SOME LEVIN-STEČKIN'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be an convex function on the convex set C in a linear space and $x,\ y\in C$, with $x\neq y$. In this paper we show among others that, if the function $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then

$$0\leq\int_0^1 p(t)\,dt\int_0^1 f((1-t)x+ty)\,dt-\int_0^1 p(t)\,f((1-t)x+ty)\,dt$$
$$\leq\frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right].$$

Some applications for norms and semi-inner products are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [10]).

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proved by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [10]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,\ y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:C\to\mathbb{R}$, a convex subset and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on C if and only if $g(x,y)$ is convex on $[0,1]$ for all x, $y\in C,\ x\neq y$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [3, p. 2], [4, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Since $f(x)=\|x\|^r$ ($x\in X$ and $1\le r<\infty$) is a convex function, then for any x, $y\in X$ we have the following norm inequality from (1.2) (see [12, p. 106])

$$\left\|\frac{x+y}{2}\right\|^r\le\int_0^1\|(1-t)x+ty\|^r dt\le\frac{\|x\|^r+\|y\|^r}{2}. \tag{1.3}$$

The following result is known in the literature as Levin-Stečkin's inequality [13]:

Theorem 1. *If the function* $p:[0,1]\to\mathbb{R}$ *is symmetric, namely* $p(1-t)=p(t)$ *for* $t\in[0,1]$ *and non-decreasing (non-increasing) on* $[0,1/2]$, *then for every convex function* g *on* $[0,1]$,

$$\int_0^1 p(t)\,g(t)\,dt\le(\ge)\int_0^1 p(t)\,dt\int_0^1 g(t)\,dt. \tag{LS}$$

If the function g *is concave on* $[0,1]$, *then the signs of inequalities reverse in (LS).*

For some recent results related to Levin-Stečkin's inequality, see [8], [9] and [14].

In this paper we extend Levin-Stečkin's inequality for convex functions f defined on convex subsets of linear spaces, study them in connection to Fejér's inequalities considered in this context and apply them for powers of norms in normed linear spaces. Several reverse inequalities for (LS) and their version in linear spaces with applications to norms are also given.

2. Convex Functions on Linear Spaces

We have the following weighted version of Hermite-Hadamard inequality for functions defined on convex subsets of linear spaces. It provides a generalization of Féjer's inequality for univariate functions. For completness, a simple proof is provided.

Theorem 2. *Assume that* $f:C\to\mathbb{R}$ *is convex on* C *and* x, $y\in C$. *If* p *is nonnegative, symmetric and integrable on* $[0,1]$, *then*

$$\begin{aligned} f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le\int_0^1 f[(1-t)x+ty]\,p(t)\,dt \\ &\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt. \end{aligned} \tag{2.1}$$

Proof. By the convexity of f we have for all $t\in[0,1]$ that

$$f\left(\frac{x+y}{2}\right)\le\frac{f[(1-t)x+ty]+f[(1-t)y+tx]}{2}\le\frac{f(x)+f(y)}{2}.$$

If we multiply this inequality by $p:[0,1]\to[0,\infty)$, a Lebesgue integrable function on $[0,1]$, and integrate on $[0,1]$ over $t\in[0,1]$, then we get

$$\begin{aligned} &f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \\ &\le\frac{\int_0^1 f[(1-t)x+ty]\,p(t)\,dt+\int_0^1 f[(1-t)y+tx]\,p(t)\,dt}{2} \\ &\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt. \end{aligned} \tag{2.2}$$

By changing the variable $s=1-t$, then we get

$$\int_0^1 f\left[(1-t)y+tx\right]p(t)\,dt=\int_0^1 f\left[sy+(1-s)x\right]p(1-s)\,dt$$

and by (2.2) we obtain

$$f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\leq\int_0^1 f\left[(1-t)x+ty\right]\breve{p}(t)\,dt \leq\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt, \tag{2.3}$$

where $\breve{p}(t):=\frac{1}{2}\left[p(t)+p(1-t)\right],\ t\in[0,1]$.

