EXTENSION OF FEJÈR INEQUALITY

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Abstract. In this paper, we extend the Fejèr inequality to a new class of functions where f is not necessary convex and g not satisfy the symmetric propriety and not necessary non-negative function.

1. Introduction

Throughout this paper, we denote by I and I' the real intervals $[a,b]$ and (a,b) respectively. A function f is said to be convex on I if and only if $f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y)$ for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. Conversely, if the opposite inequality holds, the function is said to be concave on $[a,b]$. A function f that is continuous on $[a,b]$ and twice differentiable on I' is convex on I if and only if $f''(x)\geqslant 0$ for all $x\in I'$. (f is concave on I if and only if $f''(x)\leqslant 0$ for all $x\in I'$). In 1883 Hermite[11], and in 1893, Hadamard [10] they established the following inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

holds for all convex function f on $[a,b]$. In 1906, Fejèr [9] established the following weighted generalization of Hermite-Hadamard inequality (1.1) and obtained the inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leqslant \int_a^b f(x)\,g(x)dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.2}$$

holds for all convex function f defined on $[a,b]$ and g is non-negative, integrable and symmetric function with respect to $\frac{a+b}{2}$.

The purpose of this work is to extend new results related to inequalities (1.1) and (1.2).

In this part we give definitions and properties of a new class of functions.

For a function $f:[a,b]\to\mathbb{R}$ we consider the symmetrical transform of f on the interval $[a,b]$, denoted by

$$F(x)=\frac{1}{2}\left(f(x)+f(a+b-x)\right).$$

Definition 1.1. [4],[8] A real-valued function f is said to be symmetrized convex on $[a,b]$ if the symmetrical transform of f is convex. Conversely, if F is concave, the function is said to be symmetrized concave on $[a,b]$.

From the above definition we get the following propositions.

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Proposition 1.2. [8] *Suppose that the function f is symmetrized convex on $[a,b]$, then we have:*

(i) The function F is symmetric to $\dfrac{a+b}{2}$ in the sense for all x on $[a,b]$, we have

$$\forall x \in [a,b]: \; F(a+b-x) = F(x).$$

(ii) For all $x \in [a,b]$,, we have

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant F(a) = F(b) = \frac{f(a)+f(b)}{2}.$$

(iii) The function F is increasing on $[\frac{a+b}{2}, b]$ and decreasing on $[a, \frac{a+b}{2}]$.

Proposition 1.3. [8] *If f is a convex function, then f is symmetrized concave on $[a,b]$. The inverse is false.*

Example 1.4. [8] Let $f : [a,b] \longrightarrow \mathbb{R} : x \mapsto x^3$ such that $a < 0 < b$ and $a+b > 0$. Then f is not convex on $[a,b]$, but $F(x) = \dfrac{(a+b-x)^3 + x^3}{2}$ is convex $(F''(x) = 3(a+b) > 0)$.

Example 1.5. [8] The function $f : [a,b] \longrightarrow \mathbb{R} : x \mapsto f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$ such that $a < 0 < b$ and $a+b > 0$ is not convex on $[a,b]$, but $F(x) = \dfrac{f(a+b-x)+f(x)}{2}$ is convex, $(F''(x) = \sinh\left(\frac{a+b}{2}\right)\cosh\left(\frac{a+b}{2} - x\right) > 0)$.

Proposition 1.6. *If f is is symmetric to $\dfrac{a+b}{2}$ then, f is convex on $[a,b]$ if and only if f is symmetrized convex on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above exemples and propositions we can conclude that $Con[a,b] \subsetneq SCon[a,b]$.

We have the following results [4],[8]:

Theorem A *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex function on $[a,b]$. Then f satisfies the Hermite-Hadamard inequality* (1.1)*.*

Theorem B *Let $f : [a,b] \to \mathbb{R}$ be a symmetrized convex function on $[a,b]$, and let $g : [a,b] \to [0,+\infty[$ be an integrable function and symmetric with respect to $\dfrac{a+b}{2}$, then the Fejèr inequality* (1.2) *holds.*

For more results in this direction, please refer [2, 3, 4, 8]

In the above results we have extend the set of functions that satisfies the Hermite-Hadamard inequality and Fijèr inequality. In fact, we replaced the condition of convex functions by symetrized convex functions.

