SOME FUNCTIONALS ASSOCIATED TO JESSEN AND HERMITE-HADAMARD INEQUALITIES

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Abstract. In this paper we study the properties of superadditivity and monotonicity of some functionals associated to Jessen and Hermite-Hadamard inequalities for isotonic functionals and convex functions. The obtained results incorporate both the case of functionals depending on weights as well as the case of dependence of subsets.

1. Introduction

Let L be a linear class of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) $f,\ g\in L$ imply $(\alpha f+\beta g)\in L$ for all $\alpha,\beta\in\mathbb{R}$;
(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1,\ t\in E$ then $f_0\in L$.

An *isotonic linear functional* $A:L\to\mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$.
(A2) If $f\in L$ and $f\geq 0$, then $A(f)\geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1})=1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [1] and [16]).

We note that common examples of such isotonic linear functionals A are given by

$$A(g)=\int_E g\,d\mu \text{ or } A(g)=\sum_{k\in E}p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second $(p_k\geq 0,k\in E)$.

We recall Jessen's inequality (see also [12]).

Theorem 1. *Let $\phi:I\subseteq\mathbb{R}\to\mathbb{R}$ (I is an interval), be a convex function and $f:E\to I$ such that $\phi\circ f,\ f\in L$. If $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(A(f))\leq A(\phi\circ f). \tag{1.1}$$

A counterpart of this result was proved by Beesack and Pečarić in [1] for compact intervals $I=[\alpha,\beta]$.

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Theorem 2. *Let $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to [\alpha, \beta]$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi \circ f) \leq \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta). \tag{1.2}$$

Remark 1. *Note that (1.2) is a generalization of the inequality*

$$A(\phi) \leq \frac{b - A(e_1)}{b - a} \phi(a) + \frac{A(e_1) - a}{b - a} \phi(b) \tag{1.3}$$

due to Lupaş [14] *(see for example* [1, Theorem A]*), which assumed $E = [a, b]$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A(\mathbf{1}) = 1$, ϕ is convex on E and $\phi \in L$, $e_1 \in L$, where $e_1(x) = x$, $x \in [a, b]$.*

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(t)\, dt \leq \frac{\varphi(a) + \varphi(b)}{2}, \tag{1.4}$$

provided that $\varphi : [a, b] \to \mathbb{R}$ is a convex function.

Using Theorem 1 and Theorem 2, we may state the following generalization of the Hermite-Hadamard inequality for isotonic linear functionals ([17] and [18]).

Theorem 3. *Let $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $e : E \to [a, b]$ with e, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional, with $A(e) = \frac{a+b}{2}$, then*

$$\varphi\left(\frac{a+b}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(a) + \varphi(b)}{2}. \tag{1.5}$$

For other results concerning convex functions and isotonic linear functionals, see [2] – [12], [20] and the recent monograph [4].

In this paper we study the properties of superadditivity and monotonicity of some functionals associated to Jessen and Hermite-Hadamard inequalities for isotonic functionals and convex functions. The obtained results incorporate both the case of functionals depending on weights as well as the case of dependence of subsets.

2. Jessen Inequality Related Functionals

We denote by $\mathcal{IL}(L)$ the set of all isotonic linear functionals defined on L and with real values. We observe that $\mathcal{IL}(L)$ is a cone in the linear space of all functionals defined on L and with real values, namely, if $A, B \in \mathcal{IL}(L)$, then $A + B \in \mathcal{IL}(L)$ and $\alpha A \in \mathcal{IL}(L)$ for all $\alpha \geq 0$. We also assume by default that $0 \in \mathcal{IL}(L)$.

We define on $\mathcal{IL}(L)$ the following order relation $A \succeq B$, where $A, B \in \mathcal{IL}(L)$ if $A - B \in \mathcal{IL}(L)$.

We also define $\mathcal{IL}(L)_{\mathbf{1}}$ the sub-cone of $\mathcal{IL}(L)$ for which all functionals A satisfy the condition $A(\mathbf{1}) > 0$. Obviously normalized functionals are in this cone.

We define the following functional associated to Jessen's inequality $\Psi_{\phi,f} : \mathcal{IL}(L)_{\mathbf{1}} \to [0, \infty)$ by

$$\Psi_{\phi,f}(A) = A(\phi \circ f) - A(\mathbf{1}) \phi\left(\frac{A(f)}{A(\mathbf{1})}\right) \geq 0, \tag{2.1}$$

where $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $f : E \to I$ is such that $\phi \circ f$, $f \in L$.

The positivity follows by Jessen's inequality (1.1) for the isotonic functional $B = \frac{1}{A(\mathbf{1})}A$.

Theorem 4. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$, be a convex function and $f : E \to I$ such that $\phi \circ f,\ f \in L$.*

(i) *For all $A,\ B \in \mathcal{IL}(L)_{\mathbf{1}}$,*

$$\Psi_{\phi,f}(A+B) \geq \Psi_{\phi,f}(A) + \Psi_{\phi,f}(B) \geq 0, \tag{2.2}$$

namely $\Psi_{\phi,f}$ is superadditive on $\mathcal{IL}(L)_{\mathbf{1}}$.

(ii) *For all $A,\ B \in \mathcal{IL}(L)_{\mathbf{1}}$ with $A \succeq B$ and $A(\mathbf{1}) > B(\mathbf{1})$,*

$$\Psi_{\phi,f}(A) - \Psi_{\phi,f}(B) \geq \Psi_{\phi,f}(A-B) \geq 0, \tag{2.3}$$

which also shows that $\Psi_{\phi,f}$ is monotonic nondecreasing on $\mathcal{IL}(L)_{\mathbf{1}}$.

