Abstract. In this paper we obtain several reverse of Levin-Stečkin inequality. We also extend this result for the more general case in which the symmetrical transform is requested to be monotonic non-increasing on the half interval $[0, \frac{1}{2}]$, namely if $p : [0, 1] \rightarrow \mathbb{R}$ is symmetric in $[0, 1]$ and monotonic non-decreasing on $[0, \frac{1}{2}]$ and $f : [0, 1] \rightarrow \mathbb{R}$ is integrable and such that $\tilde{f}$, the symmetrical transform of $f$, is monotonic non-increasing on $[0, \frac{1}{2}]$, then

$$0 \leq \int_0^1 p(t)dt \int_0^1 f(t)dt - \int_0^1 p(t)f(t)dt \leq \frac{1}{4} \left( p \left( \frac{1}{2} \right) - p(0) \right) \left[ \frac{f(0) + f(1)}{2} - \tilde{f} \left( \frac{1}{2} \right) \right].$$

Several other similar inequalities for either $p$ or $f$ is differentiable, are also provided.

1. Introduction

For two Lebesgue integrable functions $h$, $g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$C(h, g) := \frac{1}{b - a} \int_a^b h(t)g(t)dt - \frac{1}{(b - a)^2} \int_a^b h(t)dt \int_a^b g(t)dt.$$

In 1935, Grüss [5] showed that

$$|C(h, g)| \leq \frac{1}{4} (M - m) (N - n),$$

provided that there exists the real numbers $m, M, n, N$ such that

$$m \leq h(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [1], states that

$$|C(h, g)| \leq \frac{1}{12} \|h'\|_\infty \|g'\|_\infty (b - a)^2,$$

provided that $h', g'$ exist and are continuous on $[a, b]$ and $\|h'\|_\infty = \sup_{t \in [a, b]} |h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $h, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $h', g' \in L_\infty [a, b]$ while $\|h'\|_\infty = \esssup_{t \in [a, b]} |h'(t)|$.

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A mixture between Grüss’ result (1.2) and Čebyšev’s one (1.4) is the following inequality obtained by Ostrowski in 1970, [9]:

\[ |C(h, g)| \leq \frac{1}{8} (b - a) (M - m) \|g'\|_\infty, \]

provided that \( h \) is Lebesgue integrable and satisfies (1.3) while \( g \) is absolutely continuous and \( g' \in L_\infty[a, b] \). The constant \( \frac{1}{8} \) is best possible in (1.5).

The case of euclidean norms of the derivative was considered by A. Lupăş in [6] in which he proved that

\[ |C(h, g)| \leq \frac{1}{\pi^2} \|h'\|_2 \|g'\|_2 (b - a), \]

provided that \( h, g \) are absolutely continuous and \( h', g' \in L_2[a, b] \). The constant \( \frac{1}{\pi^2} \) is the best possible.

The following result is known in the literature as Levin-Stečkin’s inequality [10]:

**Theorem 1.** If the function \( p : [0, 1] \to \mathbb{R} \) is symmetric, namely \( p(1 - t) = p(t) \) for \( t \in [0, 1] \) and non-decreasing (non-increasing) on \( [0, 1/2] \), then for every convex function \( g \) on \( [0, 1] \),

\[
\text{(LS)} \quad \int_0^1 p(t) g(t) \, dt \leq (\geq) \int_0^1 p(t) \, dt \int_0^1 g(t) \, dt.
\]

If the function \( g \) is concave on \( [0, 1] \), then the signs of inequalities reverse in (LS).

For some recent results related to Levin-Stečkin’s inequality, see [7], [8] and [11].

In this paper we obtain several reverse inequalities for (LS). We also extend this result and its reverses for the more general case in which the symmetrical transform of \( g \) is requested to be monotonic on the half interval \( [0, 1/2] \). Some examples of interest are also provided.