The question arises, can we extend the above inequalities to a new class of functions? for answar this question we give the folowing definitions.

Definition 1.7. A real-valued function f is said to be symmetrized increasing on $[a,b]$, if the function F is increasing on $\left[\dfrac{a+b}{2}, b\right]$.

Remark 1.8. By part *(iii)* of proposition (1.2), if f is symmetrized convex function on $[a,b]$ then f is symmetrized increasing on $[a,b]$. The inverse is not true in general.

Example 1.9. We define on $[0,\pi]$ the function f by $f(x)=|\cos x|$, it's easy to see that f is symmetrized increasing function on $[0,\pi]$ but it's not symmetrized convex function on $[0,\pi]$.

Remark 1.10. Now, if we denote by $SIn\ [a,b]$ the class of symmetrized increasing on $[a,b]$, then from the above exemples and ramarks, we can conclude that $Con[a,b] \subsetneq SCon[a,b] \subsetneq SIn\ [a,b]$.

Definition 1.11. A real-valued function f is said to be symmetrized non-negative function on $[a,b]$, if the function F is non-negative on $[a,b]$.

Remark 1.12. If f is non-negative function on $[a,b]$ then f is symmetrized non-negative function on $[a,b]$. The inverse is not true in general.

Example 1.13. We define on $[0,2]$ the function g by $g(x)=x^2-2$. Clearly g is not positive on $[0,2]$ but it is non-negative function on $[a,b]$.

2. Main results

The aim of this part is to extend the Fejèr inequalities to a new class of functions.

Firstly, we replace the conditions of non-negative and symmetric with respect to $\dfrac{a+b}{2}$ of g by another conditions.

Theorem 2.1. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex function on $[a,b]$, if g is integrable, symetrized non-negative and*

$$\int_a^b g(a+b-x)f(x)\,dx=\int_a^b g(x)f(x)dx. \tag{2.1}$$

Then the Fejèr inequality (1.2) *holds.*

Remark 2.2. If g is symmetric with respect to $\dfrac{a+b}{2}$ clearly that (2.1) holds. The inverse is not true in general.

Example 2.3. We define on $[a,b]$ the functions f and g by

$$f(x)=\begin{cases} x^2, & \text{if } x\in\left[a,\frac{3a+b}{4}\right],\\ e^{x^2}, & \text{if } x\in\left]\frac{3a+b}{4},\frac{a+3b}{4}\right[,\\ (a+b-x)^2, & \text{if } x\in\left[\frac{a+3b}{4},b\right],\end{cases}$$

and

$$g(x)=\begin{cases} \frac{1}{1+x^2}, & \text{if } x\in\left[a,\frac{3a+b}{4}\right],\\ e^{x^2}+e^{(a+b-x)^2}, & \text{if } x\in\left]\frac{3a+b}{4},\frac{a+3b}{4}\right[,\\ \cosh(x^2), & \text{if } x\in\left[\frac{a+3b}{4},b\right].\end{cases}$$

Here g is not symmetric with respect to $\frac{a+b}{2}$ but

$$\int_a^b g(a+b-x)f(x)\,dx=\int_a^b g(x)f(x)dx.$$

Example 2.4. We define on $[0, 2]$ the functions f and g by

$$f(x) = e^x + \frac{4e^2 - e^4 + 1}{8} x$$

and $g(x) = e^x - 2,$. Clearly g is not symmetric with respect to 1 and not positive on $[0, 2]$ but

$$\int_0^2 g(a+b-x) f(x)\, dx = \int_0^2 g(x) f(x) dx.$$

Corollary 2.5. *Assume that $f : [a, b] \to \mathbb{R}$ is a symmetrized convex function on $[a, b]$, if g is non-negative, integrable and symmetric function with respect to $\frac{a+b}{2}$, then the Fejèr inequality holds.*

In the following Corollary we transform the condition symmetric with respect to $\frac{a+b}{2}$ frome the weight g to the convex function f.

