SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR EXPONENTIAL CONVEX FUNCTIONS

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Abstract. Some new Hermite-Hadamard type inequalities will be presented in this work for functions whose second derivative in absolute value at certain power is exponential type convex.

1. Introduction

The classical inequality of Hermite-Hadamard has been considered very useful in mathematical analysis being extended and generalized in many directions by authors as [7, 6, 11, 1, 16, 22, 13] and the references therein.

Using a recent concept of exponential type convex functions given in [14] some Hermite-Hadamard type inequalities will be presented for this new kind of functions.

We begin by recalling below the classical definition for the convex functions.

Definition 1. *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is said to be convex on an interval* I *if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ *and* $t \in [0,1]$*. The function* f *is said to be concave on* I *if the inequality (1) takes place in reversed direction.*

Definition 2. *(see* [14]*) A nonnegative function* $f : I \to \mathbb{R}$ *iscalled exponential type convex function if, for every* $m, n \in I$ *and* $k \in [0,1]$,

$$f(km + (1-k)n) \leq (e^k - 1)f(m) + (e^{1-k} - 1)f(n). \tag{2}$$

The class of all exponentially type convex functions on interval I *is indicated by EXPC(I).*

Definition 3. *(*[27]*) Let* $h : J \to \mathbf{R}$ *be a nonnegative function and* $h \neq 0$*. We say that* $f : I \to \mathbf{R}$ *is an* h*-convex function, or that* f *belongs to the class SX(h,I), if is nonnegative and for all* $m, n \in I,\ k \in [0,1]$ *we have*

$$f(km + (1-k)n) \leq h(k)f(m) + h(1-k)f(n).$$

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When previous inequality is reversed then f is said to be a h-concave function, i.e. $f \in SV(h, I)$. It is obvious that when $h(u) = u$ then the h-convexity becomes convexity.

We know from [14] that every nonnegative convex function is exponential type convex function and that every exponential type convex function is an h-convex function with $h(k) = e^k - 1$.

It is necessary to recall below the definition of fractionals integrals, see [9, 12, 11, 23, 24]. For other type of convexity see also [25, 19].

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Definition 4. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

Lemma 1. *(see [3]) Let $f : I \to \mathbf{R}$, $: I \subset \mathbf{R}$ be a twice differentiable function on I^o, the interior of I, where $a, b \in I,\ a < b$. If $f^{''} \in L[a,b]$, then the following equality holds:*

$$-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{128}[\int_0^1 t^2 f^{''}\left(t\frac{3a+b}{4}+(1-t)a\right)dt+\int_0^1 (t-1)^2 f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt+$$

$$+\int_0^1 t^2 f^{''}\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt+\int_0^1 (t-1)^2 f^{''}\left(tb+(1-t)\frac{a+3b}{4}\right)dt]$$

We will use also the following result given in [20]:

Lemma 2. *Let $f : I \to \mathbf{R}$, $: I \subset \mathbf{R}$ be a twice differentiable function on I^o, where $a, b \in I,\ a < b$ with $t \in [0,1]$.*

If $f^{''} \in L[a,b]$, then for all $a < b$ and $\alpha - 1 > 0$, with properties of Gamma function we have

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha-1} f(b) + J_{\left(\frac{a+b}{2}\right)^-}^{\alpha-1} f(a)] - f\left(\frac{a+b}{2}\right) =$$

$$= \frac{(b-a)^2}{\alpha 2^{2-\alpha}}[\int_0^{\frac{1}{2}} t^\alpha f^{''}(ta+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (1-t)^\alpha f^{''}(ta+(1-t)b)\,dt].$$

The following result is a generalization of Lemma 2 from [3], when $\alpha > n-1$ and $n \in \mathbb{N}$.