This inequality is of interest in itself.

If p is symmetric on $[0,1]$, namely $p(t)=p(1-t)$ for $t\in[0,1]$, then (2.3) becomes the Féjer's inequality (2.1). □

The inequality (2.1) provides the following weighted norm inequality provided that p is symmetric and integrable on $[0,1]$, $r\geq 1$.

Corollary 1. *If $(X,\|\cdot\|)$ is a normed space and p is symmetric and integrable on $[0,1]$, then for all $r\geq 1$*

$$\left\|\frac{x+y}{2}\right\|^r\int_0^1 p(t)\,dt\leq\int_0^1\|(1-t)x+ty\|^r p(t)\,dt \leq\frac{\|x\|^r+\|y\|^r}{2}\int_0^1 p(t)\,dt, \tag{2.4}$$

where $x,\ y\in X$.

Remark 1. *If we consider the symmetric weight $p(t)=\left|t-\frac{1}{2}\right|,\ t\in[0,1]$ then by (2.1) we get*

$$\frac{1}{4}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f\left[(1-t)x+ty\right]\left|t-\frac{1}{2}\right|dt\leq\frac{f(x)+f(y)}{8}, \tag{2.5}$$

while for $p(t)=t(1-t),\ t\in[0,1]$ we get

$$\frac{1}{6}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f\left[(1-t)x+ty\right]t(1-t)\,dt\leq\frac{f(x)+f(y)}{12} \tag{2.6}$$

for all $x,\ y\in C$.

We have the following version of Levin-Stečkin's inequality for convex functions on linear spaces:

Theorem 3. *Assume that $f:C\to\mathbb{R}$ is convex on C and $x,\ y\in C$. If the function $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing (non-increasing) on $[0,1/2]$, then*

$$\int_0^1 p(t)f((1-t)x+ty)\,dt\leq(\geq)\int_0^1 p(t)\,dt\int_0^1 f((1-t)x+ty)\,dt. \tag{2.7}$$

If the function $f:C\to\mathbb{R}$ is concave on C, then the inequalities reverses in (2.7).

Proof. The proof follows by applying Levin-Stečkin's inequality for the convex function $g(t)=f((1-t)x+ty),\ t\in[0,1]$. □

Corollary 2. *If* $(X, \|\cdot\|)$ *is a normed space and* p *is symmetric and non-decreasing (non-increasing) on* $[0, 1/2]$, *then for all* $r \geq 1$

$$\int_0^1 p(t) \|(1-t)x + ty\|^r dt \leq (\geq) \int_0^1 p(t) dt \int_0^1 \|(1-t)x + ty\|^r dt \tag{2.8}$$

for all $x, y \in X$.

We can derive the following refinements of Fejer's inequality:

Corollary 3. *Assume that* $f : C \to \mathbb{R}$ *is convex on* C *and* $x, y \in C$ *and* p *is nonnegative, symmetric and integrable on* $[0, 1]$.

(i) *If* p *is symmetric and non-decreasing on* $[0, 1/2]$, *then*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) \int_0^1 p(t) dt &\leq \int_0^1 f[(1-t)x + ty] p(t) dt \\ &\leq \int_0^1 p(t) dt \int_0^1 f((1-t)x + ty) dt \\ &\leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt. \end{aligned} \tag{2.9}$$

(ii) *If* p *is symmetric and non-increasing on* $[0, 1/2]$, *then*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) \int_0^1 p(t) dt &\leq \int_0^1 p(t) dt \int_0^1 f((1-t)x + ty) dt \\ &\leq \int_0^1 f[(1-t)x + ty] p(t) dt \\ &\leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt. \end{aligned} \tag{2.10}$$

The proof follows by Theorem 3 and by Hermite-Hadamard inequalities (1.2).