### 2. REVERSE INEQUALITIES

We have the following natural reverse of Levin-Stečkin’s inequality, see also [4, Corollary 1.4]:

**Theorem 2.** Assume that \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-decreasing on \( [0, 1/2] \) and \( f : [0, 1] \to \mathbb{R} \) is convex, then

\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
\leq \frac{1}{4} \left( p\left( \frac{1}{2} \right) - p(0) \right) \left( \frac{f(0) + f(1)}{2} - f\left( \frac{1}{2} \right) \right)
\leq \frac{1}{16} \left( p\left( \frac{1}{2} \right) - p(0) \right) \left[ f_+''(1) - f_+''(0) \right].
\]

**Proof.** Since \( p \) is symmetric on \( [0, 1] \), then

\[
\int_0^1 p(t) \frac{f(t) + f(1 - t)}{2} \, dt = \frac{1}{2} \left[ \int_0^1 p(t) f(t) \, dt + \int_0^1 p(t) f(1 - t) \, dt \right]
= \frac{1}{2} \left[ \int_0^1 p(t) f(t) \, dt + \int_0^1 p(1 - t) f(1 - t) \, dt \right].
\]
By changing the variable $1 - t = s$, $s \in [0, 1]$ we have

$$
\int_0^1 p(1 - t) f(1 - t) \, dt = \int_0^1 p(s) f(s) \, ds
$$

and then

$$
\int_0^1 p(t) \frac{f(t) + f(1 - t)}{2} \, dt = \int_0^1 p(t) f(t) \, dt.
$$

Also

$$
\int_0^1 f(t) + f(1 - t) \, dt = \int_0^1 f(t) \, dt.
$$

Therefore

$$
(2.2) \quad 0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
$$

$$
= \int_0^1 p(t) \, dt \int_0^1 \tilde{f}(t) \, dt - \int_0^1 p(t) \tilde{f}(t) \, dt,
$$

where

$$
\tilde{f}(t) := \frac{f(t) + f(1 - t)}{2}, \ t \in [0, 1]
$$

is the symmetrized transform of $f$.

Since $f$ is convex, then $\tilde{f}$ is symmetric and convex which implies that

$$
f \left( \frac{1}{2} \right) = \tilde{f} \left( \frac{1}{2} \right) \leq \tilde{f}(t) \leq \tilde{f}(1) = \frac{f(0) + f(1)}{2}, \ t \in [0, 1].
$$

Also $p(0) \leq p(t) \leq p \left( \frac{1}{2} \right), \ t \in [0, 1]$ and by Grüss’ inequality we get

$$
0 \leq \int_0^1 p(t) \, dt \int_0^1 \tilde{f}(t) \, dt - \int_0^1 p(t) \tilde{f}(t) \, dt
$$

$$
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{f(0) + f(1)}{2} - \tilde{f} \left( \frac{1}{2} \right) \right]
$$

and the second inequality (2.1) is thus proved.

For the last part, we use the inequality for convex functions $h : [a, b] \to \mathbb{R}$, see for instance [2] and [3] where the Hermite-Hadamard reverse inequalities were considered,

$$
0 \leq \frac{h(a) + h(b)}{2} - h \left( \frac{a + b}{2} \right) \leq \frac{1}{4} (b - a) \left[ h'(b) - h'(a) \right],
$$

in which the constant $\frac{1}{4}$ is best possible.

Further, we have:

**Theorem 3.** Assume that $p : [0, 1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0, 1/2]$ and $f : [0, 1] \to \mathbb{R}$ is convex.
(i) If \( p \) is differentiable on \((0, 1)\) with \( p' \in L_\infty(0, 1)\), then
\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \\
\leq \frac{1}{16} \left| p \left( \frac{1}{2} \right) - p(0) \right| \sup_{t \in (0, 1)} |f'(t) - f'(1 - t)|.
\]

