SOME WEIGHTED INTEGRAL INEQUALITIES FOR OPERATOR MONOTONIC FUNCTIONS ON HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let f be an operator monotonic function on I and $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I. Assume that $p : [0,1] \to \mathbb{R}$ is non-decreasing on $[0,1]$ and $w : [0,1] \to [0,\infty)$ is an integrable function with $\int_0^1 w(t)\,dt > 0$. In this paper we obtained, among others, that for $A \le B$ and f an operator monotonic function on I,

$$\begin{aligned}
0 &\le \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\,\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \\
&\le \frac{1}{2}\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left| p(t) - \frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right| w(t)\,dt \\
&\times [f(B)-f(A)] \\
&\le \frac{1}{2}\left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 p^2(t)w(t)\,dt - \left(\frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right)^2 \right)^{1/2} \\
&\times [f(B)-f(A)] \\
&\le \frac{1}{4}[p(1)-p(0)][f(B)-f(A)]
\end{aligned}$$

in the operator order.

Several other similar inequalities for either p or f is differentiable, are also provided. Applications for power function and logarithm are given as well.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x \rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0,\infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$.

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t}{t+s}\,dm(s),$$

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where $a\in\mathbb{R}$ *and* $b\geq 0$ *and a positive measure* m *on* $(0,\infty)$ *such that*

$$\int_0^\infty \frac{dm(s)}{t+s}<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$.

In [5], T. Furuta observed that for $\alpha_j\in[0,1]$, $j=1,...,n$ the functions

$$g(t):=\left(\sum_{j=1}^n t^{-\alpha_j}\right)^{-1} \text{ and } h(t)=\sum_{j=1}^n\left(1+t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0,\infty)$.

Let $f(t)$ be a continuous function on $(0,\infty)\to(0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t)=t/f(t)=:f^*(t)$ is also operator monotone, see for instance [5] or [10].

Consider the family of functions defined on $(0,\infty)$ and $p\in[-1,2]\setminus\{0,1\}$ by

$$f_p(t):=\frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$$

and

$$f_0(t):=\frac{t}{1-t}\ln t,$$

$$f_1(t):=\frac{t-1}{\ln t}\text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t)=\frac{2t}{1+t}\text{ (harmonic mean), } f_{1/2}(t)=\sqrt{t}\text{ (geometric mean).}$$

In [4] the authors showed that f_p is operator monotone for $1\leq p\leq 2$.

In the same category, we observe that the function

$$g_p(t):=\frac{t-1}{t^p-1}$$

is an operator monotone function for $p\in(0,1]$, [5].

It is well known that the logarithmic function ln is operator monotone and in [5] the author obtained that the functions

$$f(t)=t(1+t)\ln\left(1+\frac{1}{t}\right),\ g(t)=\frac{1}{(1+t)\ln\left(1+\frac{1}{t}\right)}$$

are also operator monotone functions on $(0,\infty)$.

Let f be an operator monotonic function on I and A, $B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I. Assume that $p:[0,1]\to\mathbb{R}$ is non-decreasing on $[0,1]$ and $w:[0,1]\to[0,\infty)$ is an integrable function with $\int_0^1 w(t)\,dt>0$. In this paper we obtained, among others, that for $A\leq B$ and f an operator monotonic function on I,

$$\begin{aligned}
0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \\
&\leq \frac{1}{2}\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left| p(t) - \frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right| w(t)\,dt \\
&\times [f(B) - f(A)] \\
&\leq \frac{1}{2}\left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 p^2(t)w(t)\,dt - \left(\frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right)^2 \right)^{1/2} \\
&\times [f(B) - f(A)] \\
&\leq \frac{1}{4}[p(1)-p(0)][f(B)-f(A)]
\end{aligned}$$

in the operator order.

Several other similar inequalities for either p or f is differentiable, are also provided. Applications for power function and logarithm are given as well.

2. Some Preliminary Facts

Let f be a continuous function on I. If $(A,B) \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I and $t \in [0,1]$, then the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator in $\mathcal{B}(H)$.