Proof. (i). We have for $A,\ B \in \mathcal{IL}(L)_{\mathbf{1}}$,

$$\begin{aligned} &\Psi_{\phi,f}(A+B) \\ &= (A+B)(\phi \circ f) - (A+B)(\mathbf{1})\,\phi\left(\frac{(A+B)(f)}{(A+B)(\mathbf{1})}\right) \\ &= A(\phi \circ f) + B(\phi \circ f) - (A+B)(\mathbf{1})\,\phi\left(\frac{(A+B)(f)}{(A+B)(\mathbf{1})}\right). \end{aligned} \tag{2.4}$$

By the convexity of f we have

$$\begin{aligned} \phi\left(\frac{(A+B)(f)}{(A+B)(\mathbf{1})}\right) &= \phi\left(\frac{A(f)+B(f)}{A(\mathbf{1})+B(\mathbf{1})}\right) \\ &= \phi\left(\frac{A(\mathbf{1})\frac{A(f)}{A(\mathbf{1})} + B(\mathbf{1})\frac{B(f)}{B(\mathbf{1})}}{A(\mathbf{1})+B(\mathbf{1})}\right) \\ &\leq \frac{A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) + B(\mathbf{1})\,\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)}{A(\mathbf{1})+B(\mathbf{1})}. \end{aligned}$$

Therefore

$$\begin{aligned} &-(A+B)(\mathbf{1})\,\phi\left(\frac{(A+B)(f)}{(A+B)(\mathbf{1})}\right) \\ &\geq -(A+B)(\mathbf{1})\left[\frac{A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) + B(\mathbf{1})\,\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)}{A(\mathbf{1})+B(\mathbf{1})}\right] \\ &= -\left[A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) + B(\mathbf{1})\,\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]. \end{aligned} \tag{2.5}$$

By utilising (2.4) and (2.5) we derive

$$\begin{aligned}\Psi_{\phi,f}(A+B) &\geq A(\phi\circ f)+B(\phi\circ f)\\ &-\left[A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)+B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]\\ &=A(\phi\circ f)-A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\\ &+B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\\ &=\Psi_{\phi,f}(A)+\Psi_{\phi,f}(B),\end{aligned}$$

which proves (2.2).

(ii) Follows by (2.2) by the superadditivity of $\Psi_{\phi,f}$ applied for the functionals $A-B$ and B which are in $\mathcal{IL}(L)_{\mathbf{1}}$. □

Corollary 1. *Let $\phi:I\subseteq\mathbb{R}\to\mathbb{R}$, be a convex function and $f:E\to I$ such that $\phi\circ f,\ f\in L$. Assume that $M>m>0$ and $A,\ B\in\mathcal{IL}(L)_{\mathbf{1}}$ with*

$$MB\succeq A\succeq mB, \tag{2.6}$$

then

$$\begin{aligned}M\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right] &\geq A(\phi\circ f)-A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\\ &\geq m\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right].\end{aligned} \tag{2.7}$$

Proof. By (2.3) we get for $A\succeq mB$,

$$\begin{aligned}\Psi_{\phi,f}(A) &= A(\phi\circ f)-A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\\ &\geq \Psi_{\phi,f}(mB)=mB(\phi\circ f)-mB(\mathbf{1})\phi\left(\frac{mB(f)}{mB(\mathbf{1})}\right)\\ &=m\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right],\end{aligned}$$

which proves the second inequality in (2.7).

The first inequality goes in a similar way. □

Remark 2. *Let $\phi:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $f:E\to I$, $w:E\to[0,\infty)$ be such that $w(\phi\circ f)$, wf, $w\in L$. Let $U\in\mathcal{IL}(L)$ with $U(w)>0$. The weighted functional $U_w(f):=U(wf)$ is in $\mathcal{IL}(L)_{\mathbf{1}}$ with $U_w(\mathbf{1})=U(w)$. For given ϕ, f, U as above, we define $L_{\phi,U,f}$ to be the set of all nonnegative weights from L which satisfy the previous conditions.*

We define the functional of weights $\Delta_{\phi,U,f}:L_{\phi,U,f}\to[0,\infty)$ by

$$\Delta_{\phi,U,f}(w)=U(w(\phi\circ f))-U(w)\phi\left(\frac{U(wf)}{U(w)}\right)=\Psi_{\phi,f}(U_w)\geq 0. \tag{2.8}$$

If $u,\ v\in L_{\phi,U,f}$, then by (2.2) we get the superadditivity property as a function of weights

$$\Delta_{\phi,U,f}(u+v)\geq\Delta_{\phi,U,f}(u)+\Delta_{\phi,U,f}(v)\geq 0. \tag{2.9}$$

If $u, v \in L_{\phi,U,f}$ with $u \geq v$ in L then by (2.3) we have the monotonicity property

$$\Delta_{\phi,U,f}(u) \geq \Delta_{\phi,U,f}(v) \geq 0. \tag{2.10}$$

Also if $Mv \geq u \geq mv$ with $M > m > 0$ we have the bounds

$$M\Delta_{\phi,U,f}(v) \geq \Delta_{\phi,U,f}(u) \geq m\Delta_{\phi,U,f}(v) \geq 0. \tag{2.11}$$

The weights superadditivity and monotonicity properties in the discrete case were obtained in [13] *while the boundedness in the same case was established in* [6]*. The integral case for all three properties was investigated in* [8] *while the case of weighted positive isotonic functionals was considered in* [15]*.*

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;
(ii) $x \in C$, $\alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is called *superadditive (subadditive)* on C if

(iii) $h(x + y) \geq (\leq) h(x) + h(y)$ for any $x, y \in C$

and *nonnegative (strictly positive)* on C if, obviously, it satisfies

(iv) $h(x) \geq (>) 0$ for each $x \in C$.

The functional h is *s-positive homogeneous* on C, for a given $s > 0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

The following result provides the quasilinearity property of a composite functional when one of the components is additive while the other is superadditive [7].

Lemma 1. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $p \geq 1$ then the functional*

$$\mathcal{F}_p : C \to [0, \infty), \ \mathcal{F}_p(x) = v^{1-\frac{1}{p}}(x)\, h(x) \tag{2.12}$$

is superadditive on C.

We have:

Corollary 2. *Assume that X, C and v are as in Lemma 1. If $h : C \to [0, \infty)$ is a superadditive functional on C and $p, q \geq 1$ then the two parameters functional*

$$\mathcal{F}_{p,q} : C \to [0, \infty), \ \mathcal{F}_{p,q}(x) = v^{q\left(1-\frac{1}{p}\right)}(x)\, h^q(x) \tag{2.13}$$

is superadditive on C.

