligned}$$

By using the first inequality in (2.1) we have

$$\begin{aligned}&-\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\ &\leq-\min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right),\end{aligned}$$

which implies that

$$\begin{aligned}&f\left(m+M-\sum_{k=1}^{n} p_k x_k\right)-\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\ &\leq f\left(m+M-\sum_{k=1}^{n} p_k x_k\right) \\ &-\min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right). \end{aligned} \tag{2.5}$$

By making use of (2.4) and (2.5) we get the second and third inequalities in (2.2). □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}&\frac{1}{2}\left[f\left(m+M-\sum_{k=1}^{n} p_k x_k\right)+\sum_{k=1}^{n} p_k f\left(x_k\right)\right]-f\left(\frac{m+M}{2}\right) \\ &\geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\ &-\frac{1}{2} \min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right) \\ &\geq 0\end{aligned} \tag{2.6}$$

for all $x_k \in [m, M]$, $p_k \geq 0$, *for* $k \in \{1, ..., n\}$ *and such that* $\sum_{k=1}^{n} p_k = 1$.

When more conditions like twice differentiability is imposed, then we can state the following alternative result:

Theorem 2. *Let $f:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on (m,M), $x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^{n}p_k=1$. Then*

$$
\begin{aligned}
(2.7)\qquad & f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\
&\geq f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\
&-\frac{1}{2}\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\inf_{s\in[0,1]}f''\left((1-s)\sum_{i=1}^{n}p_ix_i+sx_k\right)\\
&\geq f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\left[\sum_{k=1}^{n}p_kf(x_k)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right]\\
&\geq 2f\left(\frac{m+M}{2}\right)-\sum_{k=1}^{n}p_kf(x_k).
\end{aligned}
$$

Proof. Let $f:I\to\mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n\geq 1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct x, $a\in\mathring{I}$ we have the following Taylor expansion with integral remainder

$$
\begin{aligned}
(2.8)\qquad f(x)=&\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^k\\
&+\frac{1}{n!}(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds.
\end{aligned}
$$

For $n=1$ we get

$$
(2.9)\qquad f(x)=f(a)+(x-a)f'(a)+(x-a)^2\int_0^1 f''((1-s)a+sx)(1-s)\,ds
$$

or each distinct x, $a\in\mathring{I}$.

Now if we apply the identity (2.9) for the interval $[m,M]$ we get

$$
\begin{aligned}
(2.10)\qquad f(x_k)=&f\left(\sum_{i=1}^{n}p_ix_i\right)+\left(x_k-\sum_{i=1}^{n}p_ix_i\right)f'\left(\sum_{i=1}^{n}p_ix_i\right)\\
&+\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\int_0^1 f''\left((1-s)\sum_{i=1}^{n}p_ix_i+sx_k\right)(1-s)\,ds
\end{aligned}
$$

for $k\in\{1,...,n\}$.

If we multiply with $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\begin{aligned} \sum_{k=1}^{n} p_k f(x_k) &= f\left(\sum_{i=1}^{n} p_i x_i\right) \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i\right) f'\left(\sum_{i=1}^{n} p_i x_i\right) \\ &+ \sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \int_0^1 f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right)(1-s)\,ds \\ &= f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &+ \sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \int_0^1 f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right)(1-s)\,ds. \end{aligned} \tag{2.11}$$

This implies that

$$\begin{aligned} &\sum_{k=1}^{n} p_k f(x_k) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &= \sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \int_0^1 f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right)(1-s)\,ds. \end{aligned} \tag{2.12}$$

Observe that

$$\begin{aligned} &\int_0^1 f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right)(1-s)\,ds \\ &\geq \inf_{s\in[0,1]} f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right) \int_0^1 (1-s)\,ds \\ &= \frac{1}{2} \inf_{s\in[0,1]} f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right) \end{aligned}$$

and by (2.12) we get

$$\begin{aligned} &\sum_{k=1}^{n} p_k f(x_k) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\geq \frac{1}{2} \sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \inf_{s\in[0,1]} f''\left((1-s)\sum_{i=1}^{n} p_i x_i + s x_k\right). \end{aligned}$$