(ii) If \( f \) is differentiable on \((0, 1)\), then
\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \\
\leq \frac{1}{24} \|p'\|_{\infty,(0,1)} \sup_{t \in (0,1)} |f'(t) - f'(1 - t)|.
\]

(iii) If \( p \) and \( f \) are differentiable on \((0, 1)\) with \( p' \in L_\infty(0, 1)\), then
\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \\
\leq \frac{1}{2\pi^2} \|p'\|_2 \left( \int_0^1 |f'(t) - f'(1 - t)|^2 dt \right)^{1/2}.
\]

Proof. We consider the function \( g : [0, 1] \to \mathbb{R}, \)
\[
g(t) = \frac{f(t) + f(1 - t)}{2}.
\]
If \( f \) is differentiable on \((0, 1)\), then
\[
g'(t) = \frac{f'(t) - f'(1 - t)}{2}, \quad t \in (0, 1).
\]

By applying Čebyšev, Ostrowski and Lupuș inequalities for the appropriate choices of \( h \) and \( g \) we derive the corresponding inequalities (2.3)-(2.5).

\[ \square \]

Corollary 1. Assume that \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-decreasing on \([0, 1/2]\) and \( f : [0, 1] \to \mathbb{R} \) is differentiable convex with the derivative \( f' \) \( L\)-Lipschitzian on \((0, 1)\), namely
\[
|f'(u) - f'(u)| \leq L |u - v| \quad \text{for all } u, v \in (0, 1).
\]

Then
\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \leq \frac{1}{16} \left[ p \left( \frac{1}{2} \right) - p(0) \right].
\]

If \( p \) is differentiable with \( p' \in L_\infty(0, 1)\), then
\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \leq \frac{1}{24} L \|p'\|_{\infty,(0,1)}.
\]
and if $p' \in L_2 [a, b]$, then

$$0 \leq \int_0^1 p(t) dt \int_0^1 f(t) dt - \int_0^1 p(t) f(t) dt \leq \frac{\sqrt{3}}{6\pi^2} L \|p'\|_2. \quad (2.9)$$

**Proof.** Since $f'$ is $L$-Lipschitzian on $(0, 1)$, then

$$|f'(t) - f'(1 - t)| \leq L |1 - 2t| = 2L \left| t - \frac{1}{2} \right| \leq L$$

for all $t \in (0, 1)$.

Therefore

$$\sup_{t \in (0, 1)} |f'(t) - f'(1 - t)| \leq L,$$

and

$$\int_0^1 |f'(t) - f'(1 - t)|^2 dt \leq 4L^2 \int_0^1 \left| t - \frac{1}{2} \right|^2 dt = \frac{1}{3} L^2.$$

By utilising Theorem 3 we derive the desired results. \hfill \Box

### 3. Some Extensions

For a function $f : [a, b] \to \mathbb{C}$ we consider the symmetrical transform of $f$ on the interval $[a, b]$, denoted by $\hat{f}_{[a,b]}$, or simply $\hat{f}$, when the interval $[a, b]$ is implicit, as defined by

$$\hat{f}(t) := \frac{1}{2} \left[ f(t) + f(a + b - t) \right], \quad t \in [a, b]. \quad (3.1)$$

We recall that the pair of functions $(f, g)$ defined on $[a, b]$ are called synchronous (asynchronous) on $[a, b]$ if

$$\max_{t \in [a, b]} (f(t) - f(s)) (g(t) - g(s)) \geq 0 \quad (\leq 0) \quad (3.2)$$

for any $t, s \in [a, b]$. It is clear that if both functions $(f, g)$ are monotonic nondecreasing (nonincreasing) on $[a, b]$ then they are synchronous on $[a, b]$. There are also functions that change monotonicity on $[a, b]$, but as a pair they are still synchronous. For instance if $a < 0 < b$ and $f, g : [a, b] \to \mathbb{R}$, $f(t) = t^2$ and $g(t) = t^4$, then

$$(f(t) - f(s)) (g(t) - g(s)) = (t^2 - s^2) (t^4 - s^4) = (t^2 - s^2)^2 (t^2 + s^2) \geq 0$$

for any $t, s \in [a, b]$, which show that $(f, g)$ is synchronous.