Remark 2. *If* p *is symmetric and non-decreasing on* $[0, 1/2]$, *then by (2.9) for power of norm, we get*

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^r \int_0^1 p(t) dt &\leq \int_0^1 \|(1-t)x + ty\|^r p(t) dt \\ &\leq \int_0^1 p(t) dt \int_0^1 \|(1-t)x + ty\|^r dt \\ &\leq \frac{\|x\|^r + \|y\|^r}{2} \int_0^1 p(t) dt \end{aligned} \tag{2.11}$$

for all $x, y \in X$.

If p *is symmetric and non-increasing on* $[0, 1/2]$, *then*

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^r \int_0^1 p(t) dt &\leq \int_0^1 p(t) dt \int_0^1 \|(1-t)x + ty\|^r dt \\ &\leq \int_0^1 \|(1-t)x + ty\|^r p(t) dt \\ &\leq \frac{\|x\|^r + \|y\|^r}{2} \int_0^1 p(t) dt \end{aligned} \tag{2.12}$$

for all $x, y \in X$.

Remark 3. *If we consider the symmetric weight* $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ *then by (2.1) we get*

$$\begin{aligned} \frac{1}{4} f\left(\frac{x+y}{2}\right) &\le \frac{1}{4}\int_0^1 f((1-t)x + ty)\,dt \\ &\le \int_0^1 f[(1-t)x+ty]\left|t - \frac{1}{2}\right| dt \le \frac{f(x)+f(y)}{8}, \end{aligned} \tag{2.13}$$

while for $p(t) = t(1-t)$, $t \in [0,1]$ *we get*

$$\begin{aligned} \frac{1}{6} f\left(\frac{x+y}{2}\right) &\le \int_0^1 f[(1-t)x+ty]\,t(1-t)\,dt \\ &\le \frac{1}{6}\int_0^1 f((1-t)x+ty)\,dt \le \frac{f(x)+f(y)}{12} \end{aligned} \tag{2.14}$$

for all $x, y \in C$.

3. Reverse Inequalities

For two *Lebesgue integrable* functions $h, \ell : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,\ell) := \frac{1}{b-a}\int_a^b h(t)\ell(t)dt - \frac{1}{(b-a)^2}\int_a^b h(t)dt\int_a^b \ell(t)dt. \tag{3.1}$$

In 1935, Grüss [6] showed that

$$|C(h,\ell)| \le \frac{1}{4}(M-m)(N-n), \tag{3.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le h(t) \le M \quad \text{and} \quad n \le \ell(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible in (3.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(h,\ell)| \le \frac{1}{12}\|h'\|_\infty \|\ell'\|_\infty (b-a)^2, \tag{3.4}$$

provided that h', ℓ' exist and are continuous on $[a,b]$ and $\|h'\|_\infty = \sup_{t\in[a,b]}|h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (3.4) also holds if $h, \ell : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $h', \ell' \in L_\infty[a,b]$ while $\|h'\|_\infty = \operatorname{essup}_{t\in[a,b]}|h'(t)|$.

A mixture between Grüss' result (3.2) and Čebyšev's one (3.4) is the following inequality obtained by Ostrowski in 1970, [11]:

$$|C(h,\ell)| \le \frac{1}{8}(b-a)(M-m)\|\ell'\|_\infty, \tag{3.5}$$

provided that h is *Lebesgue integrable* and satisfies (3.3) while ℓ is absolutely continuous and $\ell' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (3.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [7] in which he proved that

$$|C(h,\ell)| \le \frac{1}{\pi^2}\|h'\|_2 \|\ell'\|_2 (b-a), \tag{3.6}$$

provided that $h,\ \ell$ are absolutely continuous and $h',\ \ell' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Lemma 1. *Assume that* $p:[0,1]\to\mathbb{R}$ *is symmetric and non-decreasing on* $[0,1/2]$ *and* $g:[0,1]\to\mathbb{R}$ *is convex, then*

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 g(t)\,dt - \int_0^1 p(t)\,g(t)\,dt \\ &\le \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{g(0)+g(1)}{2}-g\left(\frac{1}{2}\right)\right]. \end{aligned} \tag{3.7}$$

