SOME LEVIN-STEČKIN'S TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS ON HILBERT SPACES

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Abstract. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I. Assume that $p:[0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$. In this paper we obtained, among others, that

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dtdt - \int_0^1 p(t) f(tA+(1-t)B)\,dt \\ &\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right] \end{aligned}$$

in the operator order.

Several other similar inequalities for either p or f is differentiable, are also provided. Applications for power function and logarithm are given as well.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B) \le (\ge)\,(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \le B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \le f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [12] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \le r \le 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1 \le r \le 2$ or $-1 \le r \le 0$ and is operator concave on $(0,\infty)$ if $0 \le r \le 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

In [8] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f: I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \le \int_0^1 f((1-s)A+sB)\,ds \le \frac{f(A)+f(B)}{2}, \tag{1.2}$$

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where A, B are selfadjoint operators with spectra included in I.

From the operator convexity of the function f we have

$$f\left(\frac{A+B}{2}\right)\leq\frac{1}{2}\left[f\left(\left(1-s\right)A+sB\right)+f\left(sA+\left(1-s\right)B\right)\right]\leq\frac{f\left(A\right)+f\left(B\right)}{2}\tag{1.3}$$

for all $s\in[0,1]$ and A, B selfadjoint operators with spectra included in I.

If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p\left(1-s\right)=p\left(s\right)$ for all $s\in[0,1]$, then by multiplying (1.3) with $p\left(s\right)$, integrating on $[0,1]$ and taking into account that

$$\int_0^1 p\left(s\right)f\left(\left(1-s\right)A+sB\right)ds=\int_0^1 p\left(s\right)f\left(sA+\left(1-s\right)B\right)ds,$$

we get the weighted version of (1.2) for A, B selfadjoint operators with spectra included in I

$$\left(\int_0^1 p\left(s\right)ds\right)f\left(\frac{A+B}{2}\right)\leq\int_0^1 p\left(s\right)f\left(sA+\left(1-s\right)B\right)ds\leq\left(\int_0^1 p\left(s\right)ds\right)\frac{f\left(A\right)+f\left(B\right)}{2},\tag{1.4}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[2], [4], [6]-[14], and [21]-[26].

The following result is known in the literature as Levin-Stečkin's inequality [16]:

Theorem 1. *If the function* $p:[0,1]\to\mathbb{R}$ *is symmetric, namely* $p\left(1-t\right)=p\left(t\right)$ *for* $t\in[0,1]$ *and non-decreasing (non-increasing) on* $[0,1/2]$, *then for every convex function* g *on* $[0,1]$,

$$\int_0^1 p\left(t\right)g\left(t\right)dt\leq(\geq)\int_0^1 p\left(t\right)dt\int_0^1 g\left(t\right)dt.\tag{LS}$$

If the function g *is concave on* $[0,1]$, *then the signs of inequalities reverse in (LS).*

For some recent results related to Levin-Stečkin's inequality, see [18], [19] and [27].

Motivated by the above operator inequalities, we provide in this paper the operator version of Levin-Stečkin's inequality as well as several reverses. Applications for power function and logarithm are also given.

2. Operator Inequalities

Let f be an operator convex function on I. For $A,B\in\mathcal{SA}_I\left(H\right)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{SA}\left(H\right)$, the class of all selfadjoint operators on H, defined by

$$\varphi_{(A,B)}\left(t\right):=f\left(\left(1-t\right)A+tB\right).\tag{2.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}\left(t\right):=\left\langle\varphi_{(A,B)}\left(t\right)x,x\right\rangle=\left\langle f\left(\left(1-t\right)A+tB\right)x,x\right\rangle.\tag{2.2}$$

We have the following basic fact [10]:

Lemma 1. *Let f be an operator convex function on I. For any A, $B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B) \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0} \frac{g(A+sB)-g(A)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

We also have [10]:

Lemma 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.4}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.6}$$

and

Lemma 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A) \leq \nabla g_{(1-t_2)A+t_2B}(B-A) \tag{2.7}$$

in the operator order.

