A VERSION OF JENSEN-MERCER INEQUALITY FOR CONVEX FUNCTIONS, SELFADJOINT OPERATORS AND VECTORS IN HILBERT SPACES

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Abstract. In this paper we obtain a version of Mercer's inequality for convex functions, selfadjoint operators and vectors in Hilbert spaces and provide some examples for particular functions of interest.

1. Introduction

he Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of (1.1) and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [20] obtained the following inequality for convex functions of a real variable $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ and the finite sequences $x_k \in [m, M]$, and $p_k \ge 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$,

$$f\left(m + M - \sum_{k=1}^{n} p_k x_k\right) \le f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) \tag{1.2}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [18], [21], [30], in relation with majorization theory [29], for convex functions of selfadjoint operators in Hilbert spaces [17], [19], [22], [23] and for operator convex functions in Hilbert spaces [24] and [30].

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\mathrm{Sp}(A)$, an the

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C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [16, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \mathrm{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(\mathrm{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \mathrm{Sp}(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [16] and the references therein. For other results, see [28], [23] and [25].

The following result that provides an *operator-vector version* for the Jensen's inequality is due to Mond & Pečarić [26] (see also [16, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [26]). *Let B be a selfadjoint operator on the Hilbert space H and assume that $\mathrm{Sp}(B) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Bx, x\rangle) \leq \langle f(B)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

In this paper we obtain a version of Mercer's inequality for convex functions defined on closed intervals of real numbers, selfadjoint operators and vectors in Hilbert spaces and provide some examples for particular functions of interest.

2. Main Results

We have:

Theorem 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, then*

$$\begin{aligned} f(m + M - \langle Ax, x\rangle) &\leq \langle f(m1_H + M1_H - A)x, x\rangle \\ &\leq \frac{\langle Ax, x\rangle - m}{M - m} f(m) + \frac{M - \langle Ax, x\rangle}{M - m} f(M) \\ &\leq f(m) + f(M) - \langle f(A)x, x\rangle \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} 0 &\leq \frac{\langle Ax, x\rangle - m}{M - m} f(m) + \frac{M - \langle Ax, x\rangle}{M - m} f(M) - \langle f(m1_H + M1_H - A)x, x\rangle \\ &\leq f(m) + f(M) - \langle f(A)x, x\rangle - f(m + M - \langle Ax, x\rangle) \end{aligned} \tag{2.2}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. If $\operatorname{Sp}(A) \subseteq [m, M]$ then also $\operatorname{Sp}(m1_H + M1_H - A) \subseteq [m, M]$ and by Jensen's inequality (MP) for the operator $B = m1_H + M1_H - A$ we have for $x \in H$ with $\|x\| = 1$

$$\begin{aligned} f(m + M - \langle Ax, x\rangle) &= f(\langle (m1_H + M1_H - A)x, x\rangle) \\ &= f(\langle Bx, x\rangle) \le \langle f(B)x, x\rangle \\ &= \langle f(m1_H + M1_H - A)x, x\rangle, \end{aligned}$$

which proves the first inequality in (2.1).

Since f is convex on $[m, M]$, then

$$\begin{aligned} f(s) &= f\left(\frac{M-s}{M-m}m + \frac{s-m}{M-m}M\right) \\ &\le \frac{M-s}{M-m}f(m) + \frac{s-m}{M-m}f(M) \end{aligned} \tag{2.3}$$

for all $s \in [m, M]$.

For all $t \in [m, M]$, $s = m + M - t \in [m, M]$ and by (2.3) we get

$$\begin{aligned} f(m + M - t) &\le \frac{M - m - M + t}{M-m}f(m) + \frac{m + M - t - m}{M-m}f(M) \\ &= \frac{t-m}{M-m}f(m) + \frac{M-t}{M-m}f(M). \end{aligned}$$

Using the continuous functional calculus for selfadjoint operators, we derive in the operator order that

$$f(m1_H + M1_H - A) \le \frac{A - m1_H}{M-m}f(m) + \frac{M1_H - A}{M-m}f(M),$$

which implies for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} &\langle f(m1_H + M1_H - A)x, x\rangle \\ &\le \left\langle \left[\frac{A - m1_H}{M-m}f(m) + \frac{M1_H - A}{M-m}f(M)\right]x, x\right\rangle \\ &= \frac{\langle Ax, x\rangle - m}{M-m}f(m) + \frac{M - \langle Ax, x\rangle}{M-m}f(M), \end{aligned}$$

which proves the second inequality in (2.1).