We can introduce the following concept as well:

**Definition 1.** We say that the pair of functions $(f, g)$ defined on $[a, b]$ is called symmetrized synchronous (asynchronous) on $[a, b]$ if the pair of symmetrized transforms $(\hat{f}, \hat{g})$ is synchronous (asynchronous) on $[a, b]$, namely

$$(\hat{f}(t) - \hat{f}(s)) (\hat{g}(t) - \hat{g}(s)) \geq 0 \quad (\leq 0) \quad (3.3)$$

for any $t, s \in [a, b]$.

Now, assume that the function $x : [a, b] \to I$, where $I$ is an interval of real numbers, and $(\phi, \psi)$ is a pair of synchronous (asynchronous) functions defined on the interval $I$. Consider the functions $f, g : [a, b] \to \mathbb{R}$ defined by $f = \phi \circ x$ and $g = \psi \circ x$.

Then the functions $f$ and $g$ are symmetrical on $[a, b]$ and $\hat{f} = \phi \circ x$ and $\hat{g} = \psi \circ x$. Since $(\phi, \psi)$ is a pair of synchronous (asynchronous) functions, it follows that $(\hat{f}, \hat{g})$ is synchronous (asynchronous) on $[a, b]$, namely the pair of functions $(f, g)$ defined
on \([a, b]\) is symmetrized synchronous (asynchronous) on \([a, b]\). Therefore, we can give many example of symmetrized synchronous (asynchronous) functions on \([a, b]\). For instance, if \((\phi, \psi)\) is a pair of synchronous (asynchronous) functions defined on the interval \([0, \infty)\), then the functions \(f, g : [a, b] \rightarrow \mathbb{R}\) defined by 
\[ f(t) = \phi \left( |t - \frac{a + b}{2}|^p \right) \]
and 
\[ g(t) = \psi \left( |t - \frac{a + b}{2}|^p \right) \]
with \(p > 0\) are symmetrized synchronous (asynchronous) on \([a, b]\).

One of the most important results for synchronous (asynchronous) and integrable functions \(f, g\) on \([a, b]\) is the well-known Čebyšev’s inequality:

\[ \frac{1}{b - a} \int_a^b f(t) g(t) \, dt \geq \left( \frac{1}{b - a} \int_a^b f(t) \, dt \right) \left( \frac{1}{b - a} \int_a^b g(t) \, dt \right). \]

We have the following Čebyšev’s type result:

**Lemma 1.** Assume that the pair of integrable functions \((f, g)\) defined on \([a, b]\) is symmetrized synchronous (asynchronous) on \([a, b]\), then

\[ \frac{1}{b - a} \int_a^b \tilde{f}(t) g(t) \, dt \geq \left( \frac{1}{b - a} \int_a^b \tilde{f}(t) \, dt \right) \left( \frac{1}{b - a} \int_a^b g(t) \, dt \right). \]

**Proof.** Since \((\tilde{f}, \tilde{g})\) is synchronous (asynchronous) on \([a, b]\), then by Čebyšev’s inequality (3.4) we have

\[ \frac{1}{b - a} \int_a^b \tilde{f}(t) \tilde{g}(t) \, dt \geq \left( \frac{1}{b - a} \int_a^b \tilde{f}(t) \, dt \right) \left( \frac{1}{b - a} \int_a^b \tilde{g}(t) \, dt \right). \]

