SOME REVERSES AND EXTENSIONS OF LEVIN-STEČKIN INEQUALITY

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Abstract. In this paper we obtain several reverse of *Levin-Stečkin inequality*. We also extend this result for the more general case in which the symetrical transform is requested to be monotonic non-increasing on the half interval $[0,1/2]$, namely if $p:[0,1]\to\mathbb{R}$ is symmetric in $[0,1]$ and monotonic non-decreasing on $\left[0,\frac{1}{2}\right]$ and $f:[0,1]\to\mathbb{R}$ is integrable and such that $\check{f}$, the *symmetrical transform* of f, is monotonic non-increasing on $\left[0,\frac{1}{2}\right]$, then

$$\begin{aligned} 0 &\leq \int_0^1 p(t)\,dt\int_0^1 f(t)\,dt-\int_0^1 p(t)f(t)\,dt \\ &\leq \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right]. \end{aligned}$$

Several other similar inequalities for either p or f is differentiable, are also provided.

1. Introduction

For two *Lebesgue integrable* functions $h,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b h(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [5] showed that

$$|C(h,g)|\leq\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m\leq h(t)\leq M \quad\text{and}\quad n\leq g(t)\leq N \quad \text{for a.e. } t\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [1], states that

$$|C(h,g)|\leq\frac{1}{12}\|h'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

provided that h', g' exist and are continuous on $[a,b]$ and $\|h'\|_\infty=\sup_{t\in[a,b]}|h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $h,\ g:[a,b]\to\mathbb{R}$ are assumed to be *absolutely continuous* and $h',\ g'\in L_\infty[a,b]$ while $\|h'\|_\infty=\operatorname{essup}_{t\in[a,b]}|h'(t)|$.

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A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [9]:

$$|C(h,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{1.5}$$

provided that h is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [6] in which he proved that

$$|C(h,g)| \le \frac{1}{\pi^2}\|h'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that h, g are absolutely continuous and h', $g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

The following result is known in the literature as Levin-Stečkin's inequality [10]:

Theorem 1. *If the function $p:[0,1]\to\mathbb{R}$ is symmetric, namely $p(1-t)=p(t)$ for $t\in[0,1]$ and non-decreasing (non-increasing) on $[0,1/2]$, then for every convex function g on $[0,1]$,*

$$\int_0^1 p(t)\,g(t)\,dt \le (\ge) \int_0^1 p(t)\,dt \int_0^1 g(t)\,dt. \tag{LS}$$

If the function g is concave on $[0,1]$, then the signs of inequalities reverse in (LS).

For some recent results related to Levin-Stečkin's inequality, see [7], [8] and [11].

In this paper we obtain several reverse inequalities for (LS). We also extend this result and its reverses for the more general case in which the *symmetrical transform* of g is requested to be monotonic on the half interval $[0,1/2]$. Some examples of interest are also provided.

2. Reverse Inequalities

We have the following natural reverse of Levin-Stečkin's inequality, see also [4, Corollary 1.4]:

Theorem 2. *Assume that $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$ and $f:[0,1]\to\mathbb{R}$ is convex, then*

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t)\,f(t)\,dt \\ &\le \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(0)+f(1)}{2} - f\left(\frac{1}{2}\right)\right] \\ &\le \frac{1}{16}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[f'_-(1)-f'_+(0)\right]. \end{aligned} \tag{2.1}$$

Proof. Since p is symmetric on $[0,1]$, then

$$\begin{aligned} \int_0^1 p(t)\frac{f(t)+f(1-t)}{2}dt &= \frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt + \int_0^1 p(t)\,f(1-t)\,dt\right] \\ &= \frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt + \int_0^1 p(1-t)\,f(1-t)\,dt\right]. \end{aligned}$$