We also have

$$\nabla f_A(B-A) \leq \nabla g_{(1-t_1)A+t_1B}(B-A) \tag{2.8}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A). \tag{2.9}$$

In particular, we observe that:

Corollary 1. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for all $t \in (0,1)$ we have*

$$\nabla f_A(B-A) \leq \nabla f_{(1-t)A+tB}(B-A) \leq \nabla f_B(B-A). \tag{2.10}$$

For two *Lebesgue integrable* functions $h,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b h(t)dt\int_a^b g(t)dt. \tag{2.11}$$

In 1935, Grüss [15] showed that

$$|C(h,g)|\le\frac{1}{4}(M-m)(N-n), \tag{2.12}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m\le h(t)\le M\quad\text{and}\quad n\le g(t)\le N\quad\text{for a.e. } t\in[a,b]. \tag{2.13}$$

The constant $\frac{1}{4}$ is best possible in (2.11) in the sense that it cannot be replaced by a smaller quantity.

We have the following operator inequalities:

Theorem 2. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$. Assume that $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then we have the operator inequality*

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1 f(tA+(1-t)B)\,dt-\int_0^1 p(t)f(tA+(1-t)B)\,dt\\&\le\frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right].\end{aligned} \tag{2.14}$$

If $p:[0,1]\to\mathbb{R}$ is symmetric and non-increasing on $[0,1/2]$, then

$$\begin{aligned}0&\le\int_0^1 p(t)f(tA+(1-t)B)\,dt-\int_0^1 p(t)\,dt\int_0^1 f(tA+(1-t)B)\,dtdt\\&\le\frac{1}{4}\left[p(0)-p\left(\frac{1}{2}\right)\right]\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right].\end{aligned} \tag{2.15}$$

Proof. For $x\in H$ we consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)\,x,x\right\rangle=\langle f((1-t)A+tB)\,x,x\rangle.$$

Since p is symmetric on $[0,1]$, then

$$\begin{aligned}&\int_0^1 p(t)\frac{\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)}{2}dt\\&=\frac{1}{2}\left[\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt+\int_0^1 p(t)\,\varphi_{(A,B);x}(1-t)\,dt\right]\\&=\frac{1}{2}\left[\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt+\int_0^1 p(1-t)\,\varphi_{(A,B);x}(1-t)\,dt\right].\end{aligned}$$

By changing the variable $1-t=s,\ s\in[0,1]$ we have

$$\int_0^1 p(1-t)\,\varphi_{(A,B);x}(1-t)\,dt=\int_0^1 p(s)\,\varphi_{(A,B);x}(s)\,ds$$

and then

$$\int_0^1 p(t)\frac{\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)}{2}dt=\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt.$$

Also

$$\int_0^1 \frac{\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)}{2}dt=\int_0^1 \varphi_{(A,B);x}(t)\,dt.$$

Therefore

$$\begin{aligned}\int_0^1 p(t)\,dt\int_0^1 \varphi_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt \\ =\int_0^1 p(t)\,dt\int_0^1 \check{\varphi}_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\check{\varphi}_{(A,B);x}(t)\,dt,\end{aligned} \tag{2.16}$$

where

$$\check{\varphi}_{(A,B);x}(t):=\frac{\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)}{2},\ t\in[0,1]$$

is the symmetrical transform of $\varphi_{(A,B);x}$ on the interval $[0,1]$.

Now, if we use the Levin-Stečkin's inequality for the symmetric function p and the convex function $g=\varphi_{(A,B);x}$, then we obtain

$$0\le\int_0^1 p(t)\,dt\int_0^1 \varphi_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt, \tag{2.17}$$

for all $x\in H$.

Since, by Lemma 1, $\varphi_{(A,B);x}$ is convex, then $\check{\varphi}_{(A,B);x}$ is symmetric and convex, which implies that

$$\begin{aligned}\varphi_{(A,B);x}\left(\frac{1}{2}\right)=\check{\varphi}_{(A,B);x}\left(\frac{1}{2}\right)\le\check{\varphi}_{(A,B);x}(t) \\ \le\check{\varphi}_{(A,B);x}(1)=\frac{\varphi_{(A,B);x}(0)+\varphi_{(A,B);x}(1)}{2},\ t\in[0,1],\end{aligned}$$

for all $x\in H$.