Corollary 2.6. *Assume that $f : [a, b] \to \mathbb{R}$ is a convex function and symmetric with respect to $\frac{a+b}{2}$ and g is non-negative integrable function on $[a, b]$, then the Fejèr inequality holds.*

Now we extend the Hermite- Hadamard and Fijèr inequalities for SIn set and we obtain the following:

Theorem 2.7. *Assume that $f : [a, b] \to \mathbb{R}$ is symmetrized increasing function on $[a, b]$, if g is integrable function symetrized non-negative and satisfies the condition (2.1). Then the Fejèr inequality holds.*

In Theorem 2.7, if we take $g \equiv 1$, we get the following Corollary.

Corollary 2.8. *Assume that $f : [a, b] \to \mathbb{R}$ is symmetrized increasing function on $[a, b]$. Then, f satisfies the Hermite-Hadamard inequality.*

Remark 2.9. Theorem 2.7 is a generalization of Theorems A, B and Theorem 1.

3. Proof of Theorems and Corollaries

Proof of Theorem 2.1 Assume that f is symmetrized convex function on the interval $[a, b]$, then by Lemma 1.2, we have for all $x \in [a, b]$

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant \frac{F(a) + F(b)}{2}. \tag{3.1}$$

Multiplying both sides of (3.1) by $G(x)$ and we use the fact that G is a no-negative function we get

$$G(x) F\left(\frac{a+b}{2}\right) \leqslant G(x) F(x) dx \leqslant \frac{F(a) + F(b)}{2} G(x). \tag{3.2}$$

Then integrating the resulting inequality with respect to x over $[a, b]$ we get

$$F\left(\frac{a+b}{2}\right) \int_a^b G(x)\, dx \leqslant \int_a^b G(x) F(x)\, dx \leqslant \frac{F(a) + F(b)}{2} \int_a^b G(x)\, dx. \tag{3.3}$$

Substituting F and G in (3.3) we get

$$f\left(\frac{a+b}{2}\right)\int_a^b \frac{g(x)+g(a+b-x)}{2}dx \leqslant \int_a^b \frac{g(x)+g(a+b-x)}{4}f(a+b-x)dx + \int_a^b \frac{g(x)+g(a+b-x)}{4}f(x)dx \leqslant \frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)+g(a+b-x)}{2}dx. \tag{3.4}$$

By the change of variable $t = a+b-x$, $x \in [a,b]$

$$\int_a^b g(a+b-t)\,dx = \int_a^b g(x)\,dx \tag{3.5}$$

Also,

$$2f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leqslant \frac{1}{4}\int_a^b g(a+b-x)f(x)dx + \int_a^b g(x)f(x)dx + \frac{1}{4}\int_a^b g(x)f(x)dx + \int_a^b g(a+b-x)f(x)dx \leqslant \frac{f(b)+f(a)}{2}\int_a^b g(x)\,dx. \tag{3.4}$$

we use the condition (2.1) we get

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leqslant \int_a^b g(x)f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{3.6}$$

Proof of Corollary 2.5 By $f:[a,b]\to\mathbb{R}$ is a convex function and symmetric to $\frac{a+b}{2}$ we get

$$\int_a^b g(x)f(x)dx = \int_a^b g(x)f(a+b-x)\,dx.$$

Let $y = a+b-x$, thus

$$\int_a^b g(x)f(x)dx = \int_a^b g(a+b-x)f(x)\,dx.$$

Then the conditions of Theorem 2.1 are holds.

Proof of Theorem 2.7 Assume that f is symmetrized ingreasing function on the interval $[a,b]$, then clearly that, we have for all $x\in[a,b]$

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant \frac{F(a)+F(b)}{2}.$$

By same prove of theorem 1 we can prove Theorem 2.

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