Remark 3. *If we consider the functional $\psi_p(x) := v^{p-1}(x)\, h^p(x)$ then for $p \geq 1$ and $h : C \to [0, \infty)$ a superadditive functional on C, the functional ψ_p is also superadditive on C.*

The following result providing upper and lower bounds for a value of the functional $\mathcal{F}_{p,q}$ in the case when the composite functionals are homogeneous can be stated as well:

Corollary 3. *Let $x, y \in C$, $h : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C and v an additive, strictly positive and positive homogeneous functional on C. If $p, q \geq 1$ and $M \geq m \geq 0$ are such that $x - my$ and $My - x \in C$, then*

$$M^{sq+q\left(1-\frac{1}{p}\right)}\mathcal{F}_{p,q}(y) \geq \mathcal{F}_{p,q}(x) \geq m^{sq+q\left(1-\frac{1}{p}\right)}\mathcal{F}_{p,q}(y). \tag{2.14}$$

In particular,

$$M^{(s+1)p-1}\psi_p(y) \geq \psi_p(x) \geq m^{(s+1)p-1}\psi_p(y) \tag{2.15}$$

and

$$M^{s+1-\frac{1}{p}}\digamma_p(y) \geq \digamma_p(x) \geq m^{s+1-\frac{1}{p}}\digamma_p(y), \tag{2.16}$$

where ψ_p *and* $\digamma_p$ *are defined as above.*

We consider the functional $h : \mathcal{IL}(L)_{\mathbf{1}} \to [0, \infty)$ defined by

$$h(A) = \Psi_{\phi,f}(A) = A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) \geq 0, \tag{2.17}$$

and $v : \mathcal{IL}(L)_{\mathbf{1}} \to (0, \infty)$, $v(A) = A(\mathbf{1}) > 0$. Then h is nonnegative and a superadditive functional on $\mathcal{IL}(L)_{\mathbf{1}}$. Also v is positive and additive on $\mathcal{IL}(L)_{\mathbf{1}}$.

Using the above results, we can also state the following facts for composite functionals:

Corollary 4. *Let* $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function and* $f : E \to I$ *such that* $\phi \circ f,\ f \in L$.

The functional $\digamma_{p,\phi,f} : \mathcal{IL}(L)_{\mathbf{1}} \to [0, \infty)$

$$\digamma_{p,\phi,f}(A) = A^{1-\frac{1}{p}}(\mathbf{1})\left[A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right] \tag{2.18}$$

is superadditive on $\mathcal{IL}(L)_{\mathbf{1}}$ *for all* $p \geq 1$.

The functional $\digamma_{p,q,\phi,f} : \mathcal{IL}(L)_{\mathbf{1}} \to [0, \infty)$

$$\digamma_{p,q,\phi,f}(A) = A^{q\left(1-\frac{1}{p}\right)}(\mathbf{1})\left[A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right]^q \tag{2.19}$$

is superadditive on $\mathcal{IL}(L)_{\mathbf{1}}$ *for all* $p,\ q \geq 1$.

Remark 4. *If we consider the functional* $\psi_{p,\phi,f} : \mathcal{IL}(L)_{\mathbf{1}} \to [0, \infty)$,

$$\psi_{p,\phi,f}(A) := A^{p-1}(\mathbf{1})\left[A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right]^p,$$

then $\psi_{p,\phi,f}$ *is superadditive on* $\mathcal{IL}(L)_{\mathbf{1}}$ *for* $p \geq 1$.

Corollary 5. *Let* $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function and* $f : E \to I$ *such that* $\phi \circ f,\ f \in L$. *Assume that* $M > m > 0$ *and* $A,\ B \in \mathcal{IL}(L)_{\mathbf{1}}$ *with*

$$MB \succeq A \succeq mB,$$

then

$$\begin{aligned} &M^{q\left(2-\frac{1}{p}\right)}B^{q\left(1-\frac{1}{p}\right)}(\mathbf{1})\left[B(\phi \circ f) - B(\mathbf{1})\,\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]^q \\ &\geq A^{q\left(1-\frac{1}{p}\right)}(\mathbf{1})\left[A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right]^q \\ &\geq m^{\left(2-\frac{1}{p}\right)}B^{q\left(1-\frac{1}{p}\right)}(\mathbf{1})\left[B(\phi \circ f) - B(\mathbf{1})\,\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]^q, \end{aligned} \tag{2.20}$$

$$\begin{aligned}(2.21)\qquad & M^{2p-1}B^{p-1}(\mathbf{1})\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]^{p} \\ & \geq A^{p-1}(\mathbf{1})\left[A(\phi\circ f)-A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right]^{p} \\ & \geq m^{2p-1}B^{p-1}(\mathbf{1})\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right]^{p}\end{aligned}$$

and

$$\begin{aligned}(2.22)\qquad & M^{2-\frac{1}{p}}B^{1-\frac{1}{p}}(\mathbf{1})\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right] \\ & \geq A^{1-\frac{1}{p}}(\mathbf{1})\left[A(\phi\circ f)-A(\mathbf{1})\phi\left(\frac{A(f)}{A(\mathbf{1})}\right)\right] \\ & \geq m^{2-\frac{1}{p}}B^{1-\frac{1}{p}}(\mathbf{1})\left[B(\phi\circ f)-B(\mathbf{1})\phi\left(\frac{B(f)}{B(\mathbf{1})}\right)\right].\end{aligned}$$

3. HH-Inequality Related Functionals

The following lemma holds [18]:

Lemma 2. *Let X be a real linear space and C its convex subset. Then the following statements are equivalent for a mapping $\phi : X \to \mathbb{R}$:*

- *(i) ϕ is convex on C;*
- *(ii) for all $x,\ y \in C$ the mapping $g_{x,y} : [0,1] \to \mathbb{R}$, $g_{x,y}(t) := \phi(tx + (1-t)y)$ is convex on $[0,1]$.*

The following generalization of Hermite-Hadamard's inequality for isotonic linear functionals was obtained in [18]:

Proposition 1. *Let $\phi : C \subseteq X \to \mathbb{R}$ be a convex function on C, L and A satisfy conditions L1, L2 and A1, A2, and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1,\ h \in L$ is such that $g_{x,y} \circ h \in L$ for $x,\ y$ given in C. If $A(\mathbf{1}) = 1$, then we have the inequality*

$$\begin{aligned}(3.1)\qquad \phi(A(h)x + (1-A(h))y) &\leq A[\phi(hx + (\mathbf{1}-h)y)] \\ &\leq A(h)\phi(x) + (1-A(h))\phi(y).\end{aligned}$$

Remark 5. *If $h : E \to [0,1]$ is such that $A(h) = \frac{1}{2}$, we get from the inequality* (3.1) *that*

$$(3.2)\qquad \phi\left(\frac{x+y}{2}\right) \leq A[\phi(hx + (\mathbf{1}-h)y)] \leq \frac{\phi(x)+\phi(y)}{2},$$

for all $x,\ y$ in C.

We consider the functional $\Psi_{\phi,h,x,y} : \mathcal{IL}(L)_{\mathbf{1}} \to [0,\infty)$ defined by

$$(3.3)\quad \Psi_{\phi,h,x,y}(A) := A[\phi(hx + (\mathbf{1}-h)y)] - A(\mathbf{1})\phi\left(\frac{A(h)x + A(\mathbf{1}-h)y}{A(\mathbf{1})}\right) \geq 0,$$

where $\phi : C \subseteq X \to \mathbb{R}$ is a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1,\ h \in L$ is such that $g_{x,y} \circ h \in L$ for $x,\ y$ given in C.

Theorem 5. *Let $\phi : C \subseteq X \to \mathbb{R}$ be a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1,\ h \in L$ is such that $g_{x,y} \circ h \in L$ for $x,\ y$ given in C.*

(i) *For all* $A,\ B\in\mathcal{IL}(L)_{\mathbf{1}}$,

$$\Psi_{\phi,h,x,y}(A+B)\geq\Psi_{\phi,h,x,y}(A)+\Psi_{\phi,h,x,y}(B)\geq 0, \tag{3.4}$$

namely $\Psi_{\phi,h,x,y}$ *is superadditive on* $\mathcal{IL}(L)_{\mathbf{1}}$.

(ii) *For all* $A,\ B\in\mathcal{IL}(L)_{\mathbf{1}}$ *with* $A\succeq B$ *and* $A(\mathbf{1})>B(\mathbf{1})$,

$$\Psi_{\phi,h,x,y}(A)-\Psi_{\phi,h,x,y}(B)\geq\Psi_{\phi,h,x,y}(A-B)\geq 0, \tag{3.5}$$

which also shows that $\Psi_{\phi,h,x,y}$ *is monotonic nondecreasing on* $\mathcal{IL}(L)_{\mathbf{1}}$.

Proof. (i). We have for $A,\ B\in\mathcal{IL}(L)_{\mathbf{1}}$,

$$\begin{aligned}
&\Psi_{\phi,h,x,y}(A+B)\\
&=(A+B)[\phi(hx+(\mathbf{1}-h)y)]\\
&-(A+B)(\mathbf{1})\phi\left(\frac{(A+B)(h)x+(A+B)(\mathbf{1}-h)y}{(A+B)(\mathbf{1})}\right)\\
&=A[\phi(hx+(\mathbf{1}-h)y)]+B[\phi(hx+(\mathbf{1}-h)y)]\\
&-(A(\mathbf{1})+B(\mathbf{1}))\phi\left(\frac{A(h)x+A(\mathbf{1}-h)y+B(h)x+B(\mathbf{1}-h)y}{A(\mathbf{1})+B(\mathbf{1})}\right)\\
&=A[\phi(hx+(\mathbf{1}-h)y)]+B[\phi(hx+(\mathbf{1}-h)y)]\\
&-(A(\mathbf{1})+B(\mathbf{1}))\phi\left(\frac{A(\mathbf{1})\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}+B(\mathbf{1})\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}}{A(\mathbf{1})+B(\mathbf{1})}\right).
\end{aligned}$$

By the convexity of ϕ and since

$$\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})},\ \frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}\in C$$

and

$$\frac{A(\mathbf{1})\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}+B(\mathbf{1})\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}}{A(\mathbf{1})+B(\mathbf{1})}\in C,$$

hence

$$\begin{aligned}
&\phi\left(\frac{A(\mathbf{1})\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}+B(\mathbf{1})\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}}{A(\mathbf{1})+B(\mathbf{1})}\right)\\
&\leq\frac{A(\mathbf{1})\phi\left(\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}\right)+B(\mathbf{1})\phi\left(\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}\right)}{A(\mathbf{1})+B(\mathbf{1})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Psi_{\phi,h,x,y}(A+B)-\Psi_{\phi,h,x,y}(A)-\Psi_{\phi,h,x,y}(B)\\
&=A(\mathbf{1})\,\phi\left(\frac{A(h)\,x+A(\mathbf{1}-h)\,y}{A(\mathbf{1})}\right)+B(\mathbf{1})\,\phi\left(\frac{B(h)\,x+B(\mathbf{1}-h)\,y}{B(\mathbf{1})}\right)\\
&-(A(\mathbf{1})+B(\mathbf{1}))\,\phi\left(\frac{A(\mathbf{1})\,\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}+B(\mathbf{1})\,\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}}{A(\mathbf{1})+B(\mathbf{1})}\right).\\
&\geq A(\mathbf{1})\,\phi\left(\frac{A(h)\,x+A(\mathbf{1}-h)\,y}{A(\mathbf{1})}\right)+B(\mathbf{1})\,\phi\left(\frac{B(h)\,x+B(\mathbf{1}-h)\,y}{B(\mathbf{1})}\right)\\
&-(A(\mathbf{1})+B(\mathbf{1}))\left[\frac{A(\mathbf{1})\,\phi\left(\frac{A(h)x+A(\mathbf{1}-h)y}{A(\mathbf{1})}\right)+B(\mathbf{1})\,\phi\left(\frac{B(h)x+B(\mathbf{1}-h)y}{B(\mathbf{1})}\right)}{A(\mathbf{1})+B(\mathbf{1})}\right]\\
&=0,
\end{aligned}$$

which proves (3.4).