By changing the variable $1-t=s,\ s\in[0,1]$ we have

$$\int_0^1 p(1-t)\,f(1-t)\,dt=\int_0^1 p(s)\,f(s)\,ds$$

and then

$$\int_0^1 p(t)\,\frac{f(t)+f(1-t)}{2}dt=\int_0^1 p(t)\,f(t)\,dt.$$

Also

$$\int_0^1 \frac{f(t)+f(1-t)}{2}dt=\int_0^1 f(t)\,dt.$$

Therefore

$$\begin{aligned}(2.2)\qquad 0&\le\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt-\int_0^1 p(t)\,f(t)\,dt\\ &=\int_0^1 p(t)\,dt\int_0^1 \breve{f}(t)\,dt-\int_0^1 p(t)\,\breve{f}(t)\,dt,\end{aligned}$$

where

$$\breve{f}(t):=\frac{f(t)+f(1-t)}{2},\ t\in[0,1]$$

is the symmetrized transform of f.

Since f is convex, then $\breve{f}$ is symmetric and convex which implies that

$$f\left(\frac{1}{2}\right)=\breve{f}\left(\frac{1}{2}\right)\le\breve{f}(t)\le\breve{f}(1)=\frac{f(0)+f(1)}{2},\ t\in[0,1].$$

Also $p(0)\le p(t)\le p\left(\frac{1}{2}\right)$, $t\in[0,1]$ and by Grüss' inequality we get

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1 \breve{f}(t)\,dt-\int_0^1 p(t)\,\breve{f}(t)\,dt\\ &\le\frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right]\end{aligned}$$

and the second inequality (2.1) is thus proved.

For the last part, we use the inequality for convex functions $h:[a,b]\to\mathbb{R}$, see for instance [2] and [3] where the Hermite-Hadamard reverse inequalities were considered,

$$0\le\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\le\frac{1}{4}(b-a)\left[h'_-(b)-h'_+(a)\right],$$

in which the constant $\frac{1}{4}$ is best possible. □

Further, we have:

Theorem 3. *Assume that* $p:[0,1]\to\mathbb{R}$ *is symmetric and non-decreasing on* $[0,1/2]$ *and* $f:[0,1]\to\mathbb{R}$ *is convex.*

(i) *If p is differentiable on $(0,1)$ with $p' \in L_{\infty}(0,1)$, then*

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \tag{2.3}$$
$$\leq \frac{1}{8} \|p'\|_{\infty,(0,1)} \left[\frac{f(0)+f(1)}{2} - f\left(\frac{1}{2}\right)\right]$$
$$\leq \frac{1}{32} \|p'\|_{\infty,(0,1)} \left[f'_{-}(1) - f'_{+}(0)\right].$$

(ii) *If f is differentiable on $(0,1)$, then*

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \tag{2.4}$$
$$\leq \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right] \sup_{t\in(0,1)} |f'(t) - f'(1-t)|.$$

(iii) *If p and f are differentiable on $(0,1)$ with $p' \in L_{\infty}(0,1)$, then*

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \tag{2.5}$$
$$\leq \frac{1}{24} \|p'\|_{\infty,(0,1)} \sup_{t\in(0,1)} |f'(t) - f'(1-t)|.$$

(iv) *If p and f are differentiable on $(0,1)$ with $p',\ f' \in L_2[a,b]$, then*

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \tag{2.6}$$
$$\leq \frac{1}{2\pi^2} \|p'\|_2 \left(\int_0^1 |f'(t) - f'(1-t)|^2\,dt\right)^{1/2}.$$

Proof. We consider the function $g : [0,1] \to \mathbb{R}$,

$$g(t) = \frac{f(t) + f(1-t)}{2}.$$

If f is differentiable on $(0,1)$, then

$$g'(t) = \frac{f'(t) - f'(1-t)}{2},\ t \in (0,1).$$

By applying Čebyšev, Ostrowski and Lupaş inequalities for the appropriate choices of h and g we derive the corresponding inequalities (2.3)-(2.5). □

Corollary 1. *Assume that $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$ and $f : [0,1] \to \mathbb{R}$ is differentiable convex with the derivative f' L-Lipschitzian on $(0,1)$, namely*

$$|f'(u) - f'(u)| \leq L|u-v| \text{ for all } u,\ v \in (0,1).$$

Then

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \leq \frac{1}{16} L \left[p\left(\frac{1}{2}\right) - p(0)\right]. \tag{2.7}$$

