EXTENSION OF FEJÈR INEQUALITY

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Abstract. In this paper, we extend the Fejèr inequality to a new class of functions where \( f \) is not necessary convex and \( g \) not satisfy the symmetric propriety and not necessary non-negative function.

1. Introduction

Throughout this paper, we denote by \( I \) and \( I' \) the real intervals \([a, b]\) and \((a, b)\) respectively. A function \( f \) is said to be convex on \( I \) if and only if
\[
\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)
\]
for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \). Conversely, if the opposite inequality holds, the function is said to be concave on \([a, b]\). A function \( f \) that is continuous on \([a, b]\) and twice differentiable on \( I_0 \) is convex on \( I \) if and only if
\[
f''(x) > 0 \quad \text{for all} \quad x \in I_0.
\]
In 1883 Hermite[11], and in 1893, Hadamard [10] they established the following inequality:
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\]
holds for all convex function \( f \) on \([a, b]\). In 1906, Fejèr [9] established the following weighted generalization of Hermite-Hadamard inequality (1.1) and obtained the inequality:
\[
f\left(\frac{a+b}{2}\right) \int_a^b g(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx,
\]
holds for all convex function \( f \) defined on \([a, b]\) and \( g \) is non-negative, integrable and symmetric function with respect to \( \frac{a+b}{2} \).

The purpose of this work is to extend new results related to inequalities (1.1) and (1.2).

In this part we give definitions and properties of a new class of functions.

For a function \( f : [a, b] \to \mathbb{R} \) we consider the symmetrical transform of \( f \) on the interval \([a, b]\), denoted by
\[
F(x) = \frac{1}{2} (f(x) + f(a + b - x)).
\]

Definition 1.1. [4],[8] A real-valued function \( f \) is said to be symmetrized convex on \([a, b]\) if the symmetrical transform of \( f \) is convex. Conversely, if \( F \) is concave, the function is said to be symmetrized concave on \([a, b]\).

From the above definition we get the following propositions.

2000 Mathematics Subject Classification. Primary 52A40, 5241.

Key words and phrases. Symmetrized convex functions, Symmetrized increasing functions, Hermite-Hadamard integral inequality, Fejèr inequality.
Proposition 1.2. [8] Suppose that the function $f$ is symmetrized convex on $[a, b]$, then we have:

(i) The function $F$ is symmetric to $\frac{a + b}{2}$ in the sense for all $x$ on $[a, b]$, we have
$$\forall x \in [a, b]: F(a + b - x) = F(x).$$

(ii) For all $x \in [a, b]$, we have
$$F\left(\frac{a + b}{2}\right) \leq F(x) \leq F\left(a + b\right) = \frac{f(a) + f(b)}{2}.$$  

(iii) The function $F$ is increasing on $[\frac{a + b}{2}, b]$ and decreasing on $[a, \frac{a + b}{2}]$.

Proposition 1.3. [8] If $f$ is a convex function, then $f$ is symmetrized concave on $[a, b]$. The inverse is false.

Example 1.4. [8] Let $f : [a, b] \to \mathbb{R} : x \mapsto x^3$ such that $a < 0 < b$ and $a + b > 0$. Then $f$ is not convex on $[a, b]$, but $F(x) = \frac{(a + b - x)^3 + x^3}{2}$ is convex ($F''(x) = 3(a + b) > 0$).

Example 1.5. [8] The function $f : [a, b] \to \mathbb{R} : x \mapsto f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$ such that $a < 0 < b$ and $a + b > 0$ is not convex on $[a, b]$, but $F(x) = f(a + b - x) + f(x)$ is convex, $(F''(x) = \cosh(\frac{a + b}{2}) \cosh(\frac{a + b}{2} - x) > 0)$.

Proposition 1.6. If $f$ is is symmetric to $\frac{a + b}{2}$ then, $f$ is convex on $[a, b]$ if and only if $f$ is symmetrized convex on $[a, b]$.

Now, if we denote by $\text{Con}[a, b]$ the closed convex cone of convex functions defined on $[a, b]$ and by $\text{SCon}[a, b]$ the class of symmetrized convex functions, then from the above examples and propositions we can conclude that $\text{Con}[a, b] \not\subset \text{SCon}[a, b]$.

We have the following results [4],[8]:

Theorem A Assume that $f : [a, b] \to \mathbb{R}$ is symmetrized convex function on $[a, b]$. Then $f$ satisfies the Hermite-Hadamard inequality (1.1).