Lemma 3. *(see [5]) Let $n \in \mathbb{N}^*$, $n \geq 2$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a,b \in I^0$, $0 < a < b$, $x \in [a,b]$, $\lambda \in (0,1)$. Then the following identity holds:*

$$\mathcal{I}(f,x,a,b,\lambda,\alpha,n) = (1-\lambda)(x-a)\int_0^1 t^\alpha f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)dt+$$
$$+\lambda(x-a)\int_0^1 (1-t)^\alpha f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))dt+$$
$$+(1-\lambda)(b-x)\int_0^1 t^\alpha f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)dt+$$
$$+\lambda(b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))dt =$$
$$= \sum_{k=2}^n \alpha(\alpha-1)...(\alpha-k+2)[(\frac{(-1)^{k-1}}{(1-\lambda)^{k-1}} - \frac{1}{\lambda^{k-1}})(\frac{f^{(n-k)}(\lambda a+(1-\lambda)x)}{(x-a)^{k-1}}+$$
$$+\frac{f^{(n-k)}(\lambda x+(1-\lambda)b)}{(b-x)^{k-1}})] + \Gamma(\alpha+1)\{\frac{(-1)^n}{(1-\lambda)^\alpha(x-a)^\alpha} J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^-} f(a)+$$
$$+\frac{1}{\lambda^\alpha(b-x)^\alpha} J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^+} f(b) + \frac{1}{\lambda^\alpha(x-a)^\alpha} J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^+} f(x)+$$
$$+\frac{(-1)^n}{(1-\lambda)^\alpha(b-x)^\alpha} J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^-} f(x)\},$$

where $\alpha > n-1$.

Some Hermite-Hadamard type inequalities will be presented in this work in Theorem 1, 2, 3, 4, 5 and 6 for functions whose second derivative in absolute value at certain power is exponential type convex.

2. On Hermite-Hadamard type inequalities for exponential type convex functions

The aim of this section is to present new inequalities that refine Hermite-Hadamard inequality for functions whose second derivative in absolute value at certain power is exponential type convex.

Theorem 1. *Let $f : I \to \mathbf{R}$, $I \subseteq \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $|f^{''}|$ is an exponential type convex function on $[a,b]$ then the following inequality*

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} + \frac{1}{b-a}\int_a^b f(x)dx| \leq$$
$$\leq \frac{(b-a)^2}{64}\{(e-\frac{7}{3})A\left(|f^{''}\left(\frac{3a+b}{4}\right)|,|f^{''}\left(\frac{a+3b}{4}\right)|\right)+$$

$$+(e-\frac{8}{3})A\left(|f^{''}(a)|,|f^{''}(b)|\right)+2(e-\frac{8}{3})|f^{''}\left(\frac{a+b}{2}\right)|\}$$

holds.

Proof. By using Lemma 1 and the properties of the modulus we will find,

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)^2}{128}[\int_0^1 t^2|f^{''}\left(t\frac{3a+b}{4}+(1-t)a\right)|dt+$$

$$+\int_0^1(t-1)^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)|dt+$$

$$+\int_0^1 t^2|f^{''}\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)|dt+\int_0^1(t-1)^2|f^{''}\left(tb+(1-t)\frac{a+3b}{4}\right)|dt]$$

where we also used the definition of the exponential type convex functions in last inequality.

By calculus, we get:

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)^2}{128}[|f^{''}\left(\frac{3a+b}{4}\right)|\int_0^1 t^2(e^t-1)dt+|f^{''}(a)|\int_0^1 t^2(e^{1-t}-1)dt+$$

$$+|f^{''}\left(\frac{a+b}{2}\right)|\int_0^1(t-1)^2(e^t-1)dt+|f^{''}\left(\frac{3a+b}{4}\right)|\int_0^1(t-1)^2(e^{1-t}-1)dt+$$

$$+|f^{''}\left(\frac{a+3b}{4}\right)|\int_0^1 t^2(e^t-1)dt+|f^{''}\left(\frac{a+b}{2}\right)|\int_0^1 t^2(e^{1-t}-1)dt+$$

$$+|f^{''}(b)|\int_0^1(t-1)^2(e^t-1)dt+|f^{''}\left(\frac{a+3b}{4}\right)|\int_0^1(t-1)^2(e^{1-t}-1)dt].$$

