INEQUALITIES RELATED TO LEVIN-STEČKIN AND WITKOWSKI'S RESULTS

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Abstract. In this paper we obtain some lower and upper bounds for the quantity

$$\int_0^1 p(t) f(t) dt - \int_0^1 q(t) p(t) dt \int_0^1 f(t) dt$$

when $f : [0,1] \to \mathbb{R}$ is concave and with the property that $f(0) + f(1) \geq 0$, p is symmetric on $[0,1]$ and non-decreasing on $[0,1/2]$ while q is convex on $[0,1]$ with $\int_0^1 q(t) dt = 1$.

1. Introduction

We recall Levin-Stečkin inequality for convex (concave) functions [4]:

Theorem 1. *If $f : [0,1] \to \mathbb{R}$ is a convex function on $[0,1]$, $p : [0,1] \to \mathbb{R}$ is symmetric on $[0,1]$, namely $p(1-t) = p(t)$ for all $t \in [0,1]$ and p is non-decreasing (non-increasing) in $[0,1/2]$, then*

$$\int_0^1 p(t) f(t) dt \leq (\geq) \int_0^1 p(t) dt \int_0^1 f(t) dt. \tag{1.1}$$

If f is concave and p is non-decreasing (non-increasing) in $[0,1/2]$, then

$$\int_0^1 p(t) f(t) dt \geq (\leq) \int_0^1 p(t) dt \int_0^1 f(t) dt. \tag{1.2}$$

In 1980, A. Clausing [2] obtained the following result:

Theorem 2. *Let p be a nonnegative function on $[0,1]$ which symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$ and p is non-decreasing on $[0,1/2]$. Then for any positive concave function f defined on $[0,1]$,*

$$\int_0^1 p(t) f(t) dt \leq 4 \int_0^1 \min\{t, 1-t\} p(t) dt \int_0^1 f(t) dt. \tag{1.3}$$

In the recent paper, by using a stronger version of Čebyšev's inequality that is well know in the literature to hold for two functions $g, h : [a,b] \to \mathbb{R}$ that have the same monotonicity on $[a,b]$,

$$\frac{1}{b-a} \int_a^b g(t) h(t) dt \geq \frac{1}{b-a} \int_a^b g(t) dt \frac{1}{b-a} \int_a^b h(t) dt,$$

A. Witkowski [7] obtained the following generalization of Clausing result:

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Theorem 3. *Assume that* $f:[0,1]\to\mathbb{R}$ *is concave and with the property that* $f(0)+f(1)\geq 0$. *If*

(i) p *is nonnegative, symmetric on* $[0,1]$ *and increasing on* $[0,1/2]$;
(ii) q *is nonnegative, symmetric on* $[0,1]$, *convex on* $[0,1/2]$, $q(0)=0$ *and*

$$\int_0^1 q(t)\,dt=1;$$

then

$$\int_0^1 p(t)\,f(t)\,dt\leq\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt. \tag{1.4}$$

As pointed out in [7], the function $q_0(t)=4\min\{t,1-t\}$ is a borderline between admissible qs and sample functions f. Setting $f=q_0$ in (1.4) we obtain

$$\int_0^1 q_0(t)\,p(t)\,dt\leq\int_0^1 q(t)\,p(t)\,dt,$$

which mean that q_0 provides the best bound in (1.4) in the case of increasing functions p.

In this paper we obtain some lower and upper bounds for the quantity

$$\int_0^1 p(t)\,f(t)\,dt-\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt$$

when $f:[0,1]\to\mathbb{R}$ is concave and with the property that $f(0)+f(1)\geq 0$, p is symmetric on $[0,1]$ and non-decreasing on $[0,1/2]$ while q is convex on $[0,1]$ with $\int_0^1 q(t)\,dt=1$.

2. Main Results

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{2.1}$$

The asymmetrical transform of f on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],\ t\in[a,b]. \tag{2.2}$$

For two *Lebesgue integrable* functions $h,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b h(t)dt\int_a^b g(t)dt. \tag{2.3}$$

In 1935, Grüss [3] showed that

$$|C(h,g)|\leq\frac{1}{4}(M-m)(N-n), \tag{2.4}$$

provided that there exists the real numbers m, M, n, N such that

$$m\leq h(t)\leq M\quad\text{and}\quad n\leq g(t)\leq N\quad\text{for a.e. } t\in[a,b]. \tag{2.5}$$

The constant $\frac{1}{4}$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

We have the following result that provides a reverse of Witkowski's inequality:

Theorem 4. *Assume that* $f:[0,1]\to\mathbb{R}$ *is concave and with the property that* $f(0)+f(1)\geq 0$. *If*

(i) p *is symmetric on* $[0,1]$ *and non-decreasing on* $[0,1/2]$;

(ii) q *is convex on* $[0,1]$ *and* $\int_0^1 q(t)\,dt=1$;

then

$$\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt\leq\int_0^1 p(t)\,f(t)\,dt. \tag{2.6}$$

If p *is non-increasing on* $[0,1/2]$, *then the reverse inequality holds in (2.6).*

Proof. We have by the symmetry of p on $[0,1]$ that

$$\begin{aligned}\int_0^1 p(t)\,\check{f}(t)\,dt&=\frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt+\int_0^1 p(t)\,f(1-t)\,dt\right]\\&=\frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt+\int_0^1 p(1-t)\,f(1-t)\,dt\right]\\&=\frac{1}{2}\left[\int_0^1 p(t)\,f(t)\,dt+\int_0^1 p(s)\,f(s)\,ds\right]\\&=\int_0^1 p(t)\,f(t)\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt-\int_0^1 p(t)\,f(t)\,dt\\&=\int_0^1 f(t)\,dt\int_0^1 q(t)\,p(t)\,dt-\int_0^1 p(t)\,\check{f}(t)\,dt\\&=\int_0^1\left(Kq(t)-\check{f}(t)\right)p(t)\,dt,\end{aligned} \tag{2.7}$$

where, by the concavity of f on $[0,1]$ and Hermite-Hadamard inequality

$$K:=\int_0^1 f(t)\,dt\geq\frac{f(1)+f(0)}{2}\geq 0.$$

Since $\int_0^1 q(t)\,dt=1$, hence

$$\begin{aligned}\int_0^1\left(Kq(t)-\check{f}(t)\right)dt&=K\int_0^1 q(t)\,dt-\int_0^1\check{f}(t)\,dt\\&=K-K=0.\end{aligned}$$

Because f is concave, then $\check{f}$ is concave, which implies that $Kq-\check{f}$ is convex and by Levin-Stečkin inequality (2.6),

$$\int_0^1\left(Kq(t)-\check{f}(t)\right)p(t)\,dtdt\leq\int_0^1 p(t)\,dt\int_0^1\left(Kq(t)-\check{f}(t)\right)dt=0,$$

namely

$$\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt-\int_0^1 p(t)\,f(t)\,dt\le 0,$$

which is equivalent to (2.6). □

Remark 1. *Observe that*

$$\max\{t,1-t\}=\frac{1}{2}+\left|t-\frac{1}{2}\right|,\ \ t\in[0,1],$$

which gives

$$\int_0^1 \max\{t,1-t\}\,dt=\frac{1}{2}+\int_0^1\left|t-\frac{1}{2}\right|dt=\frac{3}{4}.$$

By taking $q(t)=\frac{4}{3}\max\{t,1-t\}$, $t\in[0,1]$ *in (2.6), we get*

$$\frac{4}{3}\int_0^1 \max\{t,1-t\}\,p(t)\,dt\int_0^1 f(t)\,dt\le\int_0^1 p(t)\,f(t)\,dt. \tag{2.8}$$

provided that $f:[0,1]\to\mathbb{R}$ *is concave and with the property that* $f(0)+f(1)\ge 0$ *while* p *is symmetric on* $[0,1]$ *and non-decreasing on* $[0,1/2]$.

We have the following reverse of (2.6):

Theorem 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
0 &\le \int_0^1 p(t)\,f(t)\,dt-\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt \\
&\le \frac{1}{4}\left[f\left(\frac{1}{2}\right)-\frac{f(0)+f(1)}{2}\right. \\
&\quad \left.+\left(\frac{q(1)+q(0)}{2}-q\left(\frac{1}{2}\right)\right)\int_0^1 f(t)\,dt\right]\left[p\left(\frac{1}{2}\right)-p(0)\right] \\
&\le \frac{1}{4}\left[f\left(\frac{1}{2}\right)-\frac{f(0)+f(1)}{2}\right. \\
&\quad \left.+\left(\frac{q(1)+q(0)}{2}-q\left(\frac{1}{2}\right)\right)f\left(\frac{1}{2}\right)\right]\left[p\left(\frac{1}{2}\right)-p(0)\right].
\end{aligned} \tag{2.9}$$