Now, observe that

$$\begin{aligned} &\frac{\langle Ax, x\rangle - m}{M-m}f(m) + \frac{M - \langle Ax, x\rangle}{M-m}f(M) \\ &= f(m) + f(M) - \left[\frac{M - \langle Ax, x\rangle}{M-m}f(m) + \frac{\langle Ax, x\rangle - m}{M-m}f(M)\right] \\ &= f(m) + f(M) - f(\langle Ax, x\rangle) \\ &- \left[\frac{M - \langle Ax, x\rangle}{M-m}f(m) + \frac{\langle Ax, x\rangle - m}{M-m}f(M) - f(\langle Ax, x\rangle)\right] \\ &= f(m) + f(M) - f(\langle Ax, x\rangle) - \Phi_{f;[m,M]}(\langle Ax, x\rangle), \end{aligned} \tag{2.4}$$

where

$$\Phi_{f;[m,M]}(t) := \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) - f(t), \ t \in [m, M].$$

Since for a convex function $f:[m,M]\to R$, we have

$$\Phi_{f;[m,M]}(t)\geq 0 \text{ for all } t\in[m,M],$$

hence

$$\Phi_{f;[m,M]}(\langle Ax,x\rangle)\geq 0,$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}&f(m)+f(M)-f(\langle Ax,x\rangle)-\Phi_{f;[m,M]}(\langle Ax,x\rangle)\\&\leq f(m)+f(M)-f(\langle Ax,x\rangle)\end{aligned}$$

for $x\in H$ with $\|x\|=1$, which proves the last part of (2.1).

The inequality (2.2) follows by observing that the difference between the extreme terms is greater than the difference between the internal ones. □

First of all, we recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}(2.5)\qquad &n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right]\\&\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\\&\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right],\end{aligned}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}\subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n:=\sum_{i=1}^{n}p_i>0$.

For $n=2$ we deduce from (2.5) that

$$\begin{aligned}(2.6)\qquad &2\min\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\&\leq t\Phi(x)+(1-t)\Phi(y)-\Phi(tx+(1-t)y)\\&\leq 2\max\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in C$ and $t\in[0,1]$.

We have the following result improving the last inequality in (2.1):

Theorem 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a continuous*

convex function on $[m,M]$, *then*

$$\begin{aligned}(2.7)\qquad & f\left(m+M-\langle Ax,x\rangle\right)\\ &\leq\left\langle f\left(m1_H+M1_H-A\right)x,x\right\rangle\\ &\leq\frac{\langle Ax,x\rangle-m}{M-m}f\left(m\right)+\frac{M-\langle Ax,x\rangle}{M-m}f\left(M\right)\\ &\leq f\left(\frac{m+M}{2}\right)+\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\langle Ax,x\rangle\right)\\ &+\frac{2}{\left(M-m\right)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\leq f\left(m\right)+f\left(M\right)-\left\langle f\left(A\right)x,x\right\rangle\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1$.

Proof. From the first inequality in (2.6) we get

$$\begin{aligned}(2.8)\qquad & 2\min\left\{\frac{M-\langle Ax,x\rangle}{M-m},1-\frac{M-\langle Ax,x\rangle}{M-m}\right\}\\ &\times\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\leq\frac{M-\langle Ax,x\rangle}{M-m}f\left(m\right)+\frac{\langle Ax,x\rangle-m}{M-m}f\left(M\right)-f\left(\langle Ax,x\rangle\right)\\ &=\Phi_{f;[m,M]}\left(\langle Ax,x\rangle\right)\end{aligned}$$

for each $x\in H$ with $\|x\|=1$.

Observe that

$$\begin{aligned}&\min\left\{\frac{M-\langle Ax,x\rangle}{M-m},1-\frac{M-\langle Ax,x\rangle}{M-m}\right\}\\ &=\min\left\{\frac{M-\langle Ax,x\rangle}{M-m},\frac{\langle Ax,x\rangle-m}{M-m}\right\}\\ &=\frac{1}{2}\left(\frac{M-\langle Ax,x\rangle}{M-m}+\frac{\langle Ax,x\rangle-m}{M-m}\right)-\frac{1}{2}\left|\frac{M-\langle Ax,x\rangle}{M-m}-\frac{\langle Ax,x\rangle-m}{M-m}\right|\\ &=\frac{1}{2}-\frac{1}{\left(M-m\right)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|,\end{aligned}$$

