SOME DISCRETE INEQUALITIES RELATED TO JENSEN RESULT FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. In this paper we obtain some new discrete inequality related to Jensen's result for convex functions on convex subsets of linear spaces and provide some natural applications for norms.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\dots,p_n)$ is a probability sequence and $\mathbf{x}=(x_1,\dots,x_n)\in C^n$, then

$$f\left(\sum_{i=1}^{n}p_i x_i\right)\leq\sum_{i=1}^{n}p_i f\left(x_i\right), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [18] obtained the following inequality for convex functions of a real variable $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$ and the finite sequences $x_k\in[m,M]$, and $p_k\geq 0$, $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$,

$$f\left(m+M-\sum_{k=1}^{n}p_k x_k\right)\leq f\left(m\right)+f\left(M\right)-\sum_{k=1}^{n}p_k f\left(x_k\right) \tag{1.2}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [16], [19], [25], in relation with majorization theory [23], for convex functions of selfadjoint operators in Hilbert spaces [15], [17], [20], [21] and for operator convex functions in Hilbert spaces [22] and [25].

In the recent paper [11] we obtained the following extension of Jensen-Mercer inequality (1.2) to the case of convex functions defined on linear spaces:

1991 *Mathematics Subject Classification.* 26D15; 46B05.

Key words and phrases. Convex functions, Linear spaces, Jensen's inequality, Mercer's inequality, Norm inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 39, 8 pp. Received 2/04/20

Theorem 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C and $x, y \in C$. If $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0, 1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k f(x + y - x_k) \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x) \leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k). \tag{1.3}$$

Corollary 1. *With the assumptions of Theorem 2 we have*

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x) - \sum_{k=1}^{n} p_k f(x + y - x_k) \leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right). \tag{1.4}$$

If we write the inequality (1.3) for the convex function $f(x) = \|x\|^p$, $p \geq 1$, where $\|\cdot\|$ is a norm on X, then we get

$$\left\|x + y - \sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \|x + y - x_k\|^p \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] \|y\|^p + \left(\sum_{k=1}^{n} p_k t_k\right) \|x\|^p \leq \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p, \tag{1.5}$$

where $x, y \in X$, $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0, 1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

This implies that

$$0 \leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] \|y\|^p + \left(\sum_{k=1}^{n} p_k t_k\right) \|x\|^p - \sum_{k=1}^{n} p_k \|x + y - x_k\|^p \leq \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p - \left\|x + y - \sum_{k=1}^{n} p_k x_k\right\|^p. \tag{1.6}$$

In this paper we obtain some new discrete inequalities for convex functions defined on convex subsets in linear spaces that can be regarded as counterparts of (1.3) and provide some natural applications for powers of norms.

2. Main Results

First of all, we recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right] \\ & \leq \frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right) \\ & \leq n \max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right], \end{aligned} \tag{2.1}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

Theorem 2. *Let $f : C \to \mathbb{R}$ be a convex function on the convex subset C in the linear space X, x, y, $x_k \in C$, $p_k \geq 0$, for $k \in \{1, ..., n\}$ such that $\sum_{k=1}^{n} p_k = 1$ and $x + y - \sum_{k=1}^{n} p_kx_k \in C$. Then*

$$\begin{aligned} & f\left(x+y-\sum_{k=1}^{n}p_kx_k\right) \\ & \geq f\left(x+y-\sum_{k=1}^{n}p_kx_k\right) - \min_{i\in\{1,\dots,n\}}\{p_i\}\left(\sum_{k=1}^{n}f(x_k)-nf\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)\right) \\ & \geq f\left(x+y-\sum_{k=1}^{n}p_kx_k\right) - \left[\sum_{k=1}^{n}p_kf(x_k)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \geq 2f\left(\frac{x+y}{2}\right)-\sum_{k=1}^{n}p_kf(x_k). \end{aligned} \tag{2.2}$$