If p is differentiable with $p' \in L_{\infty}(0,1)$, then

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \leq \frac{1}{24} L \|p'\|_{\infty,(0,1)} \tag{2.8}$$

and if $p' \in L_2[a,b]$, *then*

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \leq \frac{\sqrt{3}}{6\pi^2} L \|p'\|_2 . \tag{2.9}$$

Proof. Since f' is L-Lipschitzian on $(0,1)$, then

$$|f'(t) - f'(1-t)| \leq L|1-2t| = 2L\left|t - \frac{1}{2}\right| \leq L$$

for all $t \in (0,1)$.

Therefore

$$\sup_{t\in(0,1)} |f'(t) - f'(1-t)| \leq L,$$

and

$$\int_0^1 |f'(t) - f'(1-t)|^2 dt \leq 4L^2 \int_0^1 \left|t - \frac{1}{2}\right|^2 dt = \frac{1}{3}L^2.$$

By utilising Theorem 3 we derive the desired results. □

3. Some Extensions

For a function $f:[a,b] \to \mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t) := \frac{1}{2}[f(t) + f(a+b-t)],\ t \in [a,b]. \tag{3.1}$$

We recall that the pair of functions (f,g) defined on $[a,b]$ are called *synchronous (asynchronous)* on $[a,b]$ if

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \tag{3.2}$$

for any $t,\ s \in [a,b]$. It is clear that if both functions (f,g) are monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are synchronous on $[a,b]$. There are also functions that change monotonicity on $[a,b]$, but as a pair they are still synchronous. For instance if $a < 0 < b$ and $f,g:[a,b] \to \mathbb{R}$, $f(t) = t^2$ and $g(t) = t^4$, then

$$(f(t) - f(s))(g(t) - g(s)) = \left(t^2 - s^2\right)\left(t^4 - s^4\right) = \left(t^2 - s^2\right)^2\left(t^2 + s^2\right) \geq 0$$

for any $t,\ s \in [a,b]$, which show that (f,g) is synchronous.

We can introduce the following concept as well:

Definition 1. *We say that the pair of functions* (f,g) *defined on* $[a,b]$ *is called symmetrized synchronous (asynchronous) on* $[a,b]$ *if the pair of symmetrized transforms* $\left(\breve{f},\breve{g}\right)$ *is synchronous (asynchronous) on* $[a,b]$, *namely*

$$\left(\breve{f}(t) - \breve{f}(s)\right)(\breve{g}(t) - \breve{g}(s)) \geq (\leq)\, 0 \tag{3.3}$$

for any $t,s \in [a,b]$.

Now, assume that the function $x:[a,b] \to I$, where I is an interval of real numbers, and (ϕ,ψ) is a pair of synchronous (asynchronous) functions defined on the interval I. Consider the functions $f,g:[a,b] \to \mathbb{R}$ defined by $f = \phi\circ\breve{x}$ and $g = \psi\circ\breve{x}$. Then the functions f and g are symmetrical on $[a,b]$ and $\breve{f} = \phi\circ\breve{x}$ and $\breve{g} = \psi\circ\breve{x}$. Since (ϕ,ψ) is a pair of synchronous (asynchronous) functions, it follows that $\left(\breve{f},\breve{g}\right)$ is synchronous (asynchronous) on $[a,b]$, namely the pair of functions (f,g) defined on $[a,b]$ is symmetrized synchronous (asynchronous) on $[a,b]$. Therefore, we can give many example of symmetrized synchronous (asynchronous) functions on $[a,b]$. For instance, if (ϕ,ψ) is a pair of synchronous (asynchronous) functions defined on the interval $[0,\infty)$, then the functions f, $g:[a,b]\to\mathbb{R}$ defined by $f(t)=\phi\left(\left|t-\frac{a+b}{2}\right|^p\right)$ and $g(t)=\psi\left(\left|t-\frac{a+b}{2}\right|^p\right)$ with $p>0$ are symmetrized synchronous (asynchronous) on $[a,b]$.