Observe that, by the change of variable \(s = a + b - t, \ t \in [a, b]\) we have

\[
\int_a^b \tilde{f}(t) \, dt = \frac{1}{2} \int_a^b [f(t) + f(a + b - t)] \\
= \frac{1}{2} \left[ \int_a^b f(t) \, dt + \int_a^b f(a + b - t) \, dt \right] = \int_a^b f(t) \, dt,
\]

\[
\int_a^b \tilde{g}(t) \, dt = \int_a^b g(t) \, dt.
\]
and

\[(3.7) \quad \int_a^b \hat{f}(t) \hat{g}(t) \, dt = \frac{1}{4} \int_a^b \left[ f(t) + f(a + b - t) \right] \left[ g(t) + g(a + b - t) \right] \, dt \]

\[= \frac{1}{4} \int_a^b \left[ f(t) g(t) + f(a + b - t) g(t) \right. \]
\[\left. + f(t) g(a + b - t) + f(a + b - t) g(a + b - t) \right] \, dt \]

\[= \frac{1}{4} \left[ \int_a^b f(t) g(t) \, dt + \int_a^b f(a + b - t) g(t) \, dt \right. \]
\[\left. + \int_a^b f(a + b - t) g(t) \, dt + \int_a^b f(t) g(t) \, dt \right] \]

\[= \frac{1}{2} \left[ \int_a^b f(t) g(t) \, dt + \int_a^b f(a + b - t) g(t) \, dt \right] \]

\[= \int_a^b \hat{f}(t) \hat{g}(t) \, dt \]

since,

\[\int_a^b f(a + b - t) g(t) \, dt = \int_a^b f(s) g(a + b - s) \, ds \]

and

\[\int_a^b f(a + b - t) g(a + b - t) \, dt = \int_a^b f(s) g(s) \, ds. \]

By making use of (3.6) we obtain the desired result (3.5). \qed

**Corollary 2.** Assume that the pair of integrable functions \((p, g)\) defined on \([a, b]\) is symmetrized synchronous (asynchronous) on \([a, b]\) and \(p\) is symmetric, then

\[(3.8) \quad \frac{b - a}{b - a} \int_a^b p(t) g(t) \, dt \geq (\leq) \frac{1}{b - a} \int_a^b p(t) dt \frac{1}{b - a} \int_a^b g(t) \, dt. \]

We have the following generalization of Levin-Stečkin inequality.

**Theorem 4.** Assume that \(p : [0, 1] \rightarrow \mathbb{R}\) is symmetric in \([0, 1]\) and monotonic non-decreasing (non-increasing) on \([0, \frac{1}{2}]\) and \(f : [0, 1] \rightarrow \mathbb{R}\) integrable and such that \(f\) is monotonic non-increasing on \([0, \frac{1}{2}]\); then we have

\[(3.9) \quad \int_0^1 p(t) f(t) \, dt \leq (\geq) \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt. \]

If \(f\) is monotonic non-decreasing on \([0, \frac{1}{2}]\), then the inequalities reverse in (3.9).
Proof. Observe that

\begin{equation}
\int_0^1 p(t) f(t) \, dt = \int_0^{1/2} p(t) f(t) \, dt + \int_{1/2}^1 p(t) f(t) \, dt
\end{equation}

\begin{align*}
&= \int_0^{1/2} p(t) f(t) \, dt + \int_{1/2}^1 p(1-t) f(t) \, dt \\
&= \int_0^{1/2} p(t) f(t) \, dt + \int_0^{1/2} p(s) f(1-s) \, ds \\
&= \int_0^{1/2} p(t) [f(t) + f(1-t)] \, dt = 2 \int_0^{1/2} p(t) \bar{f}(t) \, dt.
\end{align*}

Using Čebyšev’s inequality for functions with opposite monotonicity on \([0, \frac{1}{2}]\), we have

\begin{equation}
2 \int_0^{1/2} p(t) \bar{f}(t) \, dt \leq \int_0^{1/2} p(t) \, dt \cdot 2 \int_0^{1/2} \bar{f}(t) \, dt
\end{equation}

\begin{align*}
&= \int_0^{1/2} p(t) \, dt \int_0^{1/2} \bar{f}(t) \, dt = \int_0^{1/2} p(t) \, dt \int_0^{1/2} f(t) \, dt.
\end{align*}

