REVERSES OF WITKOWSKI AND CLAUSING INEQUALITIES

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Abstract. In this paper we obtain some reverses of Witkowski's inequality, namely upper bounds for the nonnegative quantity

$$\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt-\int_0^1 p(t)\,f(t)\,dt$$

when $f:[0,1]\to\mathbb{R}$ is concave and with the property that $f(0)+f(1)\geq 0$, p is nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$ while q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and $\int_0^1 q(t)\,dt=1$. The particular case $q_0(t)=4\min\{t,1-t\}$, which provides Clausing result, is also analyzed.

1. Introduction

In 1980, A. Clausing [2] obtained the following result:

Theorem 1. *Let p be a nonnegative function on $[0,1]$ which is symmetric on $[0,1]$, namely $p(1-t)=p(t)$ for all $t\in[0,1]$ and p is non-decreasing on $[0,1/2]$. Then for any positive concave function f defined on $[0,1]$,*

$$\int_0^1 p(t)\,f(t)\,dt\leq 4\int_0^1 \min\{t,1-t\}\,p(t)\,dt\int_0^1 f(t)\,dt. \tag{1.1}$$

In the recent paper [6], by using a stronger version of Čebyšev's inequality that is well know in the literature to hold for two functions g, $h:[a,b]\to\mathbb{R}$ that have the same monotonicity on $[a,b]$,

$$\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt\geq\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt,$$

A. Witkowski obtained the following generalization of Clausing result :

Theorem 2. *Assume that $f:[0,1]\to\mathbb{R}$ is concave and with the property that $f(0)+f(1)\geq 0$. If*

(i) *p is nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$;*
(ii) *q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and*

$$\int_0^1 q(t)\,dt=1,$$

then

$$\int_0^1 p(t)\,f(t)\,dt\leq\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt. \tag{1.2}$$

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As pointed out in [6], the function $q_0(t) = 4\min\{t, 1-t\}$ is a borderline between admissible qs and sample functions f. Setting $f = q_0$ in (1.2) we obtain

$$\int_0^1 q_0(t)\, p(t)\, dt \le \int_0^1 q(t)\, p(t)\, dt,$$

which mean that q_0 provides the best bound in (1.2) in the case of increasing functions p.

Motivated by the above results we obtain in this paper some reverses of Witkowski's inequality, namely upper bounds for the nonnegative quantity

$$\int_0^1 q(t)\, p(t)\, dt \int_0^1 f(t)\, dt - \int_0^1 p(t)\, f(t)\, dt$$

when $f : [0,1] \to \mathbb{R}$ is concave and with the property that $f(0) + f(1) \ge 0$, p is nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$ while q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0) = 0$ and $\int_0^1 q(t)\, dt = 1$. The particular case $q_0(t) = 4\min\{t, 1-t\}$, which provides Clausing result, is also analyzed.

2. Reverse Inequalities

For a function $f : [a,b] \to \mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\check{f}_{[a,b]}$ or simply $\check{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\check{f}(t) := \frac{1}{2}\left[f(t) + f(a+b-t)\right],\ t \in [a,b]. \tag{2.1}$$

The asymmetrical transform of f on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\tilde{f}(t) := \frac{1}{2}\left[f(t) - f(a+b-t)\right],\ t \in [a,b]. \tag{2.2}$$

For two *Lebesgue integrable* functions $h,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g) := \frac{1}{b-a}\int_a^b h(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b h(t)dt \int_a^b g(t)dt. \tag{2.3}$$

In 1935, Grüss [3] showed that

$$|C(h,g)| \le \frac{1}{4}(M-m)(N-n), \tag{2.4}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le h(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{2.5}$$

The constant $\frac{1}{4}$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

We have the following result that provides a reverse of Witkowski's inequality:

Theorem 3. *Assume that* $f : [0,1] \to \mathbb{R}$ *is concave and with the property that* $f(0) + f(1) \ge 0$. *If*

(i) *p is nonnegative, symmetric on* $[0,1]$ *and increasing on* $[0,1/2]$;