Theorem B Let $f : [a, b] \to \mathbb{R}$ be a symmetrized convex function on $[a, b]$, and let $g : [a, b] \to [0, +\infty]$ be an integrable function and symmetric with respect to $\frac{a + b}{2}$, then the Fejér inequality (1.2) holds.

For more results in this direction, please refer [2,3,4,8]

In the above results we have extend the set of functions that satisfies the Hermite-Hadamard inequality and Fejér inequality. In fact, we replaced the condition of convex functions by symmetrized convex functions.

The question arises, can we extend the above inequalities to a new class of functions? for answar this question we give the folowing definitions.

Definition 1.7. A real-valued function $f$ is said to be symmetrized increasing on $[a, b]$, if the function $F$ is increasing on $\left[\frac{a + b}{2}, b\right]$. 

Remark 1.8. By part (iii) of proposition (1.2), if \( f \) is symmetrized convex function on \([a, b]\) then \( f \) is symmetrized increasing on \([a, b]\). The inverse is not true in general.

Example 1.9. We define on \([0, \pi]\) the function \( f \) by \( f(x) = |\cos x| \), it’s easy to see that \( f \) is symmetrized increasing function on \([0, \pi]\) but it’s not symmetrized convex function on \([0, \pi]\).

Remark 1.10. Now, if we denote by \( SIn[a, b] \) the class of symmetrized increasing on \([a, b]\), then from the above examples and remarks, we can conclude that \( Con[a, b] \subset SCon[a, b] \subset SIn[a, b] \).

Definition 1.11. A real-valued function \( f \) is said to be symmetrized non-negative function on \([a, b]\), if the function \( F \) is non-negative on \([a, b]\).

Remark 1.12. If \( f \) is non-negative function on \([a, b]\) then \( f \) is symmetrized non-negative function on \([a, b]\). The inverse is not true in general.

Example 1.13. We define on \([0, 2]\) the function \( g \) by \( g(x) = x^2 - 2 \). Clearly \( g \) is not positive on \([0, 2]\) but it is non-negative function on \([a, b]\).

2. Main results

The aim of this part is to extend the Fejèr inequalities to a new class of functions.

Firstly, we replace the conditions of non-negative and symmetric with respect to \( \frac{a+b}{2} \) of \( g \) by another conditions.

Theorem 2.1. Assume that \( f : [a, b] \to \mathbb{R} \) is symmetrized convex function on \([a, b]\), if \( g \) is integrable, symetrized non-negative and

\[
\int_a^b g(a + b - x)f(x)\,dx = \int_a^b g(x)f(x)\,dx.
\]

Then the Fejèr inequality (1.2) holds.

Remark 2.2. If \( g \) is symmetric with respect to \( \frac{a+b}{2} \) clearly that (2.1) holds. The inverse is not true in general.

Example 2.3. We define on \([a, b]\) the functions \( f \) and \( g \) by

\[
f(x) = \begin{cases} 
  x^2, & \text{if } x \in [a, \frac{3a+b}{4}], \\
  e^x, & \text{if } x \in \left(\frac{3a+b}{4}, \frac{a+3b}{4}\right], \\
  (a + b - x)^2, & \text{if } x \in \left[\frac{a+3b}{4}, b\right],
\end{cases}
\]

and

\[
g(x) = \begin{cases} 
  \frac{1+x^2}{1+x^2}, & \text{if } x \in [a, \frac{3a+b}{4}], \\
  e^x + e^{(a+b-x)^2}, & \text{if } x \in \left(\frac{3a+b}{4}, \frac{a+3b}{4}\right], \\
  \cosh(x^2), & \text{if } x \in \left[\frac{a+3b}{4}, b\right].
\end{cases}
\]

Here \( g \) is not symmetric with respect to \( \frac{a+b}{2} \) but

\[
\int_a^b g(a + b - x)f(x)\,dx = \int_a^b g(x)f(x)\,dx.
\]
Example 2.4. We define on \([0, 2]\) the functions \(f\) and \(g\) by
\[
f(x) = e^x + \frac{4e^2 - e^4 + 1}{8} x
\]
and \(g(x) = e^x - 2.\) Clearly \(g\) is not symmetric with respect to 1 and not positive on \([0, 2]\) but
\[
\int_0^2 g(a + b - x)f(x)\,dx = \int_0^2 g(x)f(x)\,dx.
\]

Corollary 2.5. Assume that \(f : [a, b] \to \mathbb{R}\) is a symmetrized convex function on \([a, b]\), if \(g\) is non-negative, integrable and symmetric function with respect to \(\frac{a + b}{2}\), then the Fejér inequality holds.