Also $p(0)\le p(t)\le p\left(\frac{1}{2}\right)$, $t\in[0,1]$ and by Grüss' inequality for $h=p$ and $g=\check{\varphi}_{(A,B);x}$ we get

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1 \check{\varphi}_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\check{\varphi}_{(A,B);x}(t)\,dt \\ &\le\frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{\varphi_{(A,B);x}(0)+\varphi_{(A,B);x}(1)}{2}-\varphi_{(A,B);x}\left(\frac{1}{2}\right)\right]\end{aligned}$$

namely, by (2.16) and (2.17)

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1 \varphi_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\varphi_{(A,B);x}(t)\,dt \\ &\le\frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\frac{\varphi_{(A,B);x}(0)+\varphi_{(A,B);x}(1)}{2}-\varphi_{(A,B);x}\left(\frac{1}{2}\right)\right]\end{aligned} \tag{2.18}$$

for all $x\in H$.

The inequality (2.18) can be written in terms of inner product as

$$
\begin{aligned}
0 &\le \left\langle \left(\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \right) x, x \right\rangle \\
&- \left\langle \left(\int_0^1 p(t) f((1-t)A+tB)\,dt \right) x, x \right\rangle \\
&\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0) \right] \left[\left\langle \left(\frac{f(A)+f(B)}{2} \right) x, x \right\rangle - \left\langle f\left(\frac{A+B}{2} \right) x, x \right\rangle \right]
\end{aligned}
$$

for all $x \in H$, which is equivalent to the operator inequality (2.14). □

Remark 1. *If f is an operator concave function on I and A, $B \in \mathcal{SA}_I(H)$, while $p:[0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then*

$$
\begin{aligned}
(2.19)\quad 0 &\le \int_0^1 p(t) f(tA+(1-t)B)\,dt - \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dtdt \\
&\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0) \right] \left[f\left(\frac{A+B}{2}\right) - \frac{f(A)+f(B)}{2} \right].
\end{aligned}
$$

Also, in this case, if $p:[0,1] \to \mathbb{R}$ is symmetric and non-increasing on $[0,1/2]$, then

$$
\begin{aligned}
(2.20)\quad 0 &\le \int_0^1 p(t) f(tA+(1-t)B)\,dt - \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dtdt \\
&\le \frac{1}{4}\left[p(0) - p\left(\frac{1}{2}\right) \right] \left[f\left(\frac{A+B}{2}\right) - \frac{f(A)+f(B)}{2} \right].
\end{aligned}
$$

The following inequality obtained by Ostrowski in 1970, [20] also holds

$$
(2.21)\qquad |C(h,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty,
$$

provided that h is *Lebesgue integrable* and satisfies (2.13) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.21).

We have the following operator inequalities when some differentiability conditions are imposed.

Theorem 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$ while $p:[0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$.*

(i) *If p is differentiable on $(0,1)$, then*

$$
\begin{aligned}
(2.22)\quad 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dt - \int_0^1 p(t) f(tA+(1-t)B)\,dt \\
&\le \frac{1}{8}\|p'\|_\infty \left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right) \right].
\end{aligned}
$$

(ii) *If $f \in \mathcal{G}([A,B])$, then*

$$
\begin{aligned}
(2.23)\quad 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dt - \int_0^1 p(t) f(tA+(1-t)B)\,dt \\
&\le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0) \right] \left[\nabla f_B(B-A) - \nabla f_A(B-A) \right].
\end{aligned}
$$

Proof. The inequality (2.22) follows by (2.21) for $g = p$ and $h = \varphi_{(A,B);x}$, $x \in H$ and proceed like in the proof of Theorem 2.

Now, by Lemma 2

$$\begin{aligned}(2.24)\qquad &\left(\check{\varphi}_{(A,B);x}(t)\right)' \\ &= \frac{\left(\varphi_{(A,B);x}(t)\right)' + \left(\varphi_{(A,B);x}(1-t)\right)'}{2} \\ &= \frac{\left\langle \varphi'_{(A,B)}(t)\,x, x\right\rangle - \left\langle \varphi'_{(A,B)}(1-t)\,x, x\right\rangle}{2} \\ &= \frac{\left\langle \nabla f_{(1-t)A+tB}(B-A)\,x, x\right\rangle - \left\langle \nabla f_{tA+(1-t)B}(B-A)\,x, x\right\rangle}{2} \\ &= \frac{\left\langle \left[\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right] x, x\right\rangle}{2} \\ &= \left\langle \left[\frac{\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)}{2}\right] x, x\right\rangle\end{aligned}$$

for all $t \in (0,1)$ and any $x \in H$.