(ii) *q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and*

$$\int_0^1 q(t)\,dt = 1,$$

then

$$0 \le \int_0^1 q(t)\,p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t)\,f(t)\,dt \tag{2.6}$$
$$\le \frac{1}{4}\left[(M-m)\int_0^1 f(t)\,dt + f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right]\left[p\left(\frac{1}{2}\right) - p(0)\right]$$
$$\le \frac{1}{4}\left[(M-m+1)\,f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right]\left[p\left(\frac{1}{2}\right) - p(0)\right],$$

where

$$m := \min_{t\in[0,1/2]} q(t),\ \ M := \max_{t\in[0,1/2]} q(t).$$

Proof. We have by the symmetry of p on $[0,1]$ that

$$\int_0^1 p(t)\,\check{f}(t)\,dt = \frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt + \int_0^1 p(t)\,f(1-t)\,dt\right]$$
$$= \frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt + \int_0^1 p(1-t)\,f(1-t)\,dt\right]$$
$$= \frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt + \int_0^1 p(s)\,f(s)\,ds\right]$$
$$= \int_0^1 p(t)\,f(t)\,dt,$$

$$\int_0^1 p(t)\,\check{f}(t)\,dt = 2\int_0^{1/2} p(t)\,\check{f}(t)\,dt,$$

and

$$\int_0^1 f(t)\,dt = \int_0^1 \check{f}(t)\,dt = 2\int_0^{1/2} \check{f}(t)\,dt.$$

Since p, q are symmetric, then pq is symmetric on $[0,1]$ and

$$\int_0^1 p(t)\,q(t)\,dt = 2\int_0^{1/2} p(t)\,q(t)\,dt.$$

Therefore

$$0 \le \int_0^1 q(t)\,p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t)\,f(t)\,dt \tag{2.7}$$
$$= 2\int_0^1 f(t)\,dt \int_0^{1/2} q(t)\,p(t)\,dt - 2\int_0^{1/2} p(t)\,\check{f}(t)\,dt$$
$$= 2\int_0^{1/2} \left(Kq(t) - \check{f}(t)\right)p(t)\,dt = \frac{1}{1/2}\int_0^{1/2} \left(Kq(t) - \check{f}(t)\right)p(t)\,dt,$$

where, by the concavity of f on $[0,1]$ and Hermite-Hadamard inequality gives that

$$K := \int_0^1 f(t)\,dt \ge \frac{f(1)+f(0)}{2} \ge 0.$$

Observe that

$$m \le q(t) \le M \text{ for } t \in [0, 1/2]$$

and since $-\check{f}(t)$ is convex and symmetric on $[0,1]$, then

$$-\check{f}\left(\frac{1}{2}\right) \le -\check{f}(t) \le -\check{f}(0) \text{ for } t \in [0, 1/2]$$

namely

$$-f\left(\frac{1}{2}\right) \le -\check{f}(t) \le -\frac{f(1)+f(0)}{2} \text{ for } t \in [0, 1/2].$$

Therefore

$$Km - f\left(\frac{1}{2}\right) \le Kq(t) - \check{f}(t) \le KM - \frac{f(1)+f(0)}{2}, \; t \in [0, 1/2].$$

Observe also that

$$\begin{aligned} \int_0^{1/2} \left(Kq(t) - \check{f}(t)\right) dt &= K \int_0^{1/2} q(t)\, dt - \int_0^{1/2} \check{f}(t)\, dt \\ &= \frac{1}{2} K \int_0^1 q(t)\, dt - \frac{1}{2} \int_0^{1/2} \check{f}(t)\, dt \\ &= \frac{1}{2} K - \frac{1}{2} K = 0. \end{aligned}$$

By using Grüss' inequality, we have

$$\begin{aligned} 0 &\le \frac{1}{1/2} \int_0^{1/2} \left(Kq(t) - \check{f}(t)\right) p(t)\, dt = \frac{1}{1/2} \int_0^{1/2} \left(Kq(t) - \check{f}(t)\right) p(t)\, dt \\ &- \frac{1}{1/2} \int_0^{1/2} \left(Kq(t) - \check{f}(t)\right) \frac{1}{1/2} \int_0^{1/2} p(t)\, dt \\ &\le \frac{1}{4} \left[KM - \frac{f(1)+f(0)}{2} - Km + f\left(\frac{1}{2}\right)\right] \left[p\left(\frac{1}{2}\right) - p(0)\right] \\ &= \frac{1}{4} \left[K(M-m) + f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right] \left[p\left(\frac{1}{2}\right) - p(0)\right]. \end{aligned}$$

By utilising the identity (2.7) we then obtain the first inequality in (2.6).