One of the most important results for synchronous (asynchronous) and integrable functions f, g on $[a,b]$ is the well-known *Čebyšev's inequality*:

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \geq (\leq)\, \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.4}$$

We have the following Čebyšev's type result:

Lemma 1. *Assume that the pair of integrable functions* (f,g) *defined on* $[a,b]$ *is symmetrized synchronous (asynchronous) on* $[a,b]$, *then*

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \geq (\leq)\, \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.5}$$

Proof. Since $(\breve{f},\breve{g})$ is synchronous (asynchronous) on $[a,b]$, then by Čebyšev's inequality (3.4) we have

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt \geq (\leq)\, \frac{1}{b-a}\int_a^b \breve{f}(t)\,dt \frac{1}{b-a}\int_a^b \breve{g}(t)\,dt. \tag{3.6}$$

Observe that, by the change of variable $s=a+b-t$, $t\in[a,b]$ we have

$$\begin{aligned}\int_a^b \breve{f}(t)\,dt &= \frac{1}{2}\int_a^b [f(t)+f(a+b-t)] \\ &= \frac{1}{2}\left[\int_a^b f(t)\,dt+\int_a^b f(a+b-t)\,dt\right] = \int_a^b f(t)\,dt,\end{aligned}$$

$$\int_a^b \breve{g}(t)\,dt = \int_a^b g(t)\,dt$$

and

$$\begin{aligned}\int_a^b \check{f}(t)\,\check{g}(t)\,dt &= \frac{1}{4}\int_a^b [f(t)+f(a+b-t)]\,[g(t)+g(a+b-t)]\,dt \\ &= \frac{1}{4}\int_a^b [f(t)\,g(t)+f(a+b-t)\,g(t) \\ &+f(t)\,g(a+b-t)+f(a+b-t)\,g(a+b-t)]\,dt \\ &= \frac{1}{4}\left[\int_a^b f(t)\,g(t)\,dt+\int_a^b f(a+b-t)\,g(t)\,dt\right. \\ &+\left.\int_a^b f(a+b-t)\,g(t)\,dt+\int_a^b f(t)\,g(t)\,dt\right] \\ &= \frac{1}{2}\left[\int_a^b f(t)\,g(t)\,dt+\int_a^b f(a+b-t)\,g(t)\,dt\right] \\ &= \int_a^b \check{f}(t)\,g(t)\,dt\end{aligned} \tag{3.7}$$

since,

$$\int_a^b f(a+b-t)\,g(t)\,dt = \int_a^b f(s)\,g(a+b-s)\,ds$$

and

$$\int_a^b f(a+b-t)\,g(a+b-t)\,dt = \int_a^b f(s)\,g(s)\,ds.$$

By making use of (3.6) we obtain the desired result (3.5). □

Corollary 2. *Assume that the pair of integrable functions (p,g) defined on $[a,b]$ is symmetrized synchronous (asynchronous) on $[a,b]$ and p is symmetric, then*

$$\frac{1}{b-a}\int_a^b p(t)\,g(t)\,dt \geq (\leq)\,\frac{1}{b-a}\int_a^b p(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.8}$$

We have the following generalization of Levin-Stečkin inequality.

Theorem 4. *Assume that $p:[0,1]\to\mathbb{R}$ is symmetric in $[0,1]$ and monotonic non-decreasing (non-increasing) on $\left[0,\frac{1}{2}\right]$ and $f:[0,1]\to\mathbb{R}$ integrable and such that $\check{f}$ is monotonic non-increasing on $\left[0,\frac{1}{2}\right]$, then we have*

$$\int_0^1 p(t)\,f(t)\,dt \leq (\geq)\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt. \tag{3.9}$$

If $\check{f}$ is monotonic non-decreasing on $\left[0,\frac{1}{2}\right]$, then the inequalities reverse in (3.9).

