SOME INTEGRAL INEQUALITIES FOR OPERATOR MONOTONIC FUNCTIONS ON HILBERT SPACES

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Abstract. Let f be an operator monotonic function on I and $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I. Assume that $p : [0,1] \to \mathbb{R}$ is non-decreasing on $[0,1]$. In this paper we obtained, among others, that for $A \leq B$ and f an operator monotonic function on I,

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{1}{4}[p(1) - p(0)][f(B) - f(A)] \end{aligned}$$

in the operator order.

Several other similar inequalities for either p or f is differentiable, are also provided. Applications for power function and logarithm are given as well.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [7] had given a definitive characterization of operator monotone functions as follows:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t}{t+s}\,dm(s)$$

where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty \frac{dm(s)}{t+s} < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$.

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In [3], T. Furuta observed that for $\alpha_j \in [0,1]$, $j = 1, ..., n$ the functions

$$g(t) := \left(\sum_{j=1}^{n} t^{-\alpha_j} \right)^{-1} \text{ and } h(t) = \sum_{j=1}^{n} \left(1 + t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0, \infty)$.

Let $f(t)$ be a continuous function $(0, \infty) \to (0, \infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [3] or [8].

Consider the family of functions defined on $(0, \infty)$ and $p \in [-1, 2] \setminus \{0, 1\}$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

and

$$f_0(t) := \frac{t}{1-t} \ln t,$$

$$f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean)}.$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean)}, \ f_{1/2}(t) = \sqrt{t} \text{ (geometric mean)}.$$

In [2] the authors showed that f_p is operator monotone for $1 \le p \le 2$.

In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p - 1}$$

is an operator monotone function for $p \in (0, 1]$, [3].

It is well known that the logarithmic function ln is operator monotone and in [3] the author obtained that the functions

$$f(t) = t(1+t) \ln\left(1 + \frac{1}{t}\right), \ g(t) = \frac{1}{(1+t) \ln\left(1 + \frac{1}{t}\right)}$$

are also operator monotone functions on $(0, \infty)$.

Let f be an operator monotonic function on I and A, $B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I. Assume that $p : [0,1] \to \mathbb{R}$ is nondecreasing on $[0,1]$. In this paper we obtain, among others, that for $A \le B$ and f an operator monotonic function on I,

$$\begin{aligned} 0 &\le \int_0^1 p(t) f((1-t)A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 f((1-t)A + tB) \, dt \\ &\le \frac{1}{4} [p(1) - p(0)] [f(B) - f(A)] \end{aligned}$$

in the operator order.

Several other similar inequalities for either p or f is differentiable, are also provided. Applications for power function and logarithm are given as well.

2. Main Results

For two *Lebesgue integrable* functions $h,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)dt-\frac{1}{b-a}\int_a^b h(t)dt\frac{1}{b-a}\int_a^b g(t)dt. \tag{2.1}$$

It is well known that, if h and g have the same monotonicity on $[a,b]$, then

$$\frac{1}{b-a}\int_a^b h(t)g(t)dt\ge\frac{1}{b-a}\int_a^b h(t)dt\frac{1}{b-a}\int_a^b g(t)dt, \tag{2.2}$$

which is known in the literature as *Čebyšev's inequality.*

In 1935, Grüss [4] showed that

$$|C(h,g)|\le\frac{1}{4}(M-m)(N-n), \tag{2.3}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m\le h(t)\le M\quad\text{and}\quad n\le g(t)\le N\quad\text{for a.e. } t\in[a,b]. \tag{2.4}$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

Let f be a continuous function on I. If $(A,B)\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I and $t\in[0,1]$, then the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator in $\mathcal{B}(H)$.