The second inequality follows by the concavity of f and Hermite-Hadamard inequality

$$\int_0^1 f(t)\, dt \le f\left(\frac{1}{2}\right).$$

□

We also have the reverse of Clausing inequality:

Corollary 1. *Let p be a nonnegative function on $[0,1]$ which is symmetric and p is non-decreasing on $[0,1/2]$. Then for any positive concave function f defined on*

$[0,1]$,

$$0 \le \int_0^1 \min\{t, 1-t\}\, p(t)\, dt \int_0^1 f(t)\, dt - \frac{1}{4}\int_0^1 p(t) f(t)\, dt \tag{2.8}$$
$$\le \frac{1}{8}\left[\int_0^1 f(t)\, dt + \frac{1}{2}\left(f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right)\right]\left[p\left(\frac{1}{2}\right) - p(0)\right]$$
$$\le \frac{1}{16}\left[3f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right]\left[p\left(\frac{1}{2}\right) - p(0)\right].$$

The following inequality obtained by Ostrowski in 1970, [5] also holds

$$|C(h,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{2.9}$$

provided that h is *Lebesgue integrable* and satisfies (2.5) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.9).

Theorem 4. *Assume that f, p and q satisfy the conditions in Theorem 3. Moreover,*

(i) *If p is differentiable on $(0,1/2)$, then*

$$0 \le \int_0^1 q(t) p(t)\, dt \int_0^1 f(t)\, dt - \int_0^1 p(t) f(t)\, dt \tag{2.10}$$
$$\le \frac{1}{16}\|p'\|_{(0,1/2),\infty}\left[(M-m)\int_0^1 f(t)\, dt + f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right]$$
$$\le \frac{1}{16}\|p'\|_{(0,1/2),\infty}\left[(M-m+1) f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right].$$

(ii) *If q is differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 q(t) p(t)\, dt \int_0^1 f(t)\, dt - \int_0^1 p(t) f(t)\, dt \tag{2.11}$$
$$\le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\left(\int_0^1 f(t)\, dt\right) q' - \widetilde{(f')}\right\|_{(0,1/2),\infty}$$
$$\le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[f\left(\frac{1}{2}\right)\|q'\|_{(0,1/2),\infty} + \left\|\widetilde{(f')}\right\|_{(0,1/2),\infty}\right],$$

where $\widetilde{(f')}$ is the antisymmetrical transform of f'.

Proof. By (2.9) we have for $h = Kq - \check{f}$, $g = p$ and $(a,b) = (0,1/2)$ that

$$0 \le \frac{1}{1/2}\int_0^{1/2}\left(Kq(t) - \check{f}(t)\right) p(t)\, dt$$
$$\le \frac{1}{16}\left[(M-m)\int_0^1 f(t)\, dt + f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right]\|p'\|_{(0,1/2),\infty}$$
$$\le \frac{1}{16}\|p'\|_{(0,1/2),\infty}\left[(M-m+1) f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right],$$

which together with (2.7) gives (2.10).

By (2.9) we have for $g = Kq - \check{f}$, $h = p$ and $(a,b) = (0,1/2)$ that

$$0 \le \frac{1}{1/2}\int_0^{1/2} \left(Kq(t) - \check{f}(t)\right) p(t)\,dt \le \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\left(Kq - \check{f}\right)'\right\|_\infty$$
$$= \frac{1}{16}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|Kq' - \widetilde{(f')}\right\|_{(0,1/2),\infty}$$

since $\left(\check{f}\right)' = \widetilde{(f')}$. Hence by (2.7) we obtain the first part of (2.11).