For A, $B \in \mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)} : [0,1] \to \mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A+tB). \tag{2.1}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)\,x, x \right\rangle = \langle f((1-t)A+tB)\,x, x\rangle. \tag{2.2}$$

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0}\frac{g(A+sB)-g(A)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that $A, B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

Lemma 1. *Let f be a continuous function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B - A). \tag{2.4}$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B - A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B - A). \tag{2.6}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t + h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A + (t+h)B) - f((1-t)A + tB)}{h} \\ &= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h}. \end{aligned} \tag{2.7}$$

Since $f \in \mathcal{G}([A, B])$, hence by taking the limit over $h \to 0$ in (2.7) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \\ &= \nabla f_{(1-t)A+tB}(B - A), \end{aligned}$$

which proves (2.4).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)A + hB) - f(A)}{h} \\ &= \lim_{h\to 0+} \frac{f(A + h(B-A)) - f(A)}{h} \\ &= \nabla f_A(B - A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.5).

The equality (2.6) follows in a similar way. □

Lemma 2. *Let f be an operator monotonic function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \leq B$, $A \neq B$. If $f \in \mathcal{G}([A, B])$, then*

$$\nabla f_{(1-t)A+tB}(B - A) \geq 0 \text{ for all } t \in (0,1). \tag{2.8}$$

Also

$$\nabla f_A(B - A),\ \nabla f_B(B - A) \geq 0. \tag{2.9}$$

Proof. Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is monotonic nondecreasing in the usual sense on $[0,1]$ and differentiable on $(0,1)$, and for $t \in (0,1)$

$$\begin{aligned}
0 \leq \varphi'_{(A,B);x}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B),x}(t+h) - \varphi_{(A,B),x}(t)}{h} \\
&= \lim_{h\to 0} \left\langle \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} x, x \right\rangle \\
&= \left\langle \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} x, x \right\rangle \\
&= \left\langle \nabla f_{(1-t)A+tB}(B-A)\, x, x \right\rangle .
\end{aligned}$$

This shows that

$$\nabla f_{(1-t)A+tB}(B-A) \geq 0$$

for all $t \in (0,1)$.

The inequalities (2.9) follow by (2.5) and (2.6). □

3. Main Results

For two *Lebesgue integrable* functions $h,\ g,\ w : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$\begin{aligned}
C(h,g;w) &:= \frac{1}{\int_a^b w(t)\,dt} \int_a^b h(t)g(t)w(t)\,dt \\
&\quad - \frac{1}{\int_a^b w(t)\,dt} \int_a^b h(t)w(t)\,dt \frac{1}{\int_a^b w(t)\,dt} \int_a^b g(t)w(t)\,dt,
\end{aligned} \tag{3.1}$$

where $\int_a^b w(t)\,dt \neq 0$.

It is well known that, if h and g have the same monotonicity on $[a,b]$ and $w \geq 0$ in $[a,b]$ with $\int_a^b w(t)\,dt > 0$, then

$$\begin{aligned}
&\frac{1}{\int_a^b w(t)\,dt} \int_a^b h(t)g(t)w(t)\,dt \\
&\geq \frac{1}{\int_a^b w(t)\,dt} \int_a^b h(t)w(t)\,dt \frac{1}{\int_a^b w(t)\,dt} \int_a^b g(t)w(t)\,dt,
\end{aligned} \tag{3.2}$$

which is known in the literature as the *weighted Čebyšev's inequality.*

In 2002, see the preprint version of [1], Cerone and Dragomir obtained the following refinement of Grüss' inequality

$$
\begin{aligned}
(3.3)\quad & \left|C\left(h,g;w\right)\right| \\
& \le \frac{1}{2}\left(M-m\right)\frac{1}{\int_a^b w\left(t\right)dt}\int_a^b \left|g(t)-\frac{1}{\int_a^b w\left(s\right)ds}\int_a^b g(s)w\left(s\right)ds\right| w\left(t\right)dt \\
& \le \frac{1}{2}\left(M-m\right) \\
& \times \left(\frac{1}{\int_a^b w\left(t\right)dt}\int_a^b g^2(t)w\left(t\right)dt-\left(\frac{1}{\int_a^b w\left(s\right)ds}\int_a^b g(s)w\left(s\right)ds\right)^2\right)^{1/2} \\
& \le \frac{1}{4}\left(M-m\right)\left(N-n\right)
\end{aligned}
$$

provided that h, g are measurable and for real numbers m, M, n, N we have

$$m \le h \le M,\ n \le h \le N$$

almost everywhere on $[a,b]$.