Proof. Since $x+y-\sum_{k=1}^{n} p_kx_k$, $\frac{x+y}{2} \in C$, then by the convexity of f on C we have

$$\begin{aligned} & \frac{1}{2}\left[f\left(x+y-\sum_{k=1}^{n}p_kx_k\right)+f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \geq f\left(\frac{x+y-\sum_{k=1}^{n}p_kx_k+\sum_{k=1}^{n}p_kx_k}{2}\right) = f\left(\frac{x+y}{2}\right), \end{aligned}$$

namely

$$f\left(x+y-\sum_{k=1}^{n}p_kx_k\right)+f\left(\sum_{k=1}^{n}p_kx_k\right) \geq 2f\left(\frac{x+y}{2}\right). \tag{2.3}$$

By subtracting in both sides of (2.3) the same quantity $\sum_{k=1}^{n} p_kf(x_k)$ we get

$$\begin{aligned} & f\left(x+y-\sum_{k=1}^{n}p_kx_k\right) - \left[\sum_{k=1}^{n}p_kf(x_k)-f\left(\sum_{k=1}^{n}p_kx_k\right)\right] \\ & \geq 2f\left(\frac{x+y}{2}\right)-\sum_{k=1}^{n}p_kf(x_k). \end{aligned} \tag{2.4}$$

By using the first inequality in (2.1) we have

$$
\begin{aligned}
&-\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\
&\leq-\min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)-\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\
&\leq f\left(x+y-\sum_{k=1}^{n} p_k x_k\right) \\
&-\min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right).
\end{aligned} \tag{2.5}
$$

By making use of (2.4) and (2.5) we get the second and third inequalities in (2.2). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
&\frac{1}{2}\left[f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)+\sum_{k=1}^{n} p_k f\left(x_k\right)\right]-f\left(\frac{x+y}{2}\right) \\
&\geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\
&-\frac{1}{2} \min_{i \in\{1, \ldots, n\}}\left\{p_i\right\}\left(\sum_{k=1}^{n} f\left(x_k\right)-n f\left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)\right) \\
&\geq 0
\end{aligned} \tag{2.6}
$$

for all $x_k \in C$, $p_k \geq 0$, for $k \in \{1, \ldots, n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

We also have:

Theorem 3. *Let $f : C \to \mathbb{R}$ be a convex function on the convex subset C in the linear space X, $x, y, x_k \in C$, $p_k \geq 0$, for $k \in \{1, \ldots, n\}$ such that $\sum_{k=1}^{n} p_k = 1$ and $x + y - \sum_{k=1}^{n} p_k x_k \in C$. Then*

$$
\begin{aligned}
&\frac{1}{2}\left[f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)+\sum_{k=1}^{n} p_k f\left(x_k\right)\right]-f\left(\frac{x+y}{2}\right) \\
&\geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right]
\end{aligned} \tag{2.7}
$$

$$\geq \frac{1}{2} \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) - \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\dots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \geq 0$$

and

$$\begin{aligned} &\frac{1}{2}\left[f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)+\sum_{k=1}^{n} p_k f\left(x_k\right)\right]-f\left(\frac{x+y}{2}\right) \\ &\geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right] \\ &\geq \frac{1}{2}\left[\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(q_1 x_{i_1}+\cdots+q_k x_{i_k}\right)\right. \\ &\left.-\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right] \\ &\geq 0, \end{aligned} \tag{2.8}$$

where $q_1,\dots,q_k \geq 0$ *with* $\sum_{j=1}^{k} q_j = 1$ *and* $1 \leq k \leq n$*.*

Proof. We observe that (2.4) is equivalent to

$$\begin{aligned} &\frac{1}{2}\left[f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)+\sum_{k=1}^{n} p_k f\left(x_k\right)\right]-f\left(\frac{x+y}{2}\right) \\ &\geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k f\left(x_k\right)-f\left(\sum_{k=1}^{n} p_k x_k\right)\right], \end{aligned}$$

which proves the first inequality in (2.7).