We see that $\int_0^1 t^2(e^t-1)dt=\int_0^1(t-1)^2(e^{1-t}-1)dt=e-\frac{7}{3}$,
$\int_0^1 t^2(e^{1-t}-1)dt=\int_0^1(t-1)^2(e^t-1)dt=2e-\frac{16}{3}$ and from here we get,

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)^2}{64}\{(e-\frac{7}{3})[|f^{''}\left(\frac{3a+b}{4}\right)|+|f^{''}\left(\frac{a+3b}{4}\right)|]+2(e-\frac{8}{3})|f^{''}\left(\frac{a+b}{2}\right)|+$$

$$+(e-\frac{8}{3})[|f^{''}(a)|+|f^{''}(b)|]\}.$$

Therefore we obtain the desired inequailty by replacing these expressions with the arithmetic mean $A(u,v)$ of u and v.

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Theorem 2. *Let $f : I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L^1[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{''}|^q$ is an exponential type convex function on $[a,b]$ then the following inequality*

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$
$$\leq \frac{(b-a)^2}{128}\frac{(e-2)^{\frac{1}{q}}}{(2p+1)^{\frac{1}{p}}}\{A^{\frac{1}{q}}\left(|f^{''}(\frac{3a+b}{4})|^q, |f^{''}(a)|^q\right)+$$
$$+A^{\frac{1}{q}}\left(|f^{''}(\frac{a+b}{2})|^q, |f^{''}(\frac{3a+b}{4})|^q\right)+A^{\frac{1}{q}}\left(|f^{''}(\frac{a+3b}{4})|^q, |f^{''}(\frac{a+b}{2})|^q\right)+$$
$$+A^{\frac{1}{q}}\left(|f^{''}(\frac{a+3b}{4})|^q, |f^{''}(b)|^q\right)\}$$

holds, where $A(u,v)$ is the arithmetic mean of u and v.

Proof. This time we use Holder's inequality. We start like in previous demonstration, by using Lemma 1 and then the modulus properties and we get:

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$
$$\leq \frac{(b-a)^2}{128}[\int_0^1 t^2|f^{''}\left(t\frac{3a+b}{4}+(1-t)a\right)|dt+$$
$$+\int_0^1 (t-1)^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)|dt+$$
$$+\int_0^1 t^2|f^{''}\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)|dt+\int_0^1 (t-1)^2|f^{''}\left(tb+(1-t)\frac{a+3b}{4}\right)|dt] \leq$$
$$\leq \frac{(b-a)^2}{128}\{\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(|f^{''}\left(t\frac{3a+b}{4}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$
$$+\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)|^q dt\right)^{\frac{1}{q}}+$$
$$+\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(|f^{''}\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)|^q dt\right)^{\frac{1}{q}}+$$
$$+\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(|f^{''}\left(tb+(1-t)\frac{a+3b}{4}\right)|^q dt\right)^{\frac{1}{q}}\},$$

where we also used Holder's inequality in last inequality.

By using the definition of the exponential type convex functions for the function $|f^{''}|^q$ and calculus we obtain,

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$
$$\leq \frac{(b-a)^2}{128}\frac{1}{(2p+1)^{\frac{1}{p}}}\{[|f^{''}\left(\frac{3a+b}{4}\right)|^q\int_0^1(e^t-1)dt+|f^{''}(a)|^q\int_0^1(e^{1-t}-1)dt]^{\frac{1}{q}}+$$
$$+[|f^{''}\left(\frac{a+b}{2}\right)|^q\int_0^1(e^t-1)dt+|f^{''}(\frac{3a+b}{4})|^q\int_0^1(e^{1-t}-1)dt]^{\frac{1}{q}}+$$

$$+[|f^{''}\left(\frac{a+3b}{4}\right)|^q\int_0^1(e^t-1)dt+|f^{''}(\frac{a+b}{2})|^q\int_0^1(e^{1-t}-1)dt]^{\frac{1}{q}}+$$