By making use of (3.10) and (3.11) we deduce the desired result (3.9). □

Remark 1. We observe that if \( f : [0, 1] \to \mathbb{R} \) is convex on \([0, 1]\), then \( \bar{f} \) is monotonic non-decreasing on \([0, \frac{1}{2}]\) and by (3.9) we get the Levin-Stečkin inequality for \( p : [0, 1] \to \mathbb{R} \) that is symmetric in \([0, 1]\) and monotonic non-decreasing on \([0, \frac{1}{2}]\) while \( f : [0, 1] \to \mathbb{R} \) is convex on \([0, 1]\).

Now, consider the function \( f : [0, 1] \to \mathbb{R}, f(t) = |t - \frac{1}{2}|^r \) for \( r \in (0, 1) \). This is a symmetric function and \( \bar{f}(t) = f(t) = |t - \frac{1}{2}|^r \). For \( t \in [0, \frac{1}{2}] \) we get

\[
f'(t) = -r \left( \frac{1}{2} - t \right)^{r-1} \quad \text{and} \quad f''(t) = -r (1-r) \left( \frac{1}{2} - t \right)^{r-2},
\]

which shows that \( \bar{f} \) is monotonic decreasing on \([0, \frac{1}{2}]\) but not convex on \([0, 1]\).

Therefore Theorem 4 extends Levin-Stečkin inequality to a larger class of functions, namely functions for which the symmetrical transform is monotonic non-increasing on \([0, \frac{1}{2}]\).

We have the following reverses of (3.9):

Theorem 5. Assume that \( p : [0, 1] \to \mathbb{R} \) is symmetric in \([0, 1]\) and monotonic non-decreasing on \([0, \frac{1}{2}]\) and \( f : [0, 1] \to \mathbb{R} \) integrable and such that \( \bar{f} \) is monotonic non-increasing on \([0, \frac{1}{2}]\).

(i) Then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
\end{equation}

\begin{align*}
&\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{f(0) + f(1)}{2} - f \left( \frac{1}{2} \right) \right].
\end{align*}
(ii) If $p$ is differentiable on $(0, 1)$ with $p' \in L_\infty [0, 1]$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
\leq \frac{1}{8} \|p'\|_{\infty, (0,1)} \left[ \frac{f(0) + f(1)}{2} - f\left(\frac{1}{2}\right) \right].
\end{equation}

(iii) If $p$ and $f$ are differentiable on $(0, 1)$ with $p', f' \in L_\infty [0, 1]$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
\leq \frac{1}{16} \sup_{t \in (0,1)} |f'(t) - f'(1-t)|.
\end{equation}

(iv) If $p$ and $f$ are differentiable on $(0, 1)$ with $p', f' \in L_2 [a,b]$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt
\leq \frac{1}{2\pi^2} \|p'\|_2 \left( \int_0^1 \left|f'(t) - f'(1-t)\right|^2 \, dt \right)^{1/2}.
\end{equation}

**Proof.** Observe that, as in the proof of Theorem 4, we have

\[
\int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt = \frac{1}{1/2} \int_0^{1/2} p(t) \, dt \cdot \frac{1}{1/2} \int_0^{1/2} f(t) \, dt - \frac{1}{1/2} \int_0^{1/2} p(t) \, dt \cdot \frac{1}{1/2} \int_0^{1/2} f(t) \, dt.
\]

By using now the inequalities (1.2), (1.5), (1.4) and (1.6) for the functions $p$ and $\tilde{f}$ on the interval $[0, 1/2]$ and perform some calculations, we derive the desired results. \qed

**Corollary 3.** Assume that $p : [0, 1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0, 1/2]$ and $f : [0, 1] \to \mathbb{R}$ is differentiable with $\tilde{f}$ is monotonic non-increasing on $[0, 1/2]$ and the derivative $f'$ is $L$-Lipschitzian on $(0, 1)$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt \leq \frac{1}{16} \left[ p\left(\frac{1}{2}\right) - p(0) \right].
\end{equation}