Therefore

$$\begin{aligned}
&f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\left[\sum_{k=1}^{n}p_kf\left(x_k\right)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right]\\
&\leq f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\
&-\frac{1}{2}\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\inf_{s\in[0,1]}f''\left((1-s)\sum_{i=1}^{n}p_ix_i+sx_k\right)
\end{aligned}$$

and by making use of (2.4) and (2.5) we get the second and third inequalities in (2.7). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
(2.13)\qquad &\frac{1}{2}\left[f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)+\sum_{k=1}^{n}p_kf\left(x_k\right)\right]-f\left(\frac{m+M}{2}\right)\\
&\geq\frac{1}{2}\left[\sum_{k=1}^{n}p_kf\left(x_k\right)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right]\\
&-\frac{1}{4}\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\inf_{s\in[0,1]}f''\left((1-s)\sum_{i=1}^{n}p_ix_i+sx_k\right)\\
&\geq 0
\end{aligned}$$

for all $x_k\in[m,M]$, $p_k\geq 0$, *for* $k\in\{1,...,n\}$ *and such that* $\sum_{k=1}^{n}p_k=1$.

Corollary 3. *With the assumptions of Theorem 2 and if there exists a* $\gamma>0$ *such that* $f''(t)\geq\gamma$ *for* $t\in(m,M)$, *then*

$$\begin{aligned}
(2.14)\qquad &f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)\\
&\geq f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\frac{1}{2}\gamma\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\\
&\geq f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)-\left[\sum_{k=1}^{n}p_kf\left(x_k\right)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right]\\
&\geq 2f\left(\frac{m+M}{2}\right)-\sum_{k=1}^{n}p_kf\left(x_k\right)
\end{aligned}$$

and

$$\begin{aligned}
(2.15)\qquad &\frac{1}{2}\left[f\left(m+M-\sum_{k=1}^{n}p_kx_k\right)+\sum_{k=1}^{n}p_kf\left(x_k\right)\right]-f\left(\frac{m+M}{2}\right)\\
&\geq\frac{1}{2}\left[\sum_{k=1}^{n}p_kf\left(x_k\right)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right]-\frac{1}{4}\gamma\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\\
&\geq 0.
\end{aligned}$$

Remark 1. *Since*

$$\sum_{k=1}^{n} p_k \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 = \sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{k=1}^{n} p_k x_k \right)^2 = \sigma^2 \left(\mathbf{p}, \mathbf{x} \right)$$

is the variance of $\mathbf{x}$, *one can write the inequalities (2.14) and (2.15) in terms of* $\sigma^2 \left(\mathbf{p}, \mathbf{x} \right)$.

3. Applications

We consider the convex function $f : [m, M] \subset (0, \infty) \to \mathbb{R},\ f(x) = -\ln x$, $x_k \in [m, M]$, $p_k \geq 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^{n} p_k = 1$. By using (2.6) we get

$$\begin{aligned} &\ln\left(\frac{m+M}{2}\right) - \frac{1}{2}\left[\ln\left(m + M - \sum_{k=1}^{n} p_k x_k\right) + \sum_{k=1}^{n} p_k \ln(x_k)\right] \\ &\geq \frac{1}{2} \min_{i \in \{1,\ldots,n\}} \{p_i\} \left(\sum_{k=1}^{n} \ln(x_k) - n \ln\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right) \\ &- \frac{1}{2}\left[\sum_{k=1}^{n} p_k \ln(x_k) - \ln\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\ &\geq 0, \end{aligned} \tag{3.1}$$

which is equivalent to

$$\begin{aligned} &\frac{\frac{m+M}{2}}{\left(m + M - \sum_{k=1}^{n} p_k x_k\right)^{1/2} \left(\prod x_k^{p_k}\right)^{1/2}} \\ &\geq \left(\frac{\left[\frac{\prod_{k=1}^{n} x_k}{\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^n}\right]^{\min_{i\in\{1,\ldots,n\}}\{p_i\}}}{\frac{\prod_{k=1}^{n} x_k^{p_k}}{\sum_{k=1}^{n} p_k x_k}} \right)^{1/2} \geq 1. \end{aligned} \tag{3.2}$$