Proof. Since p is symmetric on $[0,1]$, then

$$\begin{aligned} \int_0^1 p(t)\,\frac{g(t)+g(1-t)}{2}dt &= \frac{1}{2}\left[\int_0^1 p(t)\,g(t)\,dt + \int_0^1 p(t)\,g(1-t)\,dt\right] \\ &= \frac{1}{2}\left[\int_0^1 p(t)\,g(t)\,dt + \int_0^1 p(1-t)\,g(1-t)\,dt\right]. \end{aligned}$$

By changing the variable $1-t=s,\ s\in[0,1]$ we have

$$\int_0^1 p(1-t)\,g(1-t)\,dt = \int_0^1 p(s)\,g(s)\,ds$$

and then

$$\int_0^1 p(t)\,\frac{g(t)+g(1-t)}{2}dt = \int_0^1 p(t)\,g(t)\,dt.$$

Also

$$\int_0^1 \frac{g(t)+g(1-t)}{2}dt = \int_0^1 g(t)\,dt.$$

Therefore

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 g(t)\,dt - \int_0^1 p(t)\,g(t)\,dt \\ &= \int_0^1 p(t)\,dt \int_0^1 \breve{g}(t)\,dt - \int_0^1 p(t)\,\breve{g}(t)\,dt, \end{aligned} \tag{3.8}$$

where

$$\breve{g}(t) := \frac{g(t)+g(1-t)}{2},\ \ t\in[0,1]$$

is the symmetrized transform of g.

Since g is convex, then $\breve{g}$ is symmetric and convex which implies that

$$g\left(\frac{1}{2}\right) = \breve{g}\left(\frac{1}{2}\right) \le \breve{g}(t) \le \breve{g}(1) = \frac{g(0)+g(1)}{2},\ \ t\in[0,1].$$

Also $p(0)\le p(t)\le p\left(\frac{1}{2}\right)$, $t\in[0,1]$ and by Grüss' inequality we get

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 \breve{g}(t)\,dt - \int_0^1 p(t)\,\breve{g}(t)\,dt \\ &\le \frac{1}{4}\left(p\left(\frac{1}{2}\right)-p(0)\right)\left(\frac{g(0)+g(1)}{2}-g\left(\frac{1}{2}\right)\right) \end{aligned}$$

and the second inequality (3.7) is thus proved. □

Theorem 4. *Assume that $f : C \to \mathbb{R}$ is convex on C and $x, y \in C$. If the function $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then*

$$\begin{aligned}(3.9)\qquad 0 &\leq \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt - \int_0^1 p(t) f((1-t)x+ty)\,dt \\ &\leq \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right].\end{aligned}$$

The proof follows by Lemma 1 for the function $g(t) = f((1-t)x+ty)$, $t \in [0,1]$.

Lemma 2. *Assume that $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$ and $g : [0,1] \to \mathbb{R}$ is convex. If p is differentiable on $(0,1)$ with $p' \in L_\infty(0,1)$, then*

$$\begin{aligned}(3.10)\qquad 0 &\leq \int_0^1 p(t)\,dt \int_0^1 g(t)\,dt - \int_0^1 p(t) g(t)\,dt \\ &\leq \frac{1}{8}\|p'\|_{\infty,(0,1)}\left[\frac{g(0)+g(1)}{2} - g\left(\frac{1}{2}\right)\right].\end{aligned}$$

Proof. By (3.8) we get

$$\begin{aligned}(3.11)\qquad 0 &\leq \int_0^1 p(t)\,dt \int_0^1 g(t)\,dt - \int_0^1 p(t) g(t)\,dt \\ &= \int_0^1 p(t)\,dt \int_0^1 \breve{g}(t)\,dt - \int_0^1 p(t) \breve{g}(t)\,dt.\end{aligned}$$

If we use the inequality (3.5) for $h = \breve{g}$ and $\ell = p$, then we get (3.10). □