(ii) Follows by (3.4) by the superadditivity of $\Psi_{\phi,f}$ applied for the functionals $A-B$ and B which are in $\mathcal{IL}(L)_{\mathbf{1}}$. □

Corollary 6. *With the assumptions of Theorem 5 and if $A,\ B\in\mathcal{IL}(L)_{\mathbf{1}}$ with*

$$MB\succeq A\succeq mB,$$

then

$$M\Psi_{\phi,h,x,y}(B)\geq\Psi_{\phi,h,x,y}(A)\geq m\Psi_{\phi,h,x,y}(B)\geq 0. \tag{3.6}$$

To give a symmetric generalization of the Hermite-Hadamard inequality, we present the following lemma which is interesting in itself [2].

Lemma 3. *Let X be a real linear space and C be its convex subset. If $f:C\to\mathbb{R}$ is convex on C, then for all $x,\ y$ in C the mapping $g_{x,y}:[0,1]\to\mathbb{R}$ given by*

$$g_{x,y}(t):=\frac{1}{2}\left[f(tx+(1-t)\,y)+f((1-t)\,x+ty)\right]$$

is also convex on $[0,1]$. In addition, we have the inequality

$$f\left(\frac{x+y}{2}\right)\leq g_{x,y}(t)\leq\frac{f(x)+f(y)}{2} \tag{3.7}$$

for all $x,\ y\in C$ and $t\in[0,1]$.

The following symmetric generalization of the Hermite-Hadamard inequality holds [2]:

Proposition 2. *Let $f:C\subseteq X\to\mathbb{R}$ be a convex function on the convex set C, where L and A satisfy the conditions L1, L2 and A1,A2. Also, $h:E\to\mathbb{R}$, $0\leq h(t)\leq 1\ (t\in E)$, and $h\in L$ is such that $f(hx+(\mathbf{1}-h)\,y)$, $f((\mathbf{1}-h)\,x+hy)$*

belong to L for x, y fixed in C. If $A(\mathbf{1}) = 1$, then we have the inequality:

$$
\begin{aligned}
&f\left(\frac{x+y}{2}\right) \\
&\leq \frac{1}{2}\left[f\left(A(h)x + (1 - A(h))y\right) + f\left((1 - A(h))x + A(h)y\right)\right] \\
&\leq \frac{1}{2}\left(A\left[f\left(hx + (\mathbf{1} - h)y\right)\right] + A\left[f\left((\mathbf{1} - h)x + hy\right)\right]\right) \\
&\leq \frac{f(x) + f(y)}{2}.
\end{aligned}
\tag{3.8}
$$

We consider the functional $\Upsilon_{\phi,h,x,y} : \mathcal{IL}(L)_{\mathbf{1}} \to [0,\infty)$ defined by

$$
\begin{aligned}
&\Upsilon_{\phi,h,x,y}(A) \\
&:= \frac{1}{2}\left(A\left[f\left(hx + (\mathbf{1} - h)y\right)\right] + A\left[f\left((\mathbf{1} - h)x + hy\right)\right]\right) \\
&- \frac{1}{2}A(\mathbf{1})\left[\phi\left(\frac{A(h)x + A(\mathbf{1} - h)y}{A(\mathbf{1})}\right) + \phi\left(\frac{A(\mathbf{1} - h)x + A(h)y}{A(\mathbf{1})}\right)\right] \\
&= \frac{1}{2}\left[\Psi_{\phi,h,x,y}(A) + \Psi_{\phi,h,y,x}(A)\right],
\end{aligned}
\tag{3.9}
$$

where $\phi : C \subseteq X \to \mathbb{R}$ is a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$ is such that $g_{x,y} \circ h \in L$ for x, y given in C.

From (3.8) we also have the inequalities

$$
0 \leq \Upsilon_{\phi,h,x,y}(A) \leq \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right),
$$

where $\phi : C \subseteq X \to \mathbb{R}$ is a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$ is such that $g_{x,y} \circ h \in L$ for x, y given in C.

We can also state the result:

Theorem 6. *Let $\phi : C \subseteq X \to \mathbb{R}$ be a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$ is such that $g_{x,y} \circ h \in L$ for x, y given in C.*

(i) *For all A, $B \in \mathcal{IL}(L)_{\mathbf{1}}$,*

$$
\Upsilon_{\phi,h,x,y}(A + B) \geq \Upsilon_{\phi,h,x,y}(A) + \Upsilon_{\phi,h,x,y}(B) \geq 0,
\tag{3.10}
$$

namely $\Upsilon_{\phi,h,x,y}$ is superadditive on $\mathcal{IL}(L)_{\mathbf{1}}$.

(ii) *For all A, $B \in \mathcal{IL}(L)_{\mathbf{1}}$ with $A \succeq B$ and $A(\mathbf{1}) > B(\mathbf{1})$,*

$$
\Upsilon_{\phi,h,x,y}(A) - \Upsilon_{\phi,h,x,y}(B) \geq \Upsilon_{\phi,h,x,y}(A - B) \geq 0,
\tag{3.11}
$$

which also shows that $\Upsilon_{\phi,h,x,y}$ is monotonic nondecreasing on $\mathcal{IL}(L)_{\mathbf{1}}$.

(iii) *If A, $B \in \mathcal{IL}(L)_{\mathbf{1}}$ with*

$$
MB \succeq A \succeq mB,
$$

then

$$
M\Upsilon_{\phi,h,x,y}(B) \geq \Upsilon_{\phi,h,x,y}(A) \geq m\Upsilon_{\phi,h,x,y}(B) \geq 0.
\tag{3.12}
$$

The proof follows by Theorem 5 and representation (3.9).