The following result holds for operator monotonic functions:

Theorem 2. *Let* A, $B \in \mathcal{SA}_I\left(H\right)$ *with* $A \le B$ *and* f *an operator monotonic function on* I. *If* $p : [0,1] \to \mathbb{R}$ *is monotonic nondecreasing on* $[0,1]$ *and* $w : [0,1] \to [0,\infty)$ *an integrable function with* $\int_0^1 w\left(t\right)dt > 0$, *then*

$$
\begin{aligned}
(3.4)\quad 0 & \le \frac{1}{\int_0^1 w\left(t\right)dt}\int_0^1 w\left(t\right)p\left(t\right)f\left(\left(1-t\right)A+tB\right)dt \\
& - \frac{1}{\int_0^1 w\left(t\right)dt}\int_0^1 w\left(t\right)p\left(t\right)dt\frac{1}{\int_0^1 w\left(t\right)dt}\int_0^1 w\left(t\right)f\left(\left(1-t\right)A+tB\right)dt \\
& \le \frac{1}{2}\frac{1}{\int_0^1 w\left(t\right)dt}\int_0^1 \left|p(t)-\frac{1}{\int_0^1 w\left(s\right)ds}\int_0^1 p(s)w\left(s\right)ds\right| w\left(t\right)dt \\
& \times \left[f\left(B\right)-f\left(A\right)\right] \\
& \le \frac{1}{2}\left(\frac{1}{\int_0^1 w\left(t\right)dt}\int_0^1 p^2(t)w\left(t\right)dt-\left(\frac{1}{\int_0^1 w\left(s\right)ds}\int_0^1 p(s)w\left(s\right)ds\right)^2\right)^{1/2} \\
& \times \left[f\left(B\right)-f\left(A\right)\right] \\
& \le \frac{1}{4}\left[p\left(1\right)-p\left(0\right)\right]\left[f\left(B\right)-f\left(A\right)\right].
\end{aligned}
$$

Proof. Let $0 \le t_1 < t_2 \le 1$ and $A \le B$. Then

$$\left(1-t_2\right)A+t_2B-\left(1-t_1\right)A-t_1B=\left(t_2-t_1\right)\left(B-A\right)\ge 0$$

and by operator monotonicity of f we get

$$f\left(\left(1-t_2\right)A+t_2B\right)\ge f\left(\left(1-t_1\right)A+t_1B\right),$$

which is equivalent to

$$
\begin{aligned}
\varphi_{(A,B);x}\left(t_2\right) & = \left\langle f\left(\left(1-t_2\right)A+t_2B\right)x,x\right\rangle \\
& \ge \left\langle f\left(\left(1-t_1\right)A+t_1B\right)x,x\right\rangle = \varphi_{(A,B);x}\left(t_1\right)
\end{aligned}
$$

that shows that the scalar function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ is monotonic nondecreasing for $A \le B$ and for any $x \in H$.

If we write the inequality (3.2) for the functions p, $\varphi_{(A,B);x}$ and w, then we get

$$\begin{aligned} &\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \\ &\ge \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt, \end{aligned}$$

which can be written as

$$\begin{aligned} &\left\langle \left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt\right)x,x\right\rangle \\ &\ge \left\langle \left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\right)dtx,x\right\rangle \end{aligned}$$

for $x \in H$, and the first operator inequality in (2.3) is obtained.

We also have that

$$\begin{aligned} \langle f(A)x,x\rangle &= \varphi_{(A,B);x}(0) \le \varphi_{(A,B);x}(t) = \langle f((1-t)A+tB)x,x\rangle \\ &\le \varphi_{(A,B);x}(1) = \langle f(B)x,x\rangle \end{aligned}$$

and

$$p(0) \le p(t) \le p(1)$$

for all $t \in [0,1]$.