By the concavity of f and the positivity of K, we get

$$\left\|Kq' - \widetilde{(f')}\right\|_{(0,1/2),\infty} \le K\|q'\|_{(0,1/2),\infty} + \left\|\widetilde{(f')}\right\|_{(0,1/2),\infty}$$
$$\le f\left(\frac{1}{2}\right)\|q'\|_{(0,1/2),\infty} + \left\|\widetilde{(f')}\right\|_{(0,1/2),\infty},$$

which proves the last part of (2.11). □

Corollary 2. *Assume that f and p are as in Corollary 1. Moreover,*

(i) *If p is differentiable on $(0,1/2)$, then*

$$0 \le \int_0^1 \min\{t, 1-t\}\,p(t)\,dt\int_0^1 f(t)\,dt - \frac{1}{4}\int_0^1 p(t)\,f(t)\,dt \tag{2.12}$$
$$\le \frac{1}{32}\|p'\|_{(0,1/2),\infty}\left[\int_0^1 f(t)\,dt + \frac{1}{2}\left(f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right)\right]$$
$$\le \frac{1}{64}\|p'\|_{(0,1/2),\infty}\left[3f\left(\frac{1}{2}\right) - \frac{f(1)+f(0)}{2}\right].$$

(ii) *If q is differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 \min\{t, 1-t\}\,p(t)\,dt\int_0^1 f(t)\,dt - \frac{1}{4}\int_0^1 p(t)\,f(t)\,dt \tag{2.13}$$
$$\le \frac{1}{64}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\int_0^1 f(t)\,dt - \widetilde{(f')}\right\|_{(0,1/2),\infty}$$
$$\le \frac{1}{64}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[f\left(\frac{1}{2}\right) + \left\|\widetilde{(f')}\right\|_{(0,1/2),\infty}\right].$$

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(h,g)| \le \frac{1}{12}\|h'\|_\infty\|g'\|_\infty(b-a)^2, \tag{2.14}$$

provided that h', g' exist and are continuous on $[a,b]$ and $\|h'\|_\infty = \sup_{t\in[a,b]}|h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (2.14) also holds if $h, g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and h', $g' \in L_\infty[a,b]$ while $\|h'\|_\infty = \operatorname{essup}_{t\in[a,b]}|h'(t)|$.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [4] in which he proved that

$$|C(h,g)| \le \frac{1}{\pi^2}\|h'\|_2\|g'\|_2(b-a), \tag{2.15}$$

provided that h, g are absolutely continuous and h', $g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

By the use of these inequalities we can also state the following result:

Theorem 5. *Assume that f, p and q satisfy the conditions in Theorem 3. Moreover,*

(i) *If p, q are differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 q(t)\,p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t)\,f(t)\,dt \le \frac{1}{48}\left\|p'\right\|_{(0,1/2),\infty}\left\|\left(\int_0^1 f(t)\,dt\right)q' - \widetilde{(f')}\right\|_{(0,1/2),\infty}. \tag{2.16}$$

(ii) *If p, q are differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 q(t)\,p(t)\,dt \int_0^1 f(t)\,dt - \int_0^1 p(t)\,f(t)\,dt \le \frac{1}{2\pi^2}\left\|p'\right\|_{(0,1/2),2}\left\|\left(\int_0^1 f(t)\,dt\right)q' - \widetilde{(f')}\right\|_{(0,1/2),2} \tag{2.17}$$

provided the last integrals are finite.

We also have:

Corollary 3. *Assume that f and p are as in Corollary 1. Moreover,*

(i) *If p, q are differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 \min\{t,1-t\}\,p(t)\,dt \int_0^1 f(t)\,dt - \frac{1}{4}\int_0^1 p(t)\,f(t)\,dt \le \frac{1}{192}\left\|p'\right\|_{(0,1/2),\infty}\left\|\int_0^1 f(t)\,dt - \widetilde{(f')}\right\|_{(0,1/2),\infty}. \tag{2.18}$$

(ii) *If p is differentiable on $(0,1/2)$ and f is differentiable on $(0,1)$, then*

$$0 \le \int_0^1 \min\{t,1-t\}\,p(t)\,dt \int_0^1 f(t)\,dt - \frac{1}{4}\int_0^1 p(t)\,f(t)\,dt \le \frac{1}{8\pi^2}\left\|p'\right\|_{(0,1/2),2}\left\|\int_0^1 f(t)\,dt - \widetilde{(f')}\right\|_{(0,1/2),2} \tag{2.19}$$

provided the last integrals are finite.