For $A,\ B\in\mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{2.5}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle. \tag{2.6}$$

Theorem 2. *Let $A,\ B\in\mathcal{SA}_I(H)$ with $A\le B$ and f an operator monotonic function on I. If $p:[0,1]\to\mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then*

$$\begin{aligned}0&\le\int_0^1 p(t)f((1-t)A+tB)dt-\int_0^1 p(t)dt\int_0^1 f((1-t)A+tB)dt\\&\le\frac{1}{4}[p(1)-p(0)][f(B)-f(A)].\end{aligned} \tag{2.7}$$

If $p:[0,1]\to\mathbb{R}$ is monotonic nonincreasing on $[0,1]$, then

$$\begin{aligned}0&\le\int_0^1 p(t)dt\int_0^1 f((1-t)A+tB)dt-\int_0^1 p(t)f((1-t)A+tB)dt\\&\le\frac{1}{4}[p(0)-p(1)][f(B)-f(A)].\end{aligned} \tag{2.8}$$

Proof. Let $0 \leq t_1 < t_2 \leq 1$ and $A \leq B$. Then

$$(1-t_2)A + t_2 B - (1-t_1)A - t_1 B = (t_2 - t_1)(B-A) \geq 0$$

and by operator monotonicity of f we get

$$f((1-t_2)A + t_2 B) \geq f((1-t_1)A + t_1 B),$$

which is equivalent to

$$\begin{aligned}\varphi_{(A,B);x}(t_2) &= \langle f((1-t_2)A + t_2 B)x, x\rangle \\ &\geq \langle f((1-t_1)A + t_1 B)x, x\rangle = \varphi_{(A,B);x}(t_1)\end{aligned}$$

that shows that the scalar function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ is monotonic nondecreasing for $A \leq B$ and for any $x \in H$.

If we write the inequality (2.2) for the functions p and $\varphi_{(A,B);x}$ we get

$$\int_0^1 p(t)\langle f((1-t)A + tB)x, x\rangle dt \geq \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A + tB)x, x\rangle dt,$$

which can be written as

$$\begin{aligned}&\left\langle \left(\int_0^1 p(t) f((1-t)A + tB)\,dt\right)x, x\right\rangle \\ &\geq \left\langle \left(\int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\right) dtx, x\right\rangle\end{aligned}$$

for $x \in H$, and the first inequality in (2.7) is obtained.

We also have that

$$\begin{aligned}\langle f(A)x, x\rangle &= \varphi_{(A,B);x}(0) \leq \varphi_{(A,B);x}(t) = \langle f((1-t)A + tB)x, x\rangle \\ &\leq \varphi_{(A,B);x}(1) = \langle f(B)x, x\rangle\end{aligned}$$

and

$$p(0) \leq p(t) \leq p(1)$$

for all $t \in [0,1]$.

By writing Grüss' inequality for the functions $\varphi_{(A,B);x}$ and p, we get

$$\begin{aligned}0 &\leq \int_0^1 p(t)\langle f((1-t)A + tB)x, x\rangle dt \\ &- \int_0^1 p(t)\,dt \int_0^1 \langle f((1-t)A + tB)x, x\rangle dt \\ &\leq \frac{1}{4}[p(1) - p(0)][\langle f(B)x, x\rangle - \langle f(A)x, x\rangle]\end{aligned}$$

for $x \in H$ and the second inequality in (2.7) is obtained. □

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.9}$$

If the limit (2.9) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that $A, B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

Lemma 1. *Let f be a continuous function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.10}$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.11}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.12}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A + (t+h)B) - f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h}. \end{aligned} \tag{2.13}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.13) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h} \\ &= \nabla f_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.10).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)A + hB) - f(A)}{h} \\ &= \lim_{h\to 0+} \frac{f(A + h(B-A)) - f(A)}{h} \\ &= \nabla f_A(B-A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.11).

The equality (2.12) follows in a similar way. □

Lemma 2. *Let f be an operator monotonic function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \leq B$, $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\nabla f_{(1-t)A+tB}(B-A) \geq 0 \text{ for all } t \in (0,1). \tag{2.14}$$

Also

$$\nabla f_A(B-A),\ \nabla f_B(B-A)\geq 0. \tag{2.15}$$

Proof. Let $x\in H$. The auxiliary function $\varphi_{(A,B);x}$ is monotonic nondecreasing in the usual sense on $[0,1]$ and differentiable on $(0,1)$, and for $t\in(0,1)$

$$\begin{aligned}
0 \leq \varphi'_{(A,B);x}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B),x}(t+h)-\varphi_{(A,B),x}(t)}{h}\\
&= \lim_{h\to 0}\left\langle \frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\
&= \left\langle \lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\
&= \left\langle \nabla f_{(1-t)A+tB}(B-A)x,x\right\rangle.
\end{aligned}$$

This shows that

$$\nabla f_{(1-t)A+tB}(B-A)\geq 0$$

for all $t\in(0,1)$.