Remark 6. *Let $U \in \mathcal{IL}(L)$ with $U(w) > 0$. We can consider the functional of positive weights defined by*

$$\Psi_{\phi,U,h,x,y}(w) := U[w\phi(hx + (\mathbf{1} - h)y)] - U(w)\phi\left(\frac{U(wh)x + U(w(\mathbf{1} - h))y}{U(w)}\right) \geq 0, \tag{3.13}$$

where $\phi : C \subseteq X \to \mathbb{R}$ is a convex function on C, L satisfy conditions L1, L2 and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $0 \leq w$, w, $wh \in L$ and $w(g_{x,y} \circ h) \in L$ for x, y given in C. We denote $L_{\phi,U,h,x,y}$ to be the set of all nonnegative weights from L which satisfy the previous conditions.

If u, $v \in L_{\phi,U,h,x,y}$, then by (3.4) we get the superadditivity property as a function of weights

$$\Psi_{\phi,U,h,x,y}(u + v) \geq \Psi_{\phi,U,h,x,y}(u) + \Psi_{\phi,U,h,x,y}(v) \geq 0. \tag{3.14}$$

If u, $v \in L_{\phi,U,h,x,y}$ with $u \geq v$ in L then by (3.5) we have the monotonicity property

$$\Psi_{\phi,U,h,x,y}(u) \geq \Psi_{\phi,U,h,x,y}(v) \geq 0. \tag{3.15}$$

Also if $Mv \geq u \geq mv$ with $M > m > 0$ we have by (3.6) the bounds

$$M\Psi_{\phi,U,h,x,y}(v) \geq \Psi_{\phi,U,h,x,y}(u) \geq m\Psi_{\phi,U,h,x,y}(v) \geq 0. \tag{3.16}$$

We consider the functional of weights $\Upsilon_{\phi,U,h,x,y} : L_{\phi,U,h,x,y} \to [0, \infty)$,

$$\Upsilon_{\phi,U,h,x,y}(w) := \frac{1}{2}\left[\Psi_{\phi,U,h,x,y}(w) + \Psi_{\phi,U,h,x,y}(w)\right].$$

This functional also satisfy the properties (3.14)-(3.16).

4. Some Related Results for Subsets

Let F be a nonempty subset of E. We define the set of functions

$$\mathcal{S}(L, F) := \{f \in L,\ f\chi_F \in L\}$$

where χ_F is the *characteristic function* of F, namely

$$\chi_F(t) := \begin{cases} 1 \text{ for } t \in F, \\ 0 \text{ for } t \in E \setminus F. \end{cases}$$

We observe that $\mathcal{S}(L, F)$ is a linear class, namely if f, $g \in \mathcal{S}(L, F)$ imply $(\alpha f + \beta g) \in \mathcal{S}(L, F)$ for all α, $\beta \in \mathbb{R}$.

If $U : L \to \mathbb{R}$ is an isotonic linear functionals on L we can define $U_F : \mathcal{S}(L, F) \to \mathbb{R}$ by $U_F(f) = U(f\chi_F)$. We observe that if $f \in \mathcal{S}(L, F) \cap \mathcal{S}(L, E \setminus F)$ then by the properties of *characteristic function* we have

$$\begin{aligned} U_F(f) + U_{E\setminus F}(f) &= U(f\chi_F) + U\left(f\chi_{E\setminus F}\right) = U\left(f\chi_{F\cup(E\setminus F)}\right) \\ &= U(f\chi_E) = U(f). \end{aligned} \tag{4.1}$$

For a proper subset F or E and we can define

$$\begin{aligned} \Psi_{\phi,f,F}(A) &= A_F(\phi \circ f) - A_F(\mathbf{1})\phi\left(\frac{A_F(f)}{A_F(\mathbf{1})}\right) \\ &= A((\phi \circ f)\chi_F) - A(\mathbf{1}\chi_F)\phi\left(\frac{A(f\chi_F)}{A(\mathbf{1}\chi_F)}\right), \end{aligned} \tag{4.2}$$

where $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $f : E \to I$ is such that $\phi \circ f$, $f \in \mathcal{S}(L, F)$. Here $A \in \mathcal{IL}(\mathcal{S}(L, F))_{\mathbf{1}}$, where $\mathcal{IL}(\mathcal{S}(L, F))_{\mathbf{1}}$ is the set of all isotonic functionals on L for which $A(\mathbf{1}\chi_F) > 0$.

By making use of Theorem 4 for the functionals A_F and $A_{E \setminus F}$ and using the property (4.1) we can state the following result as well:

Theorem 7. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, F a subset of E with $F \neq \emptyset$, E and $f : E \to I$ such that $\phi \circ f$, $f \in \mathcal{S}(L, F) \cap \mathcal{S}(L, E \setminus F)$. For all $A \in \mathcal{IL}(\mathcal{S}(L, F))_{\mathbf{1}} \cap \mathcal{IL}(\mathcal{S}(L, F))_{\mathbf{1}}$,*

$$A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) \geq \Psi_{\phi,f,F}(A) + \Psi_{\phi,f,E\setminus F}(A) \geq 0 \tag{4.3}$$

and, a fortiori,

$$A(\phi \circ f) - A(\mathbf{1})\,\phi\left(\frac{A(f)}{A(\mathbf{1})}\right) \geq \max\left\{\Psi_{\phi,f,F}(A), \Psi_{\phi,f,E\setminus F}(A)\right\} \geq 0. \tag{4.4}$$

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\, w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below.

Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and a measurable function $f : \Omega \to \mathbb{R}$ such that $f, \phi \circ f \in L_w(\Omega, \mu)$. For a measurable subset Θ in Ω we can consider the weighted functional

$$\begin{aligned} \Psi_{\phi,f,\Theta}(w) &:= \int_\Omega w \cdot (\phi \circ f)\, \chi_\Theta d\mu - \left(\int_\Omega w \chi_\Theta d\mu\right) \phi\left(\frac{\int_\Omega w f \chi_\Theta d\mu}{\int_\Omega w \chi_\Theta d\mu}\right) \\ &= \int_\Theta w \cdot (\phi \circ f)\, d\mu - \left(\int_\Theta w d\mu\right) \phi\left(\frac{\int_\Theta w f d\mu}{\int_\Theta w d\mu}\right). \end{aligned} \tag{4.5}$$

By making use of (4.3) we get for a measurable subset Θ in Ω, see also [9],

$$\Psi_{\phi,f,\Omega}(w) \geq \Psi_{\phi,f,\Theta}(w) + \Psi_{\phi,f,\Omega\setminus\Theta}(w) \geq 0 \tag{4.6}$$

and

$$\Psi_{\phi,f,\Omega}(w) \geq \max\left\{\Psi_{\phi,f,\Theta}(w), \Psi_{\phi,f,\Omega\setminus\Theta}(w)\right\} \geq 0. \tag{4.7}$$

For the discrete counting measure, we can consider the functional

$$\Psi_{\phi,f,K}(w) := \sum_{k\in K} w_k \cdot \phi(x_k) - \sum_{k\in K} w_k \phi\left(\frac{\sum_{k\in K} w_k x_k}{\sum_{k\in K} w_k}\right)$$

where $\emptyset \neq K \subset J$, $w_j > 0$, $x_j \in I$, $j \in J$, a finite set of indices.

By (4.6) and (4.7) we derive

$$\Psi_{\phi,f,J}(w) \geq \Psi_{\phi,f,K}(w) + \Psi_{\phi,f,J\setminus K}(w) \geq 0 \tag{4.8}$$

and

$$\Psi_{\phi,f,J}(w) \geq \max\left\{\Psi_{\phi,f,K}(w), \Psi_{\phi,f,J\setminus K}(w)\right\} \geq 0, \tag{4.9}$$

results that were obtained in [13].

5. An Example for Norms

There are many applications of Jensen's integral and discrete inequalities as one can find in [6]-[13] and the references therein. Here we only consider a weighted integral associated to norms on linear spaces and provide some related inequalities.

Let $(X, \|\cdot\|)$ be a normed linear space and $p \geq 1$, $h : [a, b] \to [0, 1]$ continuous and $w : [a, b] \to (0, \infty)$ Lebesgue integrable. For $x,\ y \in X$, we consider the functional

$$
\begin{aligned}
(5.1)\quad & \Psi_{p,h,x,y}(w) \\
& := \int_a^b w(t) \|h(t) x + (1 - h(t)) y\|^p dt \\
& - \left(\int_a^b w(t) dt\right) \left\| \frac{\left(\int_a^b w(t) h(t) dt\right) x + \left(\int_a^b w(t)(1 - h(t)) dt\right) y}{\int_a^b w(t) dt} \right\|^p \\
& = \int_a^b w(t) \|h(t) x + (1 - h(t)) y\|^p dt \\
& - \left(\int_a^b w(t) dt\right)^{1-p} \left\| \left(\int_a^b w(t) h(t) dt\right) x + \left(\int_a^b w(t)(1 - h(t)) dt\right) y \right\|^p \\
& \geq 0.
\end{aligned}
$$

By (3.14) we get

$$
\Psi_{p,h,x,y}(u + v) \geq \Psi_{p,h,x,y}(u) + \Psi_{p,h,x,y}(v) \geq 0 \tag{5.2}
$$

for all $u,\ v : [a, b] \to (0, \infty)$ Lebesgue integrable functions.

Also if $Mv \geq u \geq mv$ with $M > m > 0$ we have by (3.16) the bounds

$$
\begin{aligned}
(5.3)\quad & M\left[\int_a^b v(t) \|h(t) x + (1 - h(t)) y\|^p dt\right. \\
& \left. - \left(\int_a^b v(t) dt\right)^{1-p} \left\| \left(\int_a^b v(t) h(t) dt\right) x + \left(\int_a^b v(t)(1 - h(t)) dt\right) y \right\|^p \right] \\
& \geq \int_a^b u(t) \|h(t) x + (1 - h(t)) y\|^p dt \\
& - \left(\int_a^b u(t) dt\right)^{1-p} \left\| \left(\int_a^b u(t) h(t) dt\right) x + \left(\int_a^b u(t)(1 - h(t)) dt\right) y \right\|^p \\
& \geq m\left[\int_a^b v(t) \|h(t) x + (1 - h(t)) y\|^p dt\right. \\
& \left. - \left(\int_a^b v(t) dt\right)^{1-p} \left\| \left(\int_a^b v(t) h(t) dt\right) x + \left(\int_a^b v(t)(1 - h(t)) dt\right) y \right\|^p \right]
\end{aligned}
$$

for $x,\ y \in X$, where $h : [a, b] \to [0, 1]$ is continuous.

If $v = \mathbf{1}$, namely $M \geq u \geq m$ with $M > m > 0$, then by (5.3) we get

$$\begin{aligned}(5.4)\quad & M\left[\int_a^b \|h(t)x + (1-h(t))y\|^p dt\right. \\ & \left.- (b-a)^{1-p}\left\|\left(\int_a^b h(t)\,dt\right)x + \left(\int_a^b (1-h(t))\,dt\right)y\right\|^p\right] \\ & \geq \int_a^b u(t)\|h(t)x + (1-h(t))y\|^p dt \\ & - \left(\int_a^b u(t)\,dt\right)^{1-p}\left\|\left(\int_a^b u(t)h(t)\,dt\right)x + \left(\int_a^b u(t)(1-h(t))\,dt\right)y\right\|^p \\ & \geq m\left[\int_a^b \|h(t)x + (1-h(t))y\|^p dt\right. \\ & \left.- (b-a)^{1-p}\left\|\left(\int_a^b h(t)\,dt\right)x + \left(\int_a^b (1-h(t))\,dt\right)y\right\|^p\right]\end{aligned}$$

for $x,\ y \in X$.