Theorem 5. *Assume that $f : C \to \mathbb{R}$ is convex on C and $x, y \in C$. If the function $p : [0,1] \to \mathbb{R}$ is symmetric, non-decreasing on $[0,1/2]$, differentiable on $(0,1)$ with $p' \in L_\infty(0,1)$, then*

$$\begin{aligned}(3.12)\qquad 0 &\leq \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt - \int_0^1 p(t) f((1-t)x+ty)\,dt \\ &\leq \frac{1}{8}\|p'\|_{\infty,(0,1)}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right].\end{aligned}$$

The proof follows by Lemma 1 for the function $g(t) = f((1-t)x+ty)$, $t \in [0,1]$.

Remark 4. *If the function $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then we have the norm inequality*

$$\begin{aligned}(3.13)\qquad 0 &\leq \int_0^1 p(t)\,dt \int_0^1 \|(1-t)x+ty\|^r\,dt - \int_0^1 p(t)\|(1-t)x+ty\|^r\,dt \\ &\leq \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{\|x\|^r+\|y\|^r}{2} - \left\|\frac{x+y}{2}\right\|^r\right]\end{aligned}$$

for all $x, y \in X$.

If the function $p:[0,1]\to\mathbb{R}$ *is symmetric, non-decreasing on* $[0,1/2]$, *differentiable on* $(0,1)$ *with* $p'\in L_\infty(0,1)$, *then*

$$0\le\int_0^1 p(t)\,dt\int_0^1\|(1-t)x+ty\|^r dt-\int_0^1 p(t)\|(1-t)x+ty\|^r dt \le\frac{1}{8}\|p'\|_{\infty,(0,1)}\left[\frac{\|x\|^r+\|y\|^r}{2}-\left\|\frac{x+y}{2}\right\|^r\right] \tag{3.14}$$

for all x, $y\in X$.

In particular, if $p(t)=t(1-t)$, $t\in[0,1]$, *then by (3.14) we derive*

$$0\le\frac{1}{6}\int_0^1\|(1-t)x+ty\|^r dt-\int_0^1 t(1-t)\|(1-t)x+ty\|^r dt \le\frac{1}{8}\left[\frac{\|x\|^r+\|y\|^r}{2}-\left\|\frac{x+y}{2}\right\|^r\right] \tag{3.15}$$

for all x, $y\in X$.

4. Inequalities in Terms of Gâteaux Derivatives

Let X be a real linear space, $x,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the attached function $\varphi_{(x,y)}:[0,1]\to\mathbb{R}$, $\varphi_{(x,y)}(t):=f[(1-t)x+ty]$, $t\in[0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s)=\nabla_{\pm}f_{(1-s)x+sy}(y-x)$, $s\in[0,1)$,
(ii) $\varphi'_{+(x,y)}(0)=\nabla_{+}f_x(y-x)$,
(iii) $\varphi'_{-(x,y)}(1)=\nabla_{-}f_y(y-x)$,

where $\nabla_{\pm}f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\nabla_+f_x(y) := \lim_{h\to0+}\frac{f(x+hy)-f(x)}{h},$$
$$\nabla_-f_x(y) := \lim_{k\to0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.$$

By the properties of convex functions of real variable, we observe that $\varphi'_{+(x,y)}(s)=\varphi'_{-(x,y)}(s)$ for all $s\in[0,1]$ except a countable number of points, therefore we can write $\varphi'_{(x,y)}(s)$ for those values and we have $\varphi'_{(x,y)}(s)=\nabla f_{(1-s)x+sy}(y-x)$.