$$+[|f^{''}(b)|^q\int_0^1(e^t-1)dt+|f^{''}(\frac{a+3b}{4})|^q\int_0^1(e^{1-t}-1)dt]^{\frac{1}{q}}\}=$$

$$=\frac{(b-a)^2}{128}\frac{(e-2)^{\frac{1}{q}}}{(2p+1)^{\frac{1}{p}}}\{[|f^{''}(\frac{3a+b}{4})|^q+|f^{''}(a)|^q]^{\frac{1}{q}}+[|f^{''}(\frac{a+b}{2})|^q+|f^{''}(\frac{3a+b}{4})|^q]^{\frac{1}{q}}+$$

$$+[|f^{''}(\frac{a+3b}{4})|^q+|f^{''}(\frac{a+b}{2})|^q]^{\frac{1}{q}}+[|f^{''}(\frac{a+3b}{4})|^q+|f^{''}(b)|^q]^{\frac{1}{q}}\}$$

, i.e. the inequality from theorem.

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Theorem 3. *Let $f: I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L^1[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f^{''}|^q$ is an exponential type convex function on $[a,b]$ then the following inequality*

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)^2}{128}\frac{1}{3^{\frac{1}{p}}}\{A^{\frac{1}{q}}\left((e-\frac{7}{3})|f^{''}(\frac{3a+b}{4})|^q,2(e-\frac{8}{3})|f^{''}(a)|^q\right)+$$

$$+A^{\frac{1}{q}}\left(2(e-\frac{8}{3})|f^{''}(\frac{a+b}{2})|^q,(e-\frac{7}{3})|f^{''}(\frac{3a+b}{4})|^q\right)+$$

$$+A^{\frac{1}{q}}\left((e-\frac{7}{3})|f^{''}(\frac{a+3b}{4})|^q,2(e-\frac{8}{3})|f^{''}(\frac{a+b}{2})|^q\right)+$$

$$+A^{\frac{1}{q}}\left((e-\frac{7}{3})|f^{''}(\frac{a+3b}{4})|^q,2(e-\frac{8}{3})|f^{''}(b)|^q\right)\}$$

holds, where $A(u,v)$ is the arithmetic mean of u and v.

Proof. We use the same tools like in Theorem 2, i.e. Lemma 1, the modulus properties and the definition of the exponential type convex functions for the function $|f^{''}|^q$, but here we use the Holder's inequality like below:

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)^2}{128}\{\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2|f^{''}(t\frac{3a+b}{4}+(1-t)a)|^qdt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 (t-1)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2|f^{''}(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})|^qdt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2|f^{''}(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})|^qdt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 (t-1)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2|f^{''}(tb+(1-t)\frac{a+3b}{4})|^qdt\right)^{\frac{1}{q}}\}\leq$$

$$\leq\frac{(b-a)^2}{128}\frac{1}{3^{\frac{1}{p}}}\{[(e-\frac{7}{3})|f^{''}(\frac{3a+b}{4})|^q+2(e-\frac{8}{3})|f^{''}(a)|^q]^{\frac{1}{q}}+$$

$$+[2(e-\frac{8}{3})|f''(\frac{a+b}{2})|^q+(e-\frac{7}{3})|f''(\frac{3a+b}{4})|^q]^{\frac{1}{q}}+$$
$$+[(e-\frac{7}{3})|f''(\frac{a+3b}{4})|^q+2(e-\frac{8}{3})|f''(\frac{a+b}{2})|^q]^{\frac{1}{q}}+$$
$$+[2(e-\frac{8}{3})|f''(b)|^q+(e-\frac{7}{3})|f''(\frac{a+3b}{4})|^q]^{\frac{1}{q}}\}.$$

From here we get the desired inequality.