3. Some Examples

Consider the concave function $f:[0,1]\to[0,\infty)$, $f(t)=t(1-t)$. If p is nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$, q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and

$$\int_0^1 q(t)\,dt = 1,$$

then by (2.6)

$$0 \le \frac{1}{6}\int_0^1 q(t)\,p(t)\,dt - \int_0^1 t(1-t)\,p(t)\,dt \le \frac{1}{8}\left[\frac{1}{3}(M-m)+\frac{1}{2}\right]\left[p\left(\frac{1}{2}\right)-p(0)\right], \tag{3.1}$$

where

$$m:=\min_{t\in[0,1/2]}q(t),\ M:=\max_{t\in[0,1/2]}q(t).$$

If q is differentiable on $(0,1/2)$, then by (2.11)

$$0\le\frac{1}{6}\int_0^1 q(t)\,p(t)\,dt-\int_0^1 t(1-t)\,p(t)\,dt \le\frac{1}{16}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left\|\frac{1}{6}q'(\cdot)+2\ell-1\right\|_{(0,1/2),\infty} \tag{3.2}$$

where $\ell(t)=t$.

Consider the concave function $f:[0,1]\to[0,\infty)$,

$$f(t):=\begin{cases}0 \text{ if } t=0;\\ -t\ln t \text{ if } t\in(0,1].\end{cases}$$

If p is nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$, q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and

$$\int_0^1 q(t)\,dt=1,$$

then by (2.6)

$$0\le\int_0^1 p(t)\,t\ln t dt-\frac{1}{4}\int_0^1 q(t)\,p(t)\,dt \le\frac{1}{8}\left[\frac{1}{2}(M-m)+\ln(2)\right]\left[p\left(\frac{1}{2}\right)-p(0)\right]. \tag{3.3}$$

Another way to get particular inequalities is to take some examples of p assumed to be nonnegative, symmetric on $[0,1]$ and increasing on $[0,1/2]$.

Consider $p:[0,1]\to[0,\infty)$, $p(t)=t(1-t)$. Assume that $f:[0,1]\to\mathbb{R}$ is concave and with the property that $f(0)+f(1)\ge 0$ while q is nonnegative, symmetric on $[0,1]$, convex on $[0,1/2]$, $q(0)=0$ and

$$\int_0^1 q(t)\,dt=1,$$

then by (2.6) we get

$$\begin{aligned}0&\le\int_0^1 q(t)\,t(1-t)\,dt\int_0^1 f(t)\,dt-\int_0^1 t(1-t)\,f(t)\,dt\\&\le\frac{1}{16}\left[(M-m)\int_0^1 f(t)\,dt+f\left(\frac{1}{2}\right)-\frac{f(1)+f(0)}{2}\right]\\&\le\frac{1}{16}\left[(M-m+1)\,f\left(\frac{1}{2}\right)-\frac{f(1)+f(0)}{2}\right],\end{aligned} \tag{3.4}$$

where

$$m:=\min_{t\in[0,1/2]}q(t),\ M:=\max_{t\in[0,1/2]}q(t).$$

If q is differentiable on $(0,1/2)$, then by (2.11)

$$0 \leq \int_0^1 q(t)\, t(1-t)\, dt \int_0^1 f(t)\, dt - \int_0^1 t(1-t) f(t)\, dt \tag{3.5}$$
$$\leq \frac{1}{64} \left\| \left(\int_0^1 f(t)\, dt \right) q' - \widetilde{(f')} \right\|_{(0,1/2),\infty}$$
$$\leq \frac{1}{64} \left[f\left(\frac{1}{2}\right) \|q'\|_{(0,1/2),\infty} + \left\| \widetilde{(f')} \right\|_{(0,1/2),\infty} \right].$$

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