The inequalities (2.15) follow by (2.11) and (2.12). □

The following inequality obtained by Ostrowski in 1970, [9] also holds

$$|C(h,g)|\leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{2.16}$$

provided that h is *Lebesgue integrable* and satisfies (2.4) while g is absolutely continuous and $g'\in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.16).

Theorem 3. *Let A, $B\in\mathcal{SA}_I(H)$ with $A\leq B$, f be an operator monotonic function on I and $p:[0,1]\to\mathbb{R}$ monotonic nondecreasing on $[0,1]$.*

(i) *If p is differentiable on $(0,1)$, then*

$$\begin{aligned}
0 &\leq \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt \\
&\leq \frac{1}{8}\sup_{t\in(0,1)} p'(t)\,[f(B)-f(A)].
\end{aligned} \tag{2.17}$$

(ii) *If $f\in\mathcal{G}([A,B])$, then*

$$\begin{aligned}
0 &\leq \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt \\
&\leq \frac{1}{8}[p(1)-p(0)]\sup_{t\in(0,1)}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\| 1_H.
\end{aligned} \tag{2.18}$$

Proof. Let $x\in H$. If we use the inequality (2.16) for $g=p$ and $h=\varphi_{(A,B);x}$, then

$$\begin{aligned}
0 &\leq \int_0^1 p(t)\langle f((1-t)A+tB)x,x\rangle\,dt \\
&\quad - \int_0^1 p(t)\,dt\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,dt \\
&\leq \frac{1}{8}\sup_{t\in(0,1)} p'(t)\,[\langle f(B)x,x\rangle-\langle f(A)x,x\rangle],
\end{aligned}$$

for any $x\in H$, which is equivalent to (2.17).

If we use the inequality (2.16) for $h=p$ and $g=\varphi_{(A,B);x}$ then by Lemmas 1 and 2

$$\begin{aligned}0 &\leq \int_0^1 p(t)\langle f((1-t)A+tB)x,x\rangle dt \\ &-\int_0^1 p(t)\,dt\int_0^1 \langle f((1-t)A+tB)x,x\rangle dt \\ &\leq \frac{1}{8}[p(1)-p(0)]\sup_{t\in(0,1)}\left\langle \nabla f_{(1-t)A+tB}(B-A)x,x\right\rangle ,\end{aligned} \tag{2.19}$$

for any $x\in H$, which is an inequality of interest in itself.

Observe that for all $t\in(0,1)$,

$$\left\langle \nabla f_{(1-t)A+tB}(B-A)x,x\right\rangle \leq \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\|x\|^2$$

for any $x\in H$, which implies that

$$\sup_{t\in(0,1)}\left\langle \nabla f_{(1-t)A+tB}(B-A)x,x\right\rangle \leq \sup_{t\in(0,1)}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\langle 1_H x,x\rangle \tag{2.20}$$

for any $x\in H$.

By making use of (2.19) and (2.20) we derive

$$\begin{aligned}0 &\leq \int_0^1 p(t)\langle f((1-t)A+tB)x,x\rangle dt \\ &-\int_0^1 p(t)\,dt\int_0^1 \langle f((1-t)A+tB)x,x\rangle dt \\ &\leq \frac{1}{8}[p(1)-p(0)]\sup_{t\in(0,1)}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\langle 1_H x,x\rangle\end{aligned}$$

for any $x\in H$, which is equivalent to (2.18). □

Another, however less known result, even though it was obtained by Čebyšev in 1882, [1], states that

$$|C(h,g)|\leq \frac{1}{12}\|h'\|_\infty\|g'\|_\infty(b-a)^2, \tag{2.21}$$

provided that h', g' exist and are continuous on $[a,b]$ and $\|h'\|_\infty=\sup_{t\in[a,b]}|h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [5] in which he proved that

$$|C(h,g)|\leq \frac{1}{\pi^2}\|h'\|_2\|g'\|_2(b-a), \tag{2.22}$$

provided that h, g are absolutely continuous and h', $g'\in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Using the above inequalities (2.21) and (2.22) and a similar procedure to the one employed in the proof of Theorem 3, we can also state the following result:

Theorem 4. *Let* A, $B\in\mathcal{SA}_I(H)$ *with* $A\leq B$, f *be an operator monotonic function on* I *and* $p:[0,1]\to\mathbb{R}$ *monotonic nondecreasing on* $[0,1]$. *If* p *is differentiable* *and* $f \in \mathcal{G}([A,B])$, *then*

$$0 \leq \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \tag{2.23}$$
$$\leq \frac{1}{12} \sup_{t\in(0,1)} p'(t) \sup_{t\in(0,1)} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| 1_H$$

and

$$0 \leq \int_0^1 p(t) f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \tag{2.24}$$
$$\leq \frac{1}{\pi^2} \left(\int_0^1 [p'(t)]^2\,dt \right)^{1/2} \left(\int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\|^2 dt \right)^{1/2} 1_H,$$

provided the integrals in the second term are finite.

3. Some Examples

We consider the function $f:(0,\infty) \to \mathbb{R}$, $f(t) = -t^{-1}$ which is *operator monotone* on $(0,\infty)$.

If $0 < A \leq B$ and $p:[0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (2.7)

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \tag{3.1}$$
$$\leq \frac{1}{4}[p(1)-p(0)]\left(A^{-1}-B^{-1}\right).$$

Moreover, if p is differentiable on $(0,1)$, then by (2.17) we obtain

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \tag{3.2}$$
$$\leq \frac{1}{8} \sup_{t\in(0,1)} p'(t)\left(A^{-1}-B^{-1}\right).$$

The function $f(t) = -t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S) = T^{-1}ST^{-1}$$

for T, $S > 0$.

If $p:[0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (2.18) we get

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \tag{3.3}$$
$$\leq \frac{1}{8}[p(1)-p(0)]$$
$$\times \sup_{t\in(0,1)} \left\| ((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1} \right\| 1_H$$

for $0 < A \leq B$.

If p is monotonic nondecreasing and differentiable on $(0,1)$, then by (2.23) and (2.24) we get

$$0 \le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \le \frac{1}{12}\sup_{t\in(0,1)} p'(t) \times \sup_{t\in(0,1)} \left\| ((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1} \right\| 1_H \tag{3.4}$$

and

$$0 \le \int_0^1 p(t)\,dt \int_0^1 ((1-t)A+tB)^{-1}\,dt - \int_0^1 p(t)((1-t)A+tB)^{-1}\,dt \le \frac{1}{\pi^2}\left(\int_0^1 [p'(t)]^2\,dt\right)^{1/2} \times \left(\int_0^1 \left\| ((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1} \right\|^2 dt\right)^{1/2} 1_H, \tag{3.5}$$

for $0 < A \le B$.

We note that the function $f(t) = \ln t$ is operator monotonic on $(0,\infty)$.

If $0 < A \le B$ and $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (2.7) we have

$$0 \le \int_0^1 p(t)\ln((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A+tB)\,dt \le \frac{1}{4}[p(1)-p(0)](\ln B - \ln A). \tag{3.6}$$

Moreover, if p is differentiable on $(0,1)$, then by (2.17) we obtain

$$0 \le \int_0^1 p(t)\ln((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A+tB)\,dt \le \frac{1}{8}\sup_{t\in(0,1)} p'(t)(\ln B - \ln A). \tag{3.7}$$

The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [10, p. 155]):

$$\nabla \ln_T(S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1}\,ds \tag{3.8}$$

for T, $S > 0$.

If $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (2.18) we get

$$0 \le \int_0^1 p(t)\ln((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A+tB)\,dt \le \frac{1}{8}[p(1)-p(0)] \times \sup_{t\in(0,1)} \left\| \int_0^\infty (s1_H + (1-t)A+tB)^{-1}(B-A)(s1_H+(1-t)A+tB)^{-1}\,ds \right\| 1_H \tag{3.9}$$

and if p is differentiable on $(0,1)$, then

(3.10)

$$0 \le \int_0^1 p(t) \ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 \ln((1-t)A + tB)\,dt$$
$$\le \frac{1}{12} \sup_{t\in(0,1)} p'(t)$$
$$\times \sup_{t\in(0,1)} \left\| \int_0^\infty (s1_H + (1-t)A + tB)^{-1} (B-A) (s1_H + (1-t)A + tB)^{-1} ds \right\| 1_H$$

for $0 < A \le B$.

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