∎

Theorem 4. *Let $n\in\mathbb{N}^*$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)}\in L[a,b]$ with $a,b\in I^0$, $0<a<b$. If $|f^{(n)}|^q$ is an exponential type convex function on $[a,b]$ for some fixed $q>1$, where $\frac{1}{p}+\frac{1}{q}=1$ then the following inequality*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)|=$$
$$=|\sum_{k=2}^{n}\alpha(\alpha-1)...(\alpha-k+2)[(\frac{(-1)^{k-1}}{(1-\lambda)^{k-1}}-\frac{1}{\lambda^{k-1}})(\frac{f^{(n-k)}(\lambda a+(1-\lambda)x)}{(x-a)^{k-1}}+$$
$$+\frac{f^{(n-k)}(\lambda x+(1-\lambda)b)}{(b-x)^{k-1}})]+\Gamma(\alpha+1)\{\frac{(-1)^n}{(1-\lambda)^{\alpha}(x-a)^{\alpha}}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^-}f(a)+$$
$$+\frac{1}{\lambda^{\alpha}(b-x)^{\alpha}}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^+}f(b)+\frac{1}{\lambda^{\alpha}(x-a)^{\alpha}}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^+}f(x)+$$
$$+\frac{(-1)^n}{(1-\lambda)^{\alpha}(b-x)^{\alpha}}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^-}f(x)\}|\leq$$
$$\leq\frac{(e-2)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)A^{\frac{1}{q}}\left(|f^{(n)}(a)|^q,|f^{(n)}(\lambda a+(1-\lambda)x)|^q\right)+$$
$$+\lambda(x-a)A^{\frac{1}{q}}\left(|f^{(n)}(\lambda a+(1-\lambda)x)|^q,|f^{(n)}(x)|^q\right)+$$
$$+(1-\lambda)(b-x)A^{\frac{1}{q}}\left(|f^{(n)}(x)|^q,|f^{(n)}(\lambda x+(1-\lambda)b)|^q\right)+$$
$$+\lambda(b-x)A^{\frac{1}{q}}\left(|f^{(n)}(\lambda x+(1-\lambda)b)|^q,|f^{(n)}(b)|^q\right)\}$$

takes place, where $A(u,v)$ is the arithmetic mean of u and v.

Proof. Using now Lemma 3, the properties of modulus and Holder's inequality, we have,

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)|\leq$$
$$\leq(1-\lambda)(x-a)\int_0^1 t^{\alpha}|f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)|dt+$$
$$+\lambda(x-a)\int_0^1(1-t)^{\alpha}|f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))|dt+$$
$$+(1-\lambda)(b-x)\int_0^1 t^{\alpha}|f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)|dt+$$
$$+\lambda(b-x)\int_0^1(1-t)^{\alpha}|f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))|dt\leq$$

$$\leq (1-\lambda)(x-a)\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\lambda(x-a)\left(\int_0^1 (1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))|^q dt\right)^{\frac{1}{q}}+$$

$$+(1-\lambda)(b-x)\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)|^q dt\right)^{\frac{1}{q}}+$$

$$+\lambda(b-x)\left(\int_0^1 (1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))|^q dt\right)^{\frac{1}{q}}.$$

Now, taking into account that $|f^{(n)}|^q$ is an exponential type convex function on $[a,b]$, we get by calculus,

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)|\leq$$

$$\leq \frac{(e-2)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)[|f^{(n)}(\lambda a+(1-\lambda)x)|^q+|f^{(n)}(a)|^q]^{\frac{1}{q}}+$$

$$+\lambda(x-a)[|f^{(n)}(x)|^q+|f^{(n)}(\lambda a+(1-\lambda)x)|^q]^{\frac{1}{q}}+$$

$$+(1-\lambda)(b-x)[|f^{(n)}(\lambda x+(1-\lambda)b)|^q+|f^{(n)}(x)|^q]^{\frac{1}{q}}+$$

$$+\lambda(b-x)[|f^{(n)}(\lambda x+(1-\lambda)b)|^q+|f^{(n)}(b)|^q]^{\frac{1}{q}}\},$$

i.e. the desired inequality.