If we apply the inequality (3.5) for $h=p$ and $\ell=\check{\varphi}(x,y)$ and observe that for almost every $s\in[0,1]$

$$\ell'(s)=\frac{1}{2}\left[\varphi'_{(x,y)}(s)-\varphi'_{(x,y)}(1-s)\right] =\frac{1}{2}\left[\nabla f_{(1-s)x+sy}(y-x)-\nabla f_{sx+(1-s)y}(y-x)\right],$$

then

$$0\le\int_0^1 p(t)\,dt\int_0^1 f((1-t)x+ty)\,dt-\int_0^1 p(t)f((1-t)x+ty)\,dt \le\frac{1}{8}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x)-\nabla f_{\cdot x+(1-\cdot)y}(y-x)\right\|_\infty, \tag{4.1}$$

where $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$.

If we assume that

$$\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x)\right\|_{\infty} := \operatorname*{essup}_{s\in[0,1]}\left|\varphi'_{(x,y)}(s)\right| < \infty,$$

then we have the chain of inequalities

$$\begin{aligned}
0 &\le \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt - \int_0^1 p(t) f((1-t)x+ty)\,dt \\
&\le \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x) - \nabla f_{\cdot x+(1-\cdot)y}(y-x)\right\|_{\infty} \\
&\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x)\right\|_{\infty}.
\end{aligned} \tag{4.2}$$

If $p:[0,1]\to\mathbb{R}$ is symmetric, non-decreasing on $[0,1/2]$ and differentiable on $(0,1)$, then by Čebyšev's inequality (3.4) for $h=p$ and $\ell=\check{\varphi}(x,y)$ we get

$$\begin{aligned}
0 &\le \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt - \int_0^1 p(t) f((1-t)x+ty)\,dt \\
&\le \frac{1}{24}\left\|p'\right\|_{\infty}\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x) - \nabla f_{\cdot x+(1-\cdot)y}(y-x)\right\|_{\infty} \\
&\le \frac{1}{12}\left\|p'\right\|_{\infty}\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x)\right\|_{\infty}.
\end{aligned} \tag{4.3}$$

Also, if we use Lupaş' inequality for $h=p$ and $\ell=\check{\varphi}(x,y)$ we get

$$\begin{aligned}
0 &\le \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt - \int_0^1 p(t) f((1-t)x+ty)\,dt \\
&\le \frac{1}{2\pi^2}\left\|p'\right\|_{2}\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x) - \nabla f_{\cdot x+(1-\cdot)y}(y-x)\right\|_{2} \\
&\le \frac{1}{\pi^2}\left\|p'\right\|_{2}\left\|\nabla f_{(1-\cdot)x+\cdot y}(y-x)\right\|_{2}.
\end{aligned} \tag{4.4}$$

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := \nabla_+ f_{0,y}(x) = \lim_{t\to 0+}\frac{\|y+tx\|^2-\|y\|^2}{2t}$;

(v) $\langle x,y\rangle_i := \nabla_- f_{0,y}(x) = \lim_{s\to 0-}\frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any $x,\ y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [5]), assuming that $p,\ q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x\in X$;

(aa) $\langle \alpha x,\beta y\rangle_p = \alpha\beta\langle x,y\rangle_p$ if $\alpha,\ \beta\ge 0$ and $x,\ y\in X$;

(aaa) $\left|\langle x,y\rangle_p\right| \le \|x\|\,\|y\|$ for all $x,\ y\in X$;

(av) $\langle \alpha x+y,x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ if $x,\ y\in X$ and $\alpha\in\mathbb{R}$;

(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,\ y\in X$;

(va) $\langle x+y,z\rangle_p \le \|x\|\,\|z\| + \langle y,z\rangle_p$ for all $x,\ y,\ z\in X$;

(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);

(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\backslash\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\leq\langle y,x\rangle_s$ for all x, $y\in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all x, $y\in X$.