Proof. Observe that

$$\begin{aligned}
(3.10)\qquad \int_0^1 p(t) f(t)\,dt &= \int_0^{\frac{1}{2}} p(t) f(t)\,dt + \int_{\frac{1}{2}}^1 p(t) f(t)\,dt \\
&= \int_0^{\frac{1}{2}} p(t) f(t)\,dt + \int_{\frac{1}{2}}^1 p(1-t) f(t)\,dt \\
&= \int_0^{\frac{1}{2}} p(t) f(t)\,dt + \int_0^{\frac{1}{2}} p(s) f(1-s)\,ds \\
&= \int_0^{\frac{1}{2}} p(t)\left[f(t) + f(1-t)\right] dt = 2\int_0^{\frac{1}{2}} p(t)\check{f}(t)\,dt.
\end{aligned}$$

Using Čebyšev's inequality for functions with opposite monotonicity on $\left[0, \frac{1}{2}\right]$, we have

$$\begin{aligned}
(3.11)\qquad 2\int_0^{\frac{1}{2}} p(t)\check{f}(t)\,dt &\le 2\int_0^{\frac{1}{2}} p(t)\,dt \cdot 2\int_0^{\frac{1}{2}} \check{f}(t)\,dt \\
&= \int_0^1 p(t)\,dt \int_0^1 \check{f}(t)\,dt = \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt.
\end{aligned}$$

By making use of (3.10) and (3.11) we deduce the desired result (3.9). □

Remark 1. *We observe that if $f : [0,1] \to \mathbb{R}$ is convex on $[0,1]$, then $\check{f}$ is monotonic non-increasing on $\left[0, \frac{1}{2}\right]$ and by (3.9) we get the Levin-Stečkin inequality for $p : [0,1] \to \mathbb{R}$ that is symmetric in $[0,1]$ and monotonic non-decreasing on $\left[0, \frac{1}{2}\right]$ while $f : [0,1] \to \mathbb{R}$ is convex on $[0,1]$.*

Now, consider the function $f : [0,1] \to \mathbb{R}$, $f(t) = \left|t - \frac{1}{2}\right|^r$ for $r \in (0,1)$. This is a symmetric function and $\check{f}(t) = f(t) = \left|t - \frac{1}{2}\right|^r$. For $t \in \left[0, \frac{1}{2}\right]$ we get

$$f'(t) = -r\left(\frac{1}{2} - t\right)^{r-1} \text{ and } f''(t) = -r(1-r)\left(\frac{1}{2} - t\right)^{r-2},$$

which shows that $\check{f}$ is monotonic decreasing on $\left[0, \frac{1}{2}\right]$ but not convex on $[0,1]$. Therefore Theorem 4 extends Levin-Stečkin inequality to a larger class of functions, namely functions for which the symmetrical transform is monotonic non-increasing on $\left[0, \frac{1}{2}\right]$.

We have the following reverses of (3.9):

Theorem 5. *Assume that $p : [0,1] \to \mathbb{R}$ is symmetric in $[0,1]$ and monotonic non-decreasing on $\left[0, \frac{1}{2}\right]$ and $f : [0,1] \to \mathbb{R}$ integrable and such that $\check{f}$ is monotonic non-increasing on $\left[0, \frac{1}{2}\right]$.*

(i) *Then*

$$\begin{aligned}
(3.12)\qquad 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \\
&\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\frac{f(0) + f(1)}{2} - f\left(\frac{1}{2}\right)\right].
\end{aligned}$$

(ii) *If p is differentiable on $(0,1)$ with $p' \in L_\infty[0,1]$, then*

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{1}{8}\|p'\|_{\infty,(0,1)} \left[\frac{f(0)+f(1)}{2} - f\left(\frac{1}{2}\right)\right]. \tag{3.13}$$

(iii) *If p and f are differentiable on $(0,1)$ with $p', f' \in L_\infty[0,1]$, then*

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right] \sup_{t\in(0,1)} |f'(t) - f'(1-t)|. \tag{3.14}$$

(iv) *If p and f are differentiable on $(0,1)$ with $p', f' \in L_2[a,b]$, then*

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{1}{2\pi^2}\|p'\|_2 \left(\int_0^1 |f'(t) - f'(1-t)|^2\,dt\right)^{1/2}. \tag{3.15}$$