∎

Theorem 5. *Let $n\in\mathbb{N}^*$ and $f: I\subset\mathbb{R}\to\mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)}\in L[a,b]$ with $a,b\in I^0$, $0<a<b$ and $t\in[0,1]$. If $|f^{(n)}|^q$ is an exponential type convex function on $[a,b]$ and $\alpha-1>0$, with properties of Gamma function, then for some fixed $q>1$, where $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+}f(b)+J^{\alpha-1}_{(\frac{a+b}{2})^-}f(a)]-f\left(\frac{a+b}{2}\right)|\leq$$

$$\leq \frac{(b-a)^2}{\alpha 2^{2+\frac{1}{p}}}\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\{A^{\frac{1}{q}}\left((e^{\frac{1}{2}}-\frac{3}{2})|f^{''}(a)|^q,(e-e^{\frac{1}{2}}-\frac{1}{2})|f^{''}(b)|^q\right)+$$

$$+A^{\frac{1}{q}}\left((e^{\frac{1}{2}}-\frac{3}{2})|f^{''}(b)|^q,(e-e^{\frac{1}{2}}-\frac{1}{2})|f^{''}(a)|^q\right)\},$$

where $A(u,v)$ is the arithmetic mean of u and v.

Proof. We use now Lemma 2, the properties of modulus and then Holder's inequality and we obtain,

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+}f(b)+J^{\alpha-1}_{(\frac{a+b}{2})^-}f(a)]-f\left(\frac{a+b}{2}\right)|\leq$$

$$\leq\frac{(b-a)^2}{\alpha 2^{2-\alpha}}[\int_0^{\frac{1}{2}}t^{\alpha}|f''(ta+(1-t)b)|dt+\int_{\frac{1}{2}}^{1}(1-t)^{\alpha}|f''(ta+(1-t)b)|dt]\leq$$

$$\leq\frac{(b-a)^2}{\alpha 2^{2-\alpha}}\{\left(\int_0^{\frac{1}{2}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\}.$$

Now we apply the definition of the exponential type convex functions for $|f^{(n)}|^q$ and by calculus we get,

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+}f(b)+J^{\alpha-1}_{(\frac{a+b}{2})^-}f(a)]-f\left(\frac{a+b}{2}\right)|\leq$$

$$\leq\frac{(b-a)^2}{\alpha 2^{2+\frac{1}{p}}}\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\{[(e^{\frac{1}{2}}-\frac{3}{2})|f''(a)|^q+(e-e^{\frac{1}{2}}-\frac{1}{2})|f''(b)|^q]^{\frac{1}{p}}+$$

$$+[(e-e^{\frac{1}{2}}-\frac{1}{2})|f''(a)|^q+(e^{\frac{1}{2}}-\frac{3}{2})|f''(b)|^q]^{\frac{1}{p}}\},$$

∎

Theorem 6. *Let $n\in\mathbb{N}^*$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)}\in L[a,b]$ with $a,b\in I^0$, $0<a<b$. If $|f^{(n)}|^q$ is an exponential type convex function on $[a,b]$ for some fixed $q>1$, where $\frac{1}{p}+\frac{1}{q}=1$ then we have*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)|\leq$$

$$\leq\frac{1}{\alpha+1}\{(x-a)[({}_1F_1(\alpha+1,\alpha+2,1)-1)|f^{(n)}(x)|^q+({}_1F_1(1,\alpha+2,1)-1)|f^{(n)}(a)|^q]^{\frac{1}{q}}+$$

$$+(b-x)[({}_1F_1(\alpha+1,\alpha+2,1)-1)|f^{(n)}(x)|^q+({}_1F_1(1,\alpha+2,1)-1)|f^{(n)}(b)|^q]^{\frac{1}{q}}\},$$

where $\alpha>n-1$, and ${}_1F_1(a,b,z)=M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}du$ with $Rea>0$, $Reb>0$ is the confluent hypergeometric function.

Proof. The demonstration will use the same tools, Lemma 3, the properties of the modulus, Holder's inequality and the definition of the exponential type convex functions.

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