Proof. Since p is symmetric on $[0,1]$, then

$$\int_0^1 q(t)\,p(t)\,dt=\int_0^1 \check{q}(t)\,p(t)\,dt$$

and by (2.7) we get

$$\begin{aligned}
0 &\le \int_0^1 p(t)\,f(t)\,dt-\int_0^1 q(t)\,p(t)\,dt\int_0^1 f(t)\,dt \\
&= \int_0^1 p(t)\,\check{f}(t)\,dt-\int_0^1 f(t)\,dt\int_0^1 \check{q}(t)\,p(t)\,dt \\
&= \int_0^1 \left(\check{f}(t)-K\check{q}(t)\right)p(t)\,dt.
\end{aligned} \tag{2.10}$$

Since $\check{f}$ is concave and symmetric, then

$$\frac{f(0)+f(1)}{2}\le \check{f}(t)\le f\left(\frac{1}{2}\right),\ \ t\in[0,1]$$

and since $\check{q}$ is symmetric and convex, hence

$$-\frac{q(1)+q(0)}{2} \leq -\check{q}(t) \leq -q\left(\frac{1}{2}\right),\ t\in[0,1].$$

Therefore

$$\begin{aligned}\frac{f(0)+f(1)}{2} - K\frac{q(1)+q(0)}{2} &\leq \check{f}(t) - K\check{q}(t) \\ &\leq f\left(\frac{1}{2}\right) - Kq\left(\frac{1}{2}\right)\end{aligned}$$

for $t\in[0,1]$.

By using Grüss inequality, we have

$$\begin{aligned}0 &\leq \int_0^1 \left(\check{f}(t) - K\check{q}(t)\right) p(t)\,dt \\ &= \int_0^1 \left(\check{f}(t) - K\check{q}(t)\right) p(t)\,dt - \int_0^1 \left(\check{f}(t) - K\check{q}(t)\right) dt \int_0^1 p(t)\,dt \\ &\leq \frac{1}{4}\left[f\left(\frac{1}{2}\right) - Kq\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + K\frac{q(1)+q(0)}{2}\right] \\ &\times \left[p\left(\frac{1}{2}\right) - p(0)\right] \\ &= \frac{1}{4}\left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + K\left(\frac{q(1)+q(0)}{2} - q\left(\frac{1}{2}\right)\right)\right] \\ &\times \left[p\left(\frac{1}{2}\right) - p(0)\right],\end{aligned}$$

which proves the first inequality in (2.9).

The last part follows by Hermite-Hadamard inequality for the concave function f, $\int_0^1 f(t)\,dt \leq f\left(\frac{1}{2}\right)$. □

Remark 2. *By taking* $q(t) = \frac{4}{3}\max\{t, 1-t\}$, $t\in[0,1]$ *in (2.9), we get*

$$\begin{aligned}0 &\leq \int_0^1 p(t) f(t)\,dt - \frac{4}{3}\int_0^1 \max\{t,1-t\}\, p(t)\,dt \int_0^1 f(t)\,dt \\ &\leq \frac{1}{4}\left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + \frac{4}{3}\int_0^1 f(t)\,dt\right]\left[p\left(\frac{1}{2}\right) - p(0)\right],\end{aligned} \tag{2.11}$$

provided that $f:[0,1]\to\mathbb{R}$ *is concave and with the property that* $f(0)+f(1)\geq 0$ *while* p *is symmetric on* $[0,1]$ *and non-decreasing on* $[0,1/2]$.

The following inequality obtained by Ostrowski in 1970, [6] also holds

$$|C(h,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{2.12}$$

provided that h is *Lebesgue integrable* and satisfies (2.5) while g is absolutely continuous and $g'\in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.12).

We have:

Theorem 6. *Assume that* f, p *and* q *satisfy the conditions in Theorem 4. Moreover,*

(i) *If p is differentiable on $(0,1)$, then*

$$0 \leq \int_0^1 p(t) f(t)\,dt - \int_0^1 q(t) p(t)\,dt \int_0^1 f(t)\,dt \tag{2.13}$$
$$\leq \frac{1}{8}\|p'\|_\infty \left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + \left(\frac{q(1)+q(0)}{2} - q\left(\frac{1}{2}\right)\right)\int_0^1 f(t)\,dt \right]$$

(ii) *If q and f are differentiable on $(0,1)$, then*

$$0 \leq \int_0^1 p(t) f(t)\,dt - \int_0^1 q(t) p(t)\,dt \int_0^1 f(t)\,dt \tag{2.14}$$
$$\leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\left(\int_0^1 f(t)\,dt\right) q' - \widetilde{(f')}\right\|_\infty$$
$$\leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\left(\int_0^1 f(t)\,dt\right)\|q'\|_\infty + \left\|\widetilde{(f')}\right\|_\infty\right]$$
$$\leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[f\left(\frac{1}{2}\right)\|q'\|_\infty + \left\|\widetilde{(f')}\right\|_\infty\right],$$

where $\widetilde{(f')}$ is the antisymmetrical transform of f'.