then

$$\begin{aligned}&2\min\left\{\frac{M-\langle Ax,x\rangle}{M-m},1-\frac{M-\langle Ax,x\rangle}{M-m}\right\}\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &=2\left(\frac{1}{2}-\frac{1}{\left(M-m\right)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right)\\ &\times\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Further, we get

$$\begin{aligned}
&f(m)+f(M)-f(\langle Ax,x\rangle)-\Phi_{f;[m,M]}(\langle Ax,x\rangle)\\
&\leq f(m)+f(M)-f(\langle Ax,x\rangle)\\
&-2\left(\frac{1}{2}-\frac{1}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right)\\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&=f(m)+f(M)-f(\langle Ax,x\rangle)-\frac{f(m)+f(M)}{2}+f\left(\frac{m+M}{2}\right)\\
&+\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&=f\left(\frac{m+M}{2}\right)+\frac{f(m)+f(M)}{2}-f(\langle Ax,x\rangle)\\
&+\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}$$

which proves the third inequality in (2.7).

Since $m\leq\langle Ax,x\rangle\leq M$ for each $x\in H$ with $\|x\|=1$, hence

$$\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m),$$

which implies that

$$\begin{aligned}
&f\left(\frac{m+M}{2}\right)+\frac{f(m)+f(M)}{2}-f(\langle Ax,x\rangle)\\
&+\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\leq f\left(\frac{m+M}{2}\right)+\frac{f(m)+f(M)}{2}-f(\langle Ax,x\rangle)+\frac{f(m)+f(M)}{2}\\
&-f\left(\frac{m+M}{2}\right)\\
&=f(m)+f(M)-f(\langle Ax,x\rangle)
\end{aligned}\tag{2.9}$$

and the last part of (2.7) is proved. □

We have the following complementary inequality as well:

Theorem 4. *With the assumptions of Theorem 2,*

$$\begin{aligned}
&\left|\frac{\langle Ax,x\rangle-m}{M-m}f(m)+\frac{M-\langle Ax,x\rangle}{M-m}f(M)-f(m+M-\langle Ax,x\rangle)\right.\\
&\left.-\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\right|\\
&\leq\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\leq\frac{f(m)+f(M)}{2}-\Phi\left(\frac{m+M}{2}\right)
\end{aligned}\tag{2.10}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. We have by the second inequality in (2.6) that

$$\begin{aligned}
&\frac{\langle Ax,x\rangle - m}{M-m} f(m) + \frac{M-\langle Ax,x\rangle}{M-m} f(M) \\
&\le f\left[\frac{\langle Ax,x\rangle - m}{M-m} m + \frac{M-\langle Ax,x\rangle}{M-m} M\right] \\
&+ 2\max\left\{\frac{\langle Ax,x\rangle - m}{M-m}, 1-\frac{\langle Ax,x\rangle - m}{M-m}\right\}\left[\frac{f(m)+f(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&= f(m+M-\langle Ax,x\rangle) \\
&+ 2\max\left\{\frac{\langle Ax,x\rangle - m}{M-m}, \frac{M-\langle Ax,x\rangle}{M-m}\right\}\left[\frac{f(m)+f(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

This implies the second inequality in the following equation (2.11)

$$\begin{aligned}
(0 \le)\, &2\min\left\{\frac{\langle Ax,x\rangle - m}{M-m}, \frac{M-\langle Ax,x\rangle}{M-m}\right\} \\
&\times\left[\frac{f(m)+f(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\le \frac{\langle Ax,x\rangle - m}{M-m} f(m) + \frac{M-\langle Ax,x\rangle}{M-m} f(M) - f(m+M-\langle Ax,x\rangle) \\
&\le 2\max\left\{\frac{\langle Ax,x\rangle - m}{M-m}, \frac{M-\langle Ax,x\rangle}{M-m}\right\} \\
&\times\left[\frac{f(m)+f(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right],
\end{aligned} \tag{2.11}$$

for each $x \in H$ with $\|x\| = 1$. The first inequality in (2.11) follows by the first inequality in (2.6).