Since $\check{\varphi}_{(A,B);x}$ is convex on $(0,1)$, then

$$\left(\check{\varphi}_{(A,B);x}(t)\right)'_{t=0+} \le \left(\check{\varphi}_{(A,B);x}(t)\right)' \le \left(\check{\varphi}_{(A,B);x}(t)\right)'_{t=1-},\ t \in (0,1)$$

namely, by Lemma 3

$$\begin{aligned}&\left\langle \left[\frac{\nabla f_A(B-A) - \nabla f_B(B-A)}{2}\right] x, x\right\rangle \\ &\le \left\langle \left[\frac{\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)}{2}\right] x, x\right\rangle \\ &\le \left\langle \left[\frac{\nabla f_B(B-A) - \nabla f_A(B-A)}{2}\right] x, x\right\rangle\end{aligned}$$

for all $t \in (0,1)$ and any $x \in H$.

Therefore

$$\left|\left(\check{\varphi}_{(A,B);x}(t)\right)'\right| \le \left|\left\langle \left[\frac{\nabla f_B(B-A) - \nabla f_A(B-A)}{2}\right] x, x\right\rangle\right|$$

for all $t \in (0,1)$ and any $x \in H$, which implies that

$$\begin{aligned}\sup_{t\in(0,1)}\left|\left(\check{\varphi}_{(A,B);x}(t)\right)'\right| &\le \left|\left\langle \left[\frac{\nabla f_B(B-A) - \nabla f_A(B-A)}{2}\right] x, x\right\rangle\right| \\ &= \left\langle \left[\frac{\nabla f_B(B-A) - \nabla f_A(B-A)}{2}\right] x, x\right\rangle\end{aligned}$$

for any $x \in H$, since by Corollary 1, we have $f_B(B-A) \ge \nabla f_A(B-A)$.

If we use Ostrowski's inequality (2.21) for $h=p$ and $g=\check{\varphi}_{(A,B);x}$, then we obtain

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1\check{\varphi}_{(A,B);x}(t)\,dt-\int_0^1 p(t)\,\check{\varphi}_{(A,B);x}(t)\,dt\\&\le\frac{1}{8}\left[p\left(\frac{1}{2}\right)-p(0)\right]\sup_{t\in(0,1)}\left|\left(\check{\varphi}_{(A,B);x}(t)\right)'\right|\\&\le\frac{1}{8}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left\langle\left[\frac{\nabla f_B(B-A)-\nabla f_A(B-A)}{2}\right]x,x\right\rangle,\end{aligned}$$

namely

$$\begin{aligned}0&\le\left\langle\left(\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right)x,x\right\rangle\\&\le\frac{1}{8}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left\langle\left[\frac{\nabla f_B(B-A)-\nabla f_A(B-A)}{2}\right]x,x\right\rangle,\end{aligned}$$

which is equivalent to the operator inequality (2.23). □

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(h,g)|\le\frac{1}{12}\|h'\|_\infty\|g'\|_\infty(b-a)^2, \tag{2.25}$$

provided that h', g' exist and are continuous on $[a,b]$ and $\|h'\|_\infty=\sup_{t\in[a,b]}|h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [17] in which he proved that

$$|C(h,g)|\le\frac{1}{\pi^2}\|h'\|_2\|g'\|_2(b-a), \tag{2.26}$$

provided that h, g are absolutely continuous and h', $g'\in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Further, we have:

Theorem 4. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$ while $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$.*

(i) *If p is differentiable on $(0,1)$ and $f\in\mathcal{G}([A,B])$, then*

$$\begin{aligned}0&\le\int_0^1 p(t)\,dt\int_0^1 f(tA+(1-t)B)\,dt-\int_0^1 p(t)\,f(tA+(1-t)B)\,dt\\&\le\frac{1}{24}\|p'\|_\infty\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right].\end{aligned} \tag{2.27}$$

(ii) *If p is differentiable on $(0,1)$ with $p' \in L_2[0,1]$ and $f \in \mathcal{G}([A,B])$, then*

$$
\begin{aligned}
(2.28)\quad 0 &\le \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B)\,dt - \int_0^1 p(t) f(tA+(1-t)B)\,dt \\
&\le \frac{1}{2\pi^2}\|p'\|_2 \\
&\times \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt\right)^{1/2} 1_H \\
&\le \frac{1}{\pi^2}\|p'\|_2 \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^2 dt\right)^{1/2} 1_H
\end{aligned}
$$

provided the last integral is finite.