Proof. By (2.12) we have for $h = Kq - \check{f}$, $g = p$ and $(a,b) = (0,1)$ that

$$0 \leq \int_0^1 \left(\check{f}(t) - K\check{q}(t)\right) p(t)\,dt$$
$$\leq \frac{1}{8}\|p'\|_\infty \left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + \left(\frac{q(1)+q(0)}{2} - q\left(\frac{1}{2}\right)\right)\int_0^1 f(t)\,dt \right],$$

which together with (2.10), give (2.13).

By (2.12) we have for $g = Kq - \check{f}$, $h = p$ and $(a,b) = (0,1/2)$ that

$$0 \leq \int_0^1 \left(\check{f}(t) - K\check{q}(t)\right) p(t)\,dt \leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|\left(Kq - \check{f}\right)'\right\|_\infty$$
$$= \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left\|Kq' - \widetilde{(f')}\right\|_\infty$$

since $\left(\check{f}\right)' = \widetilde{(f')}$. Hence by (2.10) we obtain the first part of (2.14).

By the concavity of f and the positivity of K, we get

$$\left\|Kq' - \widetilde{(f')}\right\|_\infty \leq K\|q'\|_\infty + \left\|\widetilde{(f')}\right\|_\infty \leq f\left(\frac{1}{2}\right)\|q'\|_\infty + \left\|\widetilde{(f')}\right\|_\infty,$$

which proves the last part of (2.10). □

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(h,g)| \leq \frac{1}{12}\|h'\|_\infty \|g'\|_\infty (b-a)^2, \tag{2.15}$$

provided that h', g' exist and are continuous on $[a,b]$ and $\|h'\|_\infty = \sup_{t\in[a,b]} |h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (2.15) also holds if $h, g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $h', g' \in L_\infty[a,b]$ while $\|h'\|_\infty = \operatorname{essup}_{t\in[a,b]} |h'(t)|$.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [5] in which he proved that

$$|C(h,g)| \leq \frac{1}{\pi^2} \|h'\|_2 \|g'\|_2 (b-a), \tag{2.16}$$

provided that h, g are absolutely continuous and $h', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

By the use of these inequalities we can also state the following result:

Theorem 7. *Assume that f, p and q satisfy the conditions in Theorem 4. Moreover, if p, q, f are differentiable on $(0,1)$, then*

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f(t)\,dt - \int_0^1 q(t) p(t)\,dt \int_0^1 f(t)\,dt \\ &\leq \frac{1}{12} \|p'\|_\infty \left\| \left(\int_0^1 f(t)\,dt \right) q' - \widetilde{(f')} \right\|_\infty \\ &\leq \frac{1}{12} \|p'\|_\infty \left[\left(\int_0^1 f(t)\,dt \right) \|q'\|_\infty + \left\| \widetilde{(f')} \right\|_\infty \right] \\ &\leq \frac{1}{12} \|p'\|_\infty \left[f\left(\frac{1}{2}\right) \|q'\|_\infty + \left\| \widetilde{(f')} \right\|_\infty \right] \end{aligned} \tag{2.17}$$

Also,

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f(t)\,dt - \int_0^1 q(t) p(t)\,dt \int_0^1 f(t)\,dt \\ &\leq \frac{1}{\pi^2} \|p'\|_2 \left\| \left(\int_0^1 f(t)\,dt \right) q' - \widetilde{(f')} \right\|_2 \\ &\leq \frac{1}{\pi^2} \|p'\|_2 \left[\left(\int_0^1 f(t)\,dt \right) \|q'\|_2 + \left\| \widetilde{(f')} \right\|_2 \right] \\ &\leq \frac{1}{\pi^2} \|p'\|_2 \left[f\left(\frac{1}{2}\right) \|q'\|_2 + \left\| \widetilde{(f')} \right\|_2 \right], \end{aligned} \tag{2.18}$$

provided the last integrals are finite.