J. Pečarić and the author obtained in 1989, the following refinement of Jensen's inequality (see [24]):

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\dots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \\ &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\ &\leq \cdots \leq \sum_{i=1}^{n} p_i f\left(x_i\right), \end{aligned} \tag{2.9}$$

for $k \geq 1$ and $x_k \in C$, $p_k \geq 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

From (2.9) we get

$$\begin{aligned}\sum_{k=1}^{n} p_k f(x_k) - f\left(\sum_{k=1}^{n} p_k x_k\right) &\geq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right) \\ &\quad - \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\dots p_{i_{k+1}} f\left(\frac{x_{i_1}+\dots+x_{i_{k+1}}}{k+1}\right) \\ &\geq 0,\end{aligned}$$

which proves the last inequality in (2.7).

If $q_1,\dots,q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following refinement of Jensen inequality obtained in 1994 by the author [5] also holds:

$$\begin{aligned}f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right) \\ &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(q_1 x_{i_1}+\dots+q_k x_{i_k}\right) \\ &\leq \sum_{i=1}^{n} p_i f(x_i),\end{aligned} \tag{2.10}$$

where $1 \leq k \leq n$ and $x_k \in C$, $p_k \geq 0$, for $k \in \{1,...,n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

From this inequality we derive that

$$\begin{aligned}&\sum_{k=1}^{n} p_k f(x_k) - f\left(\sum_{k=1}^{n} p_k x_k\right) \\ &\geq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(q_1 x_{i_1}+\dots+q_k x_{i_k}\right) \\ &- \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right) \\ &\geq 0\end{aligned}$$

where $1 \leq k \leq n$ and $x_k \in C$, $p_k \geq 0$, for $k \in \{1,...,n\}$ and such that $\sum_{k=1}^{n} p_k = 1$.

This prove the last inequality in (2.8). □

3. Norm Inequalities

If we write the inequality (2.2) for the convex function $f(x) = \|x\|^p$, $p \geq 1$, where $\|\cdot\|$ is a norm on X, then for all $x_k \in X$, $p_k \geq 0$, with $k \in \{1,...,n\}$ such that

$\sum_{k=1}^{n} p_k = 1$,

$$
\begin{aligned}
(3.1)\qquad & \left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p \\
& \geq \left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p - \min_{i\in\{1,\dots,n\}}\{p_i\}\left(\sum_{k=1}^{n} \|x_k\|^p - n^{1-p}\left\| \sum_{k=1}^{n} x_k \right\|^p \right) \\
& \geq \left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p - \left[\sum_{k=1}^{n} p_k \|x_k\|^p - \left\| \sum_{k=1}^{n} p_k x_k \right\|^p \right] \\
& \geq 2^{1-p}\|x+y\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p .
\end{aligned}
$$

From the inequalities (2.7) and (2.8) we obtain

$$
\begin{aligned}
(3.2)\qquad & \frac{1}{2}\left[\left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p + \sum_{k=1}^{n} p_k \|x_k\|^p \right] - \left\| \frac{x+y}{2} \right\|^p \\
& \geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k \|x_k\|^p - \left\| \sum_{k=1}^{n} p_k x_k \right\|^p \right] \\
& \geq \frac{1}{2}\left[\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} \left\| \frac{x_{i_1}+\dots+x_{i_k}}{k} \right\|^p \right. \\
& \left. - \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\dots p_{i_{k+1}} \left\| \frac{x_{i_1}+\dots+x_{i_{k+1}}}{k+1} \right\|^p \right] \geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{2}\left[\left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p + \sum_{k=1}^{n} p_k \|x_k\|^p \right] - \left\| \frac{x+y}{2} \right\|^p \\
& \geq \frac{1}{2}\left[\sum_{k=1}^{n} p_k \|x_k\|^p - \left\| \sum_{k=1}^{n} p_k x_k \right\|^p \right] \\
& \geq \frac{1}{2}\left[\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} \left\| q_1 x_{i_1}+\dots+q_k x_{i_k} \right\|^p \right. \\
& \left. - \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} \left\| \frac{x_{i_1}+\dots+x_{i_k}}{k} \right\|^p \right] \geq 0,
\end{aligned}
$$

for all $x_k \in X$, $p_k \geq 0$, with $k \in \{1,...,n\}$ such that $\sum_{k=1}^{n} p_k = 1$.

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