Now, observe that

$$\begin{aligned}
&2\min\left\{\frac{M-\langle Ax,x\rangle}{M-m}, 1-\frac{M-\langle Ax,x\rangle}{M-m}\right\}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&= \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&- \frac{2}{(M-m)}\left|\langle Ax,x\rangle - \frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&2\max\left\{\frac{M-\langle Ax,x\rangle}{M-m}, 1-\frac{M-\langle Ax,x\rangle}{M-m}\right\}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&= \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&+ \frac{2}{(M-m)}\left|\langle Ax,x\rangle - \frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

Then by (2.11) we get

$$
\begin{aligned}
&-\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\le\frac{\langle Ax,x\rangle-m}{M-m}f(m)+\frac{M-\langle Ax,x\rangle}{M-m}f(M)-f(m+M-\langle Ax,x\rangle)\\
&-\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\le\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{f(m)+f(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]
\end{aligned}
$$

for each $x\in H$ with $\|x\|=1$, which is equivalent to (2.10). □

In [11] we obtained among others the following two point Taylor's type representation:

Lemma 1. *Let $f:I\to\mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n\ge1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct $x,\ a,\ b\in\mathring{I}$ we have*

$$
\begin{aligned}
f(x)&=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+\frac{(b-x)(x-a)}{b-a}\\
&\quad\times\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\}\\
&\quad+L_n(x,a,b),
\end{aligned}\tag{2.12}
$$

where

$$
\begin{aligned}
&L_n(x,a,b)\\
&:=\frac{(b-x)(x-a)}{n!(b-a)}\left[(x-a)^n\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds\right.\\
&\left.+(-1)^{n+1}(b-x)^n\int_0^1 f^{(n+1)}((1-s)x+sb)s^n\,ds\right]
\end{aligned}\tag{2.13}
$$

The case $n=0$, namely when the function f is locally absolutely continuous on $\mathring{I}$ with the derivative f' existing almost everywhere in $\mathring{I}$ is important and produces the following simple identity for each distinct $x,\ a,\ b\in\mathring{I}$

$$
f(x)=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+L(x,a,b),\tag{2.14}
$$

where

$$
\begin{aligned}
&L(x,a,b)\\
&:=\frac{(b-x)(x-a)}{b-a}\left[\int_0^1 f'((1-s)a+sx)\,ds-\int_0^1 f'((1-s)x+sb)\,ds\right].
\end{aligned}\tag{2.15}
$$

Using these facts we can improve inequality (2.1) as follows:

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars $m,\ M$ with $m<M$. If f is a twice differentiable function on $[m,M]$ and such that*

$$
f''(t)\ge k \text{ for all } t\in(m,M)\tag{2.16}
$$

for some $k>0$, *then*

$$
\begin{aligned}
(2.17)\qquad & f\left(m+M-\langle Ax,x\rangle\right)\\
&\leq \langle f\left(m1_H+M1_H-A\right)x,x\rangle\\
&\leq \frac{\langle Ax,x\rangle-m}{M-m}f\left(m\right)+\frac{M-\langle Ax,x\rangle}{M-m}f\left(M\right)\\
&\leq f\left(m\right)+f\left(M\right)-f\left(\langle Ax,x\rangle\right)-\frac{1}{2}k\left(M-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-m\right)\\
&\leq f\left(m\right)+f\left(M\right)-f\left(\langle Ax,x\rangle\right)
\end{aligned}
$$

for each $x\in H$ *with* $\|x\|=1$.

Proof. From (2.14) and (2.15) we get

$$
\begin{aligned}
(2.18)\qquad & \frac{1}{M-m}\left[\left(M-t\right)f\left(m\right)+\left(t-m\right)f\left(M\right)\right]-f\left(t\right)\\
&=\frac{\left(M-t\right)\left(t-m\right)}{M-m}\\
&\times\left[\int_0^1 f'\left(\left(1-s\right)t+sM\right)ds-\int_0^1 f'\left(\left(1-s\right)m+st\right)ds\right]\\
&=\frac{\left(M-t\right)\left(t-m\right)}{M-m}\\
&\times\left[\int_0^1 f'\left(st+\left(1-s\right)M\right)ds-\int_0^1 f'\left(\left(1-s\right)m+st\right)ds\right].
\end{aligned}
$$

Since f is twice differentiable and satisfies the condition (2.16), hence

$$
\begin{aligned}
&\int_0^1 f'\left(st+\left(1-s\right)M\right)ds-\int_0^1 f'\left(\left(1-s\right)m+st\right)ds\\
&=\int_0^1\left(\int_{(1-s)m+st}^{st+(1-s)M} f''\left(u\right)du\right)ds\geq\int_0^1\left(\int_{(1-s)m+st}^{st+(1-s)M} kdu\right)ds\\
&=k\left(M-m\right)\int_0^1\left(1-s\right)ds=\frac{1}{2}k\left(M-m\right).
\end{aligned}
$$

Therefore

$$
\frac{1}{M-m}\left[\left(M-t\right)f\left(m\right)+\left(t-m\right)f\left(M\right)\right]-f\left(t\right)\geq\frac{1}{2}k\left(M-t\right)\left(t-m\right)
$$

for $t\in[m,M]$.