From (2.15) we get

$$\begin{aligned} &\ln\left(\frac{m+M}{2}\right) - \frac{1}{2}\left[\ln\left(m + M - \sum_{k=1}^{n} p_k x_k\right) + \sum_{k=1}^{n} p_k \ln(x_k)\right] \\ &\geq \frac{1}{2}\left[\ln\left(\sum_{k=1}^{n} p_k x_k\right) - \ln\left(\prod_{k=1}^{n} x_k^{p_k}\right)\right] - \frac{1}{4M^2}\sigma^2(\mathbf{p}, \mathbf{x}) \\ &\geq 0, \end{aligned} \tag{3.3}$$

which is equivalent to

$$\begin{aligned} &\frac{\frac{m+M}{2}}{\left(m + M - \sum_{k=1}^{n} p_k x_k\right)^{1/2} \left(\prod x_k^{p_k}\right)^{1/2}} \\ &\geq \left(\frac{\frac{\sum_{k=1}^{n} p_k x_k}{\prod_{k=1}^{n} x_k^{p_k}}}{\exp\left(\frac{1}{2M^2}\sigma^2(\mathbf{p}, \mathbf{x})\right)} \right)^{1/2} \geq 1. \end{aligned} \tag{3.4}$$

Consider the function $f:[m,M]\subset(0,\infty)\to\mathbb{R},\ f(x)=x^r$ which is convex for $r\in(-\infty,0)\cup[1,\infty)$ and concave for $r\in(0,1)$. If we use (2.6) and (2.15), then we get for $r\in(-\infty,0)\cup[1,\infty)$ that

$$\begin{aligned}
&\frac{1}{2}\left[\left(m+M-\sum_{k=1}^{n}p_kx_k\right)^r+\sum_{k=1}^{n}p_kx_k^r\right]-\left(\frac{m+M}{2}\right)^r \\
&\geq\frac{1}{2}\left[\sum_{k=1}^{n}p_kx_k^r-\left(\sum_{k=1}^{n}p_kx_k\right)^r\right] \\
&-\frac{1}{2}\min_{i\in\{1,...,n\}}\{p_i\}\left(\sum_{k=1}^{n}x_k^r-n\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^r\right) \\
&\geq 0
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[\left(m+M-\sum_{k=1}^{n}p_kx_k\right)^r+\sum_{k=1}^{n}p_kx_k^r\right]-\left(\frac{m+M}{2}\right)^r \\
&\geq\frac{1}{2}\left[\sum_{k=1}^{n}p_kx_k^r-\left(\sum_{k=1}^{n}p_kx_k\right)^r\right]-\frac{1}{4}\gamma_r\sigma^2(\mathbf{p},\mathbf{x}) \\
&\geq 0,
\end{aligned}\tag{3.6}$$

where

$$\gamma_r:=r(r-1)\times\begin{cases} m^{r-2}\text{ if } r\geq 2, \\ \\ M^{r-2}\text{if } r\in(-\infty,0)\cup[1,2),\end{cases}$$

$x_k\in[m,M]$, $p_k\geq 0$, for $k\in\{1,...,n\}$ and such that $\sum_{k=1}^{n}p_k=1$.

The interested reader may apply the above inequalities for other convex functions such as $f(x)=x\ln x$, $x\in[m,M]\subset(0,\infty)$ or $f(x)=\exp(\alpha x)$, $\alpha\in\mathbb{R}$ and $x\in[m,M]$.

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