3. Some Examples

Consider the functions $q : [0,1] \to [0,\infty)$, $q(t) = \frac{1}{4}\left|t - \frac{1}{2}\right|$. We observe that q is convex and $\int_0^1 q(t)\,dt = 1$.

Assume that $f : [0,1] \to \mathbb{R}$ is concave and with the property that $f(0)+f(1) \geq 0$. If p is symmetric on $[0,1]$ and non-decreasing on $[0,1/2]$, then from (2.6) we get

$$\frac{1}{4} \int_0^1 \left|t - \frac{1}{2}\right| p(t)\,dt \int_0^1 f(t)\,dt \leq \int_0^1 p(t) f(t)\,dt. \tag{3.1}$$

From (2.9) we get

$$
\begin{aligned}
0 &\leq \int_0^1 p(t) f(t)\,dt - \frac{1}{4}\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \int_0^1 f(t)\,dt \\
&\leq \frac{1}{4}\left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + \frac{1}{8}\int_0^1 f(t)\,dt\right]\left[p\left(\frac{1}{2}\right) - p(0)\right] \\
&\leq \frac{1}{4}\left[\frac{9}{8} f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2}\right]\left[p\left(\frac{1}{2}\right) - p(0)\right].
\end{aligned}
\tag{3.2}
$$

If p is differentiable on $(0,1)$, then by (2.13) we get

$$
\begin{aligned}
0 &\leq \int_0^1 p(t) f(t)\,dt - \frac{1}{4}\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \int_0^1 f(t)\,dt \\
&\leq \frac{1}{8}\|p'\|_\infty \left[f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2} + \frac{1}{8}\int_0^1 f(t)\,dt\right] \\
&\leq \frac{1}{8}\|p'\|_\infty \left[\frac{9}{8} f\left(\frac{1}{2}\right) - \frac{f(0)+f(1)}{2}\right].
\end{aligned}
\tag{3.3}
$$

The inequalities (2.14)-(2.18) can be extended for absolutely continuous functions q. From the last inequality in (2.14) we then get

$$
\begin{aligned}
0 &\leq \int_0^1 p(t) f(t)\,dt - \frac{1}{4}\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \int_0^1 f(t)\,dt \\
&\leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\frac{1}{4} f\left(\frac{1}{2}\right) + \left\|\widetilde{(f')}\right\|_\infty\right],
\end{aligned}
\tag{3.4}
$$

provided that $f:[0,1]\to\mathbb{R}$ is differentiable concave and with the property that $f(0)+f(1)\geq 0$ while p is symmetric on $[0,1]$ and non-decreasing on $[0,1/2]$.

From (2.18) we get

$$
\begin{aligned}
0 &\leq \int_0^1 p(t) f(t)\,dt - \frac{1}{4}\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \int_0^1 f(t)\,dt \\
&\leq \frac{1}{\pi^2}\|p'\|_2 \left[f\left(\frac{1}{2}\right) + \left\|\widetilde{(f')}\right\|_2\right].
\end{aligned}
\tag{3.5}
$$

The interested reader may get other inequalities by taking particular examples of functions such as $f(t) = -t\ln t$, $t\in(0,1]$ or $f(t) = t(1-t)$, $t\in(0,1]$.

Indeed if we take in (2.6) and (2.9)

$$
f(t) := \begin{cases} 0 \text{ if } t = 0; \\ \\ -t\ln t \text{ if } t\in(0,1], \end{cases}
$$

then we get

$$
\begin{aligned}
0 &\leq \frac{1}{4}\int_0^1 q(t)\,p(t)\,dt - \int_0^1 p(t)\,t\ln t dt \\
&\leq \frac{1}{8}\left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\ln 2 + \frac{1}{2}\left(\frac{q(1)+q(0)}{2} - q\left(\frac{1}{2}\right)\right)\right]
\end{aligned}
\tag{3.6}
$$

provided that p is symmetric on $[0,1]$, non-decreasing on $[0,1/2]$ and q is convex on $[0,1]$ with $\int_0^1 q(t)\,dt = 1$.

References

[1] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.
[2] A. Clausing, Disconjugacy and integral inequalities, *Trans. Amer. Math. Soc.* **260** (1980), 293-307.
[3] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.
[4] V. I. Levin and S. B. Stečkin, Inequalities. *Amer. Math. Soc. Transl.* **14** (1960), 1–29.
[5] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(**38**)(2) (1973), 219-222
[6] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.
[7] A. Witkowski, Inequalities of Levin-Stečkin, Clausing and Chebysev revisited, *Elem. Math.* **75** (2020), 32-36.

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