If we take $h(t) = \sin^2 t$, $t \in [0, \pi/2]$ in (5.4), then we get

$$\begin{aligned}(5.5)\quad & M\left[\int_0^{\pi/2} \left\|\left(\sin^2 t\right)x + \left(\cos^2 t\right)y\right\|^p dt - \frac{\pi}{2}\left\|\frac{x+y}{2}\right\|^p\right] \\ & \geq \int_0^{\pi/2} u(t)\left\|\left(\sin^2 t\right)x + \left(\cos^2 t\right)y\right\|^p dt \\ & - \left(\int_0^{\pi/2} u(t)\,dt\right)^{1-p}\left\|\left(\int_0^{\pi/2} u(t)\sin^2 t dt\right)x + \left(\int_0^{\pi/2} u(t)\cos^2 t dt\right)y\right\|^p \\ & \geq m\left[\int_0^{\pi/2} \left\|\left(\sin^2 t\right)x + \left(\cos^2 t\right)y\right\|^p dt - \frac{\pi}{2}\left\|\frac{x+y}{2}\right\|^p\right]\end{aligned}$$

provided $M \geq u \geq m$ with $M > m > 0$ and $x,\ y \in X$.

Further, we can consider the functional depending on intervals $[c, d] \subset [a, b]$

$$\begin{aligned}(5.6)\quad & \Psi_{p,h,x,y,[c,d]}(w) \\ & = \int_c^d w(t)\|h(t)x + (1-h(t))y\|^p dt \\ & - \left(\int_c^d w(t)\,dt\right)^{1-p}\left\|\left(\int_c^d w(t)h(t)\,dt\right)x + \left(\int_c^d w(t)(1-h(t))\,dt\right)y\right\|^p \\ & \geq 0.\end{aligned}$$

Using Theorem 4 we conclude that

$$\Psi_{p,h,x,y,[a,b]}(w) \geq \Psi_{p,h,x,y,[a,c]}(w) + \Psi_{p,h,x,y,[c,b]}(w) \geq 0$$

and

$$\Psi_{p,h,x,y,[a,b]}(w) \geq \max\left\{\Psi_{p,h,x,y,[a,c]}(w), \Psi_{p,h,x,y,[c,b]}(w)\right\} \geq 0$$

for all $c \in (a, b)$.

Now, for the discrete case, we consider $t_i \in [0,1]$, $w_i > 0$, $i \in \{1, ..., n\}$, $x, y \in X$ and define the functional

$$\begin{aligned} \Psi_{p,t,x,y,n}(w) :=& \sum_{i=1}^{n} w_i \|t_i x + (1-t_i) y\|^p \\ & - W_n^{1-p} \left\| \left(\sum_{i=1}^{n} w_i t_i \right) x + \left(\sum_{i=1}^{n} w_i (1-t_i) \right) y \right\|^p \\ & \geq 0, \end{aligned} \tag{5.7}$$

where $W_n := \sum_{i=1}^{n} w_i$.

By (3.14) we get

$$\Psi_{p,t,x,y,n}(u+v) \geq \Psi_{p,t,x,y,n}(u) + \Psi_{p,t,x,y,n}(v) \geq 0 \tag{5.8}$$

for all u, v positive weights.

Also if $Mv_i \geq u_i \geq mv_i$, $i \in \{1, ..., n\}$, with $M > m > 0$ we have by (3.16) the bounds

(5.9)

$$\begin{aligned} M & \left(\sum_{i=1}^{n} v_i \|t_i x + (1-t_i) y\|^p - V_n^{1-p} \left\| \left(\sum_{i=1}^{n} v_i t_i \right) x + \left(\sum_{i=1}^{n} v_i (1-t_i) \right) y \right\|^p \right) \\ & \geq \sum_{i=1}^{n} u_i \|t_i x + (1-t_i) y\|^p - U_n^{1-p} \left\| \left(\sum_{i=1}^{n} u_i t_i \right) x + \left(\sum_{i=1}^{n} u_i (1-t_i) \right) y \right\|^p \\ & \geq m \left(\sum_{i=1}^{n} v_i \|t_i x + (1-t_i) y\|^p - V_n^{1-p} \left\| \left(\sum_{i=1}^{n} v_i t_i \right) x + \left(\sum_{i=1}^{n} v_i (1-t_i) \right) y \right\|^p \right). \end{aligned}$$

In particular, if $M \geq u_i \geq m$, $i \in \{1, ..., n\}$, with $M > m > 0$, then

$$\begin{aligned} M & \left(\sum_{i=1}^{n} \|t_i x + (1-t_i) y\|^p - n^{1-p} \left\| \left(\sum_{i=1}^{n} t_i \right) x + \left(\sum_{i=1}^{n} (1-t_i) \right) y \right\|^p \right) \\ & \geq \sum_{i=1}^{n} u_i \|t_i x + (1-t_i) y\|^p - U_n^{1-p} \left\| \left(\sum_{i=1}^{n} u_i t_i \right) x + \left(\sum_{i=1}^{n} u_i (1-t_i) \right) y \right\|^p \\ & \geq m \left(\sum_{i=1}^{n} \|t_i x + (1-t_i) y\|^p - n^{1-p} \left\| \left(\sum_{i=1}^{n} t_i \right) x + \left(\sum_{i=1}^{n} (1-t_i) \right) y \right\|^p \right). \end{aligned} \tag{5.10}$$

For a finite set of indices $I \subset \mathbb{N}$, we consider the functional

$$\begin{aligned} \Psi_{p,t,x,y,I}(w) :=& \sum_{i \in I} w_i \|t_i x + (1-t_i) y\|^p \\ & - W_I^{1-p} \left\| \left(\sum_{i \in I} w_i t_i \right) x + \left(\sum_{i \in I} w_i (1-t_i) \right) y \right\|^p \\ & \geq 0, \end{aligned} \tag{5.11}$$

where $t_i \in [0,1]$, $w_i > 0$, $i \in I$ and $W_I := \sum_{i \in I} w_i$.

Assume that $K \subset I$, $K \neq \emptyset$, E and put $\overline{K} := I \setminus K$. Then by using Theorem 4 we conclude that

$$\Psi_{p,t,x,y,I}(w) \geq \Psi_{p,t,x,y,K}(w) + \Psi_{p,t,x,y,\overline{K}}(w) \geq 0, \tag{5.12}$$

and

$$\Psi_{p,t,x,y,I}(w) \geq \max\left\{\Psi_{p,t,x,y,K}(w), \Psi_{p,t,x,y,\overline{K}}(w)\right\} \geq 0. \tag{5.13}$$

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