Using this inequality, we get

$$
\Phi_{f;[m,M]}\left(\langle Ax,x\rangle\right)\geq\frac{1}{2}k\left(M-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-m\right)
$$

for each $x\in H$ with $\|x\|=1$.

We then get

$$
\begin{aligned}
&f\left(m\right)+f\left(M\right)-f\left(\langle Ax,x\rangle\right)-\Phi_{f;[m,M]}\left(\langle Ax,x\rangle\right)\\
&\leq f\left(m\right)+f\left(M\right)-f\left(\langle Ax,x\rangle\right)-\frac{1}{2}k\left(M-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-m\right)\\
&\leq f\left(m\right)+f\left(M\right)-f\left(\langle Ax,x\rangle\right)
\end{aligned}
$$

for each $x\in H$ with $\|x\|=1$ and by (2.4) we derive the desired result (2.17). □

Remark 1. *With the assumptions of Theorem 5, from (2.17) we derive the following inequalities as well*

$$0 \le f(m) + f(M) - f(\langle Ax, x\rangle) - \langle f(m1_H + M1_H - A)x, x\rangle - \frac{1}{2}k(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m) \le f(m) + f(M) - f(\langle Ax, x\rangle) - f(m + M - \langle Ax, x\rangle) \tag{2.19}$$

and

$$0 \le \frac{M - \langle Ax, x\rangle}{M - m} f(m) + \frac{\langle Ax, x\rangle - m}{M - m} f(M) - f(\langle Ax, x\rangle) - \frac{1}{2}k(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m) \le f(m) + f(M) - f(\langle Ax, x\rangle) - f(m + M - \langle Ax, x\rangle) \tag{2.20}$$

for each $x \in H$ with $\|x\| = 1$.

3. Some Examples

Assume that the selfadjoint operator A is such that $\mathrm{Sp}(A) \subseteq [m, M] \subset (0, \infty)$. Consider the convex function $f(x) = -\ln x$, $x > 0$, then by (2.1) and (2.2) we get

$$\ln(m + M - \langle Ax, x\rangle) \ge \langle \ln(m1_H + M1_H - A)x, x\rangle \ge \frac{\langle Ax, x\rangle - m}{M - m}\ln(m) + \frac{M - \langle Ax, x\rangle}{M - m}\ln(M) \ge \ln(m) + \ln(M) - \langle \ln(A)x, x\rangle \tag{3.1}$$

and

$$0 \le \langle \ln(m1_H + M1_H - A)x, x\rangle - \frac{\langle Ax, x\rangle - m}{M - m}\ln(m) - \frac{M - \langle Ax, x\rangle}{M - m}\ln(M) \le \langle \ln(A)x, x\rangle + \ln(m + M - \langle Ax, x\rangle) - \ln(m) - \ln(M) \tag{3.2}$$

for each $x \in H$ with $\|x\| = 1$.

These inequalities are equivalent to

$$m + M - \langle Ax, x\rangle \ge \exp\langle \ln(m1_H + M1_H - A)x, x\rangle \ge m^{\frac{\langle Ax, x\rangle - m}{M - m}} M^{\frac{M - \langle Ax, x\rangle}{M - m}} \ge \frac{mM}{\exp\langle \ln(A)x, x\rangle} \tag{3.3}$$

and

$$\frac{(m + M - \langle Ax, x\rangle)\exp\langle \ln(A)x, x\rangle}{mM} \ge \frac{\exp\langle \ln(m1_H + M1_H - A)x, x\rangle}{m^{\frac{\langle Ax, x\rangle - m}{M - m}} M^{\frac{M - \langle Ax, x\rangle}{M - m}}} \ge 1 \tag{3.4}$$

for each $x \in H$ with $\|x\| = 1$.