Proof. The inequality (2.27) follows by (2.25) for $h = p$ and $g = \varphi_{(A,B);x}$, $x \in H$ and proceed like in the proof of Theorem 2.

From (2.24) we have

$$
\begin{aligned}
(2.29)\quad &\int_0^1 \left[\left(\check{\varphi}_{(A,B);x}(t)\right)'\right]^2 dt \\
&= \int_0^1 \left|\left\langle \left[\frac{\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)}{2}\right] x, x\right\rangle\right|^2 dt \\
&\le \int_0^1 \left\|\left[\frac{\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)}{2}\right] x\right\|^2 \|x\|^2 dt \\
&\le \frac{1}{4}\|x\|^4 \int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt
\end{aligned}
$$

for all $x \in H$, implying that

$$
\begin{aligned}
(2.30)\quad &\left(\int_0^1 \left[\left(\check{\varphi}_{(A,B);x}(t)\right)'\right]^2 dt\right)^{1/2} \\
&\le \frac{1}{2}\|x\|^2 \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt\right)^{1/2}
\end{aligned}
$$

for all $x \in H$.

By using (2.26) for $h = p$ and $g = \varphi_{(A,B);x}$, $x \in H$, we derive

$$
\begin{aligned}
0 &\le \left\langle \left(\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt - \int_0^1 p(t) f((1-t)A+tB)\,dt\right) x, x\right\rangle \\
&\le \frac{1}{2\pi^2}\|p'\|_2 \|x\|^2 \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt\right)^{1/2},
\end{aligned}
$$

which is equivalent to the first inequality in (2.28).

By the triangle inequality, we have

$$\left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt\right)^{1/2}$$
$$\le \left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^2 dt\right)^{1/2} + \left(\int_0^1 \left\|\nabla f_{tA+(1-t)B}(B-A)\right\|^2 dt\right)^{1/2}$$
$$= 2\left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^2 dt\right)^{1/2},$$

which proves the last part of (2.28). □

Remark 2. *If either p is non-increasing on $[0,1/2]$ or f is an operator concave function on I, then the interested reader may state similar results to the ones in Theorem 3 and Theorem 4. We omit the details.*

3. Some Examples

The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \le r \le 2$ or $-1 \le r \le 0$. Assume that $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then we have by (2.14) the operator inequality

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 (tA+(1-t)B)^r\,dt - \int_0^1 p(t)(tA+(1-t)B)^r\,dt \\ &\le \frac{1}{4}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\frac{A^r+B^r}{2} - \left(\frac{A+B}{2}\right)^r\right] \end{aligned} \tag{3.1}$$

for all A, $B > 0$.

Moreover, if p is differentiable on $(0,1)$, then by (2.22)

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 (tA+(1-t)B)^r\,dt - \int_0^1 p(t)(tA+(1-t)B)^r\,dt \\ &\le \frac{1}{8}\|p'\|_\infty\left[\frac{A^r+B^r}{2} - \left(\frac{A+B}{2}\right)^r\right] \end{aligned} \tag{3.2}$$

for all A, $B > 0$.

The function $f(x) = x^{-1}$ is operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S) = -T^{-1}ST^{-1}$$

for T, $S > 0$.

If we use (2.23), then we get the inequality

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,dt \int_0^1 (tA+(1-t)B)^{-1}\,dt - \int_0^1 p(t)(tA+(1-t)B)^{-1}\,dt \\ &\le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \end{aligned} \tag{3.3}$$

provided that $p : [0,1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$ and A, $B > 0$.

Moreover, if p is differentiable on $(0,1)$ then by (2.27) we derive

$$0 \le \int_0^1 p(t)\,dt \int_0^1 (tA+(1-t)B)^{-1}\,dt - \int_0^1 p(t)(tA+(1-t)B)^{-1}\,d \le \frac{1}{24}\|p'\|_\infty \left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \tag{3.4}$$

for $A,\ B>0$.