Proof. Observe that, as in the proof of Theorem 4, we have

$$\int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt = \frac{1}{1/2}\int_0^{\frac{1}{2}} p(t)\,dt \cdot \frac{1}{1/2}\int_0^{\frac{1}{2}} \check{f}(t)\,dt - \frac{1}{1/2}\int_0^{\frac{1}{2}} p(t)\check{f}(t)\,dt.$$

By using now the inequalities (1.2), (1.5), (1.4) and (1.6) for the functions p and $\check{f}$ on the interval $[0,1/2]$ and perform some calculations, we derive the desired results. □

Corollary 3. *Assume that $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$ and $f : [0,1] \to \mathbb{R}$ is differentiable with $\check{f}$ is monotonic non-increasing on $\left[0,\frac{1}{2}\right]$ and the derivative f' is L-Lipschitzian on $(0,1)$, then*

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{1}{16}L\left[p\left(\frac{1}{2}\right) - p(0)\right]. \tag{3.16}$$

Moreover, if p is differentiable with $p' \in L_\infty(0,1)$, then

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{1}{24}L\|p'\|_{\infty,(0,1)} \tag{3.17}$$

and if $p' \in L_2[a,b]$, then

$$0 \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t) f(t)\,dt \le \frac{\sqrt{3}}{6\pi^2}L\|p'\|_2. \tag{3.18}$$

4. Examples

Consider the function $p:[0,1]\to[0,\infty)$, $p(t)=t(1-t)$. If $f:[0,1]\to\mathbb{R}$ is convex, then by (2.1) we get

$$0\leq\frac{1}{6}\int_0^1 f(t)\,dt-\int_0^1 t(1-t)f(t)\,dt \leq\frac{1}{16}\left[\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right]\leq\frac{1}{64}\left[f'_-(1)-f'_+(0)\right]. \tag{4.1}$$

while from (2.4) we derive

$$0\leq\frac{1}{6}\int_0^1 f(t)\,dt-\int_0^1 t(1-t)f(t)\,dt\leq\frac{1}{64}\sup_{t\in(0,1)}\left|f'(t)-f'(1-t)\right|, \tag{4.2}$$

provided that f is differentiable on $(0,1)$.

If f is differentiable convex on $(0,1)$ with $f'\in L_2[a,b]$, then by (2.6)

$$0\leq\frac{1}{6}\int_0^1 f(t)\,dt-\int_0^1 t(1-t)f(t)\,dt \leq\frac{\sqrt{3}}{6\pi^2}\left(\int_0^1\left|f'(t)-f'(1-t)\right|^2dt\right)^{1/2}. \tag{4.3}$$

since

$$\|p'\|_2=\left(\int_0^1(1-2t)^2\,dt\right)^{1/2}=\frac{\sqrt{3}}{3}.$$

If we consider the function $p:[0,1]\to[0,\infty)$, $p(t)=\left|t-\frac{1}{2}\right|$, then for $f:[0,1]\to\mathbb{R}$ convex we get by (2.1) that

$$0\leq\int_0^1\left|t-\frac{1}{2}\right|f(t)\,dt-\frac{1}{4}\int_0^1 f(t)\,dt \leq\frac{1}{8}\left[\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right]\leq\frac{1}{32}\left[f'_-(1)-f'_+(0)\right]. \tag{4.4}$$

By (2.4) we get

$$0\leq\int_0^1\left|t-\frac{1}{2}\right|f(t)\,dt-\frac{1}{4}\int_0^1 f(t)\,dt\leq\frac{1}{32}\sup_{t\in(0,1)}\left|f'(t)-f'(1-t)\right|, \tag{4.5}$$

provided that f is differentiable convex on $(0,1)$.

If f is differentiable convex on $(0,1)$ with $f'\in L_2[a,b]$, then

$$0\leq\int_0^1\left|t-\frac{1}{2}\right|f(t)\,dt-\frac{1}{4}\int_0^1 f(t)\,dt \leq\frac{1}{2\pi^2}\left(\int_0^1\left|f'(t)-f'(1-t)\right|^2dt\right)^{1/2}. \tag{4.6}$$

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