Moreover, if $p$ is differentiable with $p' \in L_\infty (0, 1)$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt \leq \frac{1}{24} L \|p'\|_{\infty, (0,1)}
\end{equation}

and if $p' \in L_2 [a,b]$, then

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(t) \, dt - \int_0^1 p(t) f(t) \, dt \leq \frac{\sqrt{3}}{2\pi^2} L \|p'\|_2.
\end{equation}
4. Examples

Consider the function $p : [0, 1] \rightarrow [0, \infty), \ p(t) = t(1-t)$. If $f : [0, 1] \rightarrow \mathbb{R}$ is convex, then by (2.1) we get

\begin{equation}
0 \leq \frac{1}{6} \int_{0}^{1} f(t) \, dt - \int_{0}^{1} t(1-t) \, f(t) \, dt \\
\leq \frac{1}{16} \left[ f(0) + f(1) \right] - f\left( \frac{1}{2} \right) \leq \frac{1}{64} \left[ f'(1) - f'(0) \right].
\end{equation}

while from (2.4) we derive

\begin{equation}
0 \leq \frac{1}{6} \int_{0}^{1} f(t) \, dt - \int_{0}^{1} t(1-t) \, f(t) \, dt \leq \frac{1}{64} \sup_{t \in (0,1)} |f'(t) - f'(1-t)|,
\end{equation}

provided that $f$ is differentiable on $(0,1)$.

If $f$ is differentiable convex on $(0,1)$ with $f' \in L_2[a,b]$, then by (2.6)

\begin{equation}
0 \leq \frac{1}{6} \int_{0}^{1} f(t) \, dt - \int_{0}^{1} t(1-t) \, f(t) \, dt \\
\leq \frac{\sqrt{3}}{6\pi^2} \left( \int_{0}^{1} |f'(t) - f'(1-t)|^2 \, dt \right)^{1/2}.
\end{equation}

since

\[ ||p'||_2 = \left( \int_{0}^{1} (1-2t)^2 \, dt \right)^{1/2} = \frac{\sqrt{3}}{3}. \]

If we consider the function $p : [0, 1] \rightarrow [0, \infty), \ p(t) = |t - \frac{1}{2}|$, then for $f : [0, 1] \rightarrow \mathbb{R}$ convex we get by (2.1) that

\begin{equation}
0 \leq \int_{0}^{1} \left| t - \frac{1}{2} \right| f(t) \, dt - \frac{1}{4} \int_{0}^{1} f(t) \, dt \\
\leq \frac{1}{8} \left[ \frac{f(0) + f(1)}{2} - f\left( \frac{1}{2} \right) \right] \leq \frac{1}{32} \left[ f'(1) - f'(0) \right].
\end{equation}

By (2.4) we get

\begin{equation}
0 \leq \int_{0}^{1} \left| t - \frac{1}{2} \right| f(t) \, dt - \frac{1}{4} \int_{0}^{1} f(t) \, dt \leq \frac{1}{32} \sup_{t \in (0,1)} |f'(t) - f'(1-t)|,
\end{equation}

provided that $f$ is differentiable convex on $(0,1)$.

If $f$ is differentiable convex on $(0,1)$ with $f' \in L_2[a,b]$, then

\begin{equation}
0 \leq \int_{0}^{1} \left| t - \frac{1}{2} \right| f(t) \, dt - \frac{1}{4} \int_{0}^{1} f(t) \, dt \\
\leq \frac{1}{2\pi^2} \left( \int_{0}^{1} |f'(t) - f'(1-t)|^2 \, dt \right)^{1/2}.
\end{equation}

References


[5] G. Grüss, Über das Maximum des absoluten Betrages von \( \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \), *Math. Z.*, 39(1935), 215-226.


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