If we use the first and last term in (2.28), then we also have

$$0 \le \int_0^1 p(t)\,dt \int_0^1 (tA+(1-t)B)^{-1}\,dt - \int_0^1 p(t)(tA+(1-t)B)^{-1}\,d \le \frac{1}{\pi^2}\|p'\|_2 \times \left(\int_0^1 \left\|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right\|^2 dt\right)^{1/2} 1_H, \tag{3.5}$$

provided that $p' \in L_2[0,1]$ and $A,\ B>0$.

The logarithmic function $f(t)=\ln t$ is operator concave on $(0,\infty)$. Assume that $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$, then we have by (2.14) the operator inequality

$$0 \le \int_0^1 p(t)\ln(tA+(1-t)B)\,dt - \int_0^1 p(t)\,dt\int_0^1 \ln(tA+(1-t)B)\,dt \le \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right], \tag{3.6}$$

for all $A,\ B>0$.

Moreover, if p is differentiable on $(0,1)$, then by (2.22)

$$0 \le \int_0^1 p(t)\ln(tA+(1-t)B)\,dt - \int_0^1 p(t)\,dt\int_0^1 \ln(tA+(1-t)B)\,dt \le \frac{1}{8}\|p'\|_\infty\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right], \tag{3.7}$$

for all $A,\ B>0$.

We note that the function $f(x)=\ln x$ is operator concave on $(0,\infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [21, p. 155]):

$$\nabla \ln_T(S) = \int_0^\infty (s1_H+T)^{-1} S (s1_H+T)^{-1}\,ds \tag{3.8}$$

for $T,\ S>0$.

If we use inequality (2.23) for ln we get for $A,\ B>0$,

$$0 \le \int_0^1 p(t)\ln(tA+(1-t)B)\,dt - \int_0^1 p(t)\,dt\int_0^1 \ln(tA+(1-t)B)\,dt \le \frac{1}{16}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\,ds - \int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}\,ds\right], \tag{3.9}$$

provided that $p:[0,1]\to\mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$.

If p is differentiable, then by (2.27) we

$$0\le\int_0^1 p(t)\ln(tA+(1-t)B)\,dt-\int_0^1 p(t)\,dt\int_0^1\ln(tA+(1-t)B)\,dt \tag{3.10}$$
$$\le\frac{1}{24}\|p'\|_\infty\left[\int_0^\infty(s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right.$$
$$\left.-\int_0^\infty(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right],$$

for $A,\ B>0$.

A similar inequality can be derive from (2.28), however the details are omitted.

The interested author can also state the corresponding operator inequalities for $f(t)=t\ln t$ that is operator convex on $(0,\infty)$.

Finally, if we take $p(t)=t(1-t)$, $t\in[0,1]$, then we observe that p is symmetric and non-decreasing on $[0,1/2]$ and by (3.1) we obtain

$$0\le\frac{1}{6}\int_0^1(tA+(1-t)B)^r\,dt-\int_0^1 t(1-t)(tA+(1-t)B)^r\,dt \tag{3.11}$$
$$\le\frac{1}{16}\left[\frac{A^r+B^r}{2}-\left(\frac{A+B}{2}\right)^r\right]$$

if either $1\le r\le 2$ or $-1\le r\le 0$ and $A,\ B>0$.

From (3.3) we derive

$$0\le\frac{1}{6}\int_0^1(tA+(1-t)B)^{-1}\,dt-\int_0^1 t(1-t)(tA+(1-t)B)^{-1}\,dt \tag{3.12}$$
$$\le\frac{1}{64}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]$$

for $A,\ B>0$.

From (3.6) we obtain the logarithmic inequality

$$0\le\int_0^1 t(1-t)\ln(tA+(1-t)B)\,dt-\frac{1}{6}\int_0^1\ln(tA+(1-t)B)\,dt \tag{3.13}$$
$$\le\frac{1}{16}\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right],$$

while from (3.9), the inequality

$$0\le\int_0^1 t(1-t)\ln(tA+(1-t)B)\,dt-\frac{1}{6}\int_0^1\ln(tA+(1-t)B)\,dt \tag{3.14}$$
$$\le\frac{1}{64}\left[\int_0^\infty(s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right.$$
$$\left.-\int_0^\infty(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right]$$

for $A,\ B>0$.

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