The function $f_r(x)=\|x\|^r$ ($x\in X$ and $1\leq r<\infty$) is also convex. Therefore, the following limits, which are related to the *superior (inferior) semi-inner products*,

$$\begin{aligned}\nabla_{\pm}f_{r,y}(x)&:=\lim_{t\to 0\pm}\frac{\|y+tx\|^r-\|y\|^r}{t}\\&=r\|y\|^{r-1}\lim_{t\to 0\pm}\frac{\|y+tx\|-\|y\|}{t}=r\|y\|^{r-2}\langle x,y\rangle_{s(i)}\end{aligned}$$

exist for all x, $y\in X$ whenever $r\geq 2$; otherwise, they exist for any $x\in X$ and nonzero $y\in X$. In particular, if $r=1$, then the following limits

$$\nabla_{\pm}f_{1,y}(x):=\lim_{t\to 0\pm}\frac{\|y+tx\|-\|y\|}{t}=\frac{\langle x,y\rangle_{s(i)}}{\|y\|}$$

exist for x, $y\in X$ and $y\neq 0$.

For the function $f_r(x)=\|x\|^r$ ($x\in X$ and $1\leq r<\infty$), we have

$$\nabla_{\pm}f_{r,(1-s)x+sy}(y-x)=r\|(1-s)x+sy\|^{r-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}$$

and by (4.1) we get for $p:[0,1]\to\mathbb{R}$ that is symmetric and non-decreasing on $[0,1/2]$,

$$\begin{aligned}(4.5)\qquad 0&\leq\int_0^1p(t)\,dt\int_0^1\|(1-t)x+ty\|^r\,dt-\int_0^1p(t)\|(1-t)x+ty\|^r\,dt\\&\leq\frac{1}{4}r\left[p\left(\frac{1}{2}\right)-p(0)\right]\\&\quad\times\sup_{s\in[0,1]}\left\{\|(1-s)x+sy\|^{r-2}\left|\langle y-x,(1-s)x+sy\rangle_{s(i)}\right|\right\}\\&\leq\frac{1}{4}r\|y-x\|\left[p\left(\frac{1}{2}\right)-p(0)\right]\sup_{s\in[0,1]}\left\{\|(1-s)x+sy\|^{r-1}\right\}\\&\leq\frac{1}{4}r\|y-x\|\left[p\left(\frac{1}{2}\right)-p(0)\right]\max\left\{\|x\|^{r-1},\|y\|^{r-1}\right\}\end{aligned}$$

for x, $y\in X$ and $y\neq 0$.

By (4.2) we get for $p:[0,1]\to\mathbb{R}$, symmetric, non-decreasing on $[0,1/2]$ and differentiable that

$$\begin{aligned}(4.6)\qquad 0&\leq\int_0^1p(t)\,dt\int_0^1\|(1-t)x+ty\|^r\,dt-\int_0^1p(t)\|(1-t)x+ty\|^r\,dt\\&\leq\frac{1}{12}r\|p'\|_{\infty}\sup_{s\in[0,1]}\left\{\|(1-s)x+sy\|^{r-2}\left|\langle y-x,(1-s)x+sy\rangle_{s(i)}\right|\right\}\\&\leq\frac{1}{12}r\|y-x\|\|p'\|_{\infty}\sup_{s\in[0,1]}\left\{\|(1-s)x+sy\|^{r-1}\right\}\\&\leq\frac{1}{12}r\|y-x\|\|p'\|_{\infty}\max\left\{\|x\|^{r-1},\|y\|^{r-1}\right\}\end{aligned}$$

for x, $y\in X$ and $y\neq 0$.

From (4.4) we also derive

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 \|(1-t)x+ty\|^r\,dt - \int_0^1 p(t)\,\|(1-t)x+ty\|^r\,dt \tag{4.7}$$
$$\leq \frac{1}{\pi^2} r \|p'\|_2 \left[\int_0^1 \|(1-s)x+sy\|^{2(r-2)} \left|\langle y-x,(1-s)x+sy\rangle_{s(i)}\right|^2 ds\right]^{1/2}$$
$$\leq \frac{1}{\pi^2} r \|y-x\| \|p'\|_2 \left(\int_0^1 \|(1-s)x+sy\|^{2(r-1)}\,ds\right)^{1/2}$$

for $x,\ y \in X$ and $y \neq 0$.

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