From (2.7) we get

$$\begin{aligned}
&\ln(m+M-\langle Ax,x\rangle) \\
&\geq \langle \ln(m1_H+M1_H-A)x,x\rangle \\
&\geq \frac{\langle Ax,x\rangle - m}{M-m}\ln(m)+\frac{M-\langle Ax,x\rangle}{M-m}\ln(M) \\
&\geq \ln\left(\frac{m+M}{2}\right)+\frac{\ln(m)+f(M)}{2}-\ln(\langle Ax,x\rangle) \\
&+\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{\ln(m)+\ln(M)}{2}-\ln\left(\frac{m+M}{2}\right)\right] \\
&\geq \ln(m)+\ln(M)-\langle f(A)x,x\rangle
\end{aligned} \tag{3.5}$$

and from (2.17) we obtain

$$\begin{aligned}
&\ln(m+M-\langle Ax,x\rangle) \\
&\geq \langle \ln(m1_H+M1_H-A)x,x\rangle \\
&\geq \frac{\langle Ax,x\rangle - m}{M-m}\ln(m)+\frac{M-\langle Ax,x\rangle}{M-m}\ln(M) \\
&\geq \ln(m)+\ln(M)-\ln(\langle Ax,x\rangle)-\frac{1}{2M^2}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m) \\
&\geq \ln(m)+\ln(M)-\ln(\langle Ax,x\rangle)
\end{aligned} \tag{3.6}$$

for each $x \in H$ with $\|x\| = 1$.

Consider the function $f : [m,M] \subset (0,\infty) \to \mathbb{R}$, $f(x) = x^r$ which is convex for $r \in (-\infty,0) \cup [1,\infty)$ and concave for $r \in (0,1)$. If we use (2.1) and (2.2), then we get for $r \in (-\infty,0) \cup [1,\infty)$

$$\begin{aligned}
(m+M-\langle Ax,x\rangle)^r &\leq \langle (m1_H+M1_H-A)^r x,x\rangle \\
&\leq \frac{\langle Ax,x\rangle - m}{M-m}m^r+\frac{M-\langle Ax,x\rangle}{M-m}M^r \\
&\leq m^r+M^r-\langle A^r x,x\rangle
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
0 &\leq \frac{\langle Ax,x\rangle - m}{M-m}m^r+\frac{M-\langle Ax,x\rangle}{M-m}M^r-\langle (m1_H+M1_H-A)^r x,x\rangle \\
&\leq m^r+M^r-\langle A^r x,x\rangle-(m+M-\langle Ax,x\rangle)^r
\end{aligned} \tag{3.8}$$

for each $x \in H$ with $\|x\| = 1$.

From (2.7) we get

$$\begin{aligned}
&(m+M-\langle Ax,x\rangle)^r \\
&\leq \langle (m1_H+M1_H-A)^r x,x\rangle \\
&\leq \frac{\langle Ax,x\rangle - m}{M-m}m^r+\frac{M-\langle Ax,x\rangle}{M-m}M^r \\
&\leq \left(\frac{m+M}{2}\right)^r+\frac{m^r+M^r}{2}-\langle Ax,x\rangle^r \\
&+\frac{2}{(M-m)}\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right] \\
&\leq m^r+M^r-\langle A^r x,x\rangle
\end{aligned} \tag{3.9}$$

for each $x \in H$ with $\|x\| = 1$, while from (2.17) we get

$$(3.10) \qquad \begin{aligned} &(m + M - \langle Ax, x\rangle)^r \\ &\leq \langle (m1_H + M1_H - A)^r x, x\rangle \\ &\leq \frac{\langle Ax, x\rangle - m}{M - m} m^r + \frac{M - \langle Ax, x\rangle}{M - m} M^r \\ &\leq m^r + M^r - \langle Ax, x\rangle^r - \frac{1}{2} k_r (M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m) \\ &\leq m^r + M^r - \langle Ax, x\rangle^r , \end{aligned}$$

where

$$\gamma_r := r(r-1) \times \begin{cases} m^{r-2} \text{ if } r \geq 2, \\ \\ M^{r-2} \text{if } r \in (-\infty, 0) \cup [1, 2). \end{cases}$$

The interested reader may apply the above inequalities for other convex functions such as $f(x) = x \ln x,\ x \in [m, M] \subset (0, \infty)$ or $f(x) = \exp(\alpha x),\ \alpha \in \mathbb{R}$ and $x \in [m, M]$.

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