In the following Corollary we transform the condition symmetric with respect to \(\frac{a + b}{2}\) frome the weight \(g\) to the convex function \(f\).

Corollary 2.6. Assume that \(f : [a, b] \to \mathbb{R}\) is a convex function and symmetric with respect to \(\frac{a + b}{2}\) and \(g\) is non-negative integrable function on \([a, b]\), then the Fejér inequality holds.

Now we extend the Hermite-Hadamard and Fejér inequalities for \(SIn\) set and we obtain the following:

Theorem 2.7. Assume that \(f : [a, b] \to \mathbb{R}\) is symmetrized increasing function on \([a, b]\), if \(g\) is integrable function symetrized non-negative and satisfies the condition (2.1). Then the Fejér inequality holds.

In Theorem 2.7, if we take \(g \equiv 1\), we get the following Corollary.

Corollary 2.8. Assume that \(f : [a, b] \to \mathbb{R}\) is symmetrized increasing function on \([a, b]\). Then, \(f\) satisfies the Hermite-Hadamard inequality.

Remark 2.9. Theorem 2.7 is a generalization of Theorems A, B and Theorem 1.

3. Proof of Theorems and Corollaries

Proof of Theorem 2.1 Assume that \(f\) is symmetrized convex function on the interval \([a, b]\), then by Lemma 1.2, we have for all \(x \in [a, b]\)
\[
F\left(\frac{a + b}{2}\right) \leq F(x) \leq \frac{F(a) + F(b)}{2}.
\]
Multiplying both sides of (3.1) by \(G(x)\) and we use the fact that \(G\) is a no-negative function we get
\[
G\left(\frac{a + b}{2}\right) \leq G(x) F(x)\,dx \leq \frac{F(a) + F(b)}{2} G(x) .
\]
Then integrating the resulting inequality with respect to \(x\) over \([a, b]\) we get
\[
F\left(\frac{a + b}{2}\right) \int_a^b G(x)\,dx \leq \int_a^b G(x) F(x)\,dx \leq \frac{F(a) + F(b)}{2} \int_a^b G(x)\,dx.
\]
Substituting $F$ and $G$ in (3.3) we get

$$f \left( \frac{a + b}{2} \right) \int_a^b g(x) + g(a + b - x) \frac{dx}{2} \leq \int_a^b \frac{g(x) + g(a + b - x)}{4} f(a + b - x)dx + \int_a^b \frac{g(x) + g(a + b - x)}{4} f(x)dx$$

$$\leq \frac{f(b) + f(a)}{2} \int_a^b \frac{g(x) + g(a + b - x)}{2} dx. \quad (3.4)$$

By the change of variable $t = a + b - x$, $x \in [a, b]$

$$\int_a^b g(a + b - t) dt = \int_a^b g(x) dx \quad (3.5)$$

Also,

$$2f \left( \frac{a + b}{2} \right) \int_a^b g(x) dx \leq \frac{1}{4} \int_a^b g(a + b - x) f(x)dx + \int_a^b g(x) f(x)dx$$

$$+ \frac{1}{4} \int_a^b g(x) f(x)dx + \int_a^b g(a + b - x) f(x)dx$$

$$\leq \frac{f(b) + f(a)}{2} \int_a^b g(x) dx. \quad (3.4)$$

we use the condition (2.1) we get

$$f \left( \frac{a + b}{2} \right) \int_a^b g(x)dx \leq \int_a^b g(x) f(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx. \quad (3.6)$$

**Proof of Corollary 2.5** By $f : [a, b] \to \mathbb{R}$ is a convex function and symmetric to $\frac{a + b}{2}$ we get

$$\int_a^b g(x)f(x)dx = \int_a^b g(x) f(a + b - x) dx.$$ 

Let $y = a + b - x$, thus

$$\int_a^b g(x)f(x)dx = \int_a^b g(a + b - x) f(x) dx.$$ 

Then the conditions of Theorem 2.1 are holds.

**Proof of Theorem 2.7** Assume that $f$ is symmetrized increasing function on the interval $[a, b]$, then clearly that, we have for all $x \in [a, b]$ 

$$F \left( \frac{a + b}{2} \right) \leq F(x) \leq \frac{F(a) + F(b)}{2}.$$ 

By same prove of theorem 1 we can prove Theorem 2.

**REFERENCES**


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