By writing the inequalities 3.3 for the functions $\varphi_{(A,B);x}$, p and w we get the second part of (3.4).

The details are omitted. □

Corollary 1. *Let A, $B \in \mathcal{SA}_I(H)$ with $A \le B$ and f an operator monotonic function on I. If $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then*

$$\begin{aligned} 0 &\le \int_0^1 p(t)\,f((1-t)A+tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \\ &\le \frac{1}{2}\left(\int_0^1 \left|p(t) - \int_0^1 p(s)ds\right| dt\right)[f(B)-f(A)] \\ &\le \frac{1}{2}\left(\int_0^1 p^2(t)dt - \left(\int_0^1 p(s)ds\right)^2\right)^{1/2}[f(B)-f(A)] \\ &\le \frac{1}{4}[p(1)-p(0)][f(B)-f(A)]. \end{aligned} \tag{3.5}$$

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(h,g)| \le \frac{1}{12}\|h'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3.6}$$

where

$$C(h,g) := \frac{1}{b-a}\int_a^b h(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b h(t)dt \int_a^b g(t)dt,$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|h'\|_{\infty} = \sup_{t\in[a,b]} |h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (3.6) also holds if h, $g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and h', $g' \in L_{\infty}[a,b]$ while $\|h'\|_{\infty} = \operatorname{essup}_{t\in[a,b]} |h'(t)| < \infty$.

A mixture between Grüss' result and Čebyšev's one (3.6) is the following inequality obtained by Ostrowski in 1970, [11]:

$$|C(h,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{3.7}$$

provided that f is *Lebesgue integrable* and satisfies

$$m \le h \le M \text{ almost everywhere on } [a,b], \tag{3.8}$$

while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (3.7).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [7] in which he proved that

$$|C(h,g)| \le \frac{1}{\pi^2}\|h'\|_2 \|g'\|_2 (b-a), \tag{3.9}$$

provided that h, g are absolutely continuous and h', $g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

In the recent paper [3] we obtained the following weighted version of Ostrowski's and Čebyšev's inequalities:

Lemma 3. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$, h *is Lebesgue integrable and satisfies the condition* $m \le h(t) \le M$ *for almost every* $t \in [a,b]$ *and* $g : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$ *with* $\frac{g'}{w}$ *is essentially bounded, namely* $\frac{g'}{w} \in L_{\infty}[a,b]$, *then we have*

$$|C(h,g;w)| \le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{3.10}$$

The constant $\frac{1}{8}$ *is best possible.*

Lemma 4. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$. *If* f, $g : [a,b] \to \mathbb{R}$ *are absolutely continuous on* $[a,b]$ *and* $\frac{f'}{w}$, $\frac{g'}{w} \in L_{\infty}[a,b]$, *then we have*

$$|C(h,g;w)| \le \frac{1}{12}\left\|\frac{h'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2. \tag{3.11}$$

The constant $\frac{1}{12}$ *is best possible.*

Under the assumptions of Lemma 3 and if there exists a constant $K > 0$ such that $|g'(t)| \le Kw(t)$ for a.e. $t \in [a,b]$, then by (3.10) we get

$$|C(h,g;w)| \le \frac{1}{8}(M-m)K\int_a^b w(s)\,ds. \tag{3.12}$$

Under the assumptions of Lemma 4 and if there exist the constants K, $L > 0$ such that $|h'(t)| \le Lw(t)$, $|g'(t)| \le Kw(t)$ for a.e. $t \in [a,b]$, then by (3.11) we get

$$|C(h,g;w)| \le \frac{1}{12}LK\left(\int_a^b w(s)\,ds\right)^2. \tag{3.13}$$

We have:

Theorem 3. *Let A, $B \in \mathcal{SA}_I(H)$ with $A \leq B$ and f be an operator monotonic function on I while $w : [0,1] \to [0,\infty)$ is an integrable function with $\int_0^1 w(t)\,dt > 0$. Assume also that $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$.*

(i) *If p is differentiable on $(0,1)$, then*

$$0 \leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \leq \frac{1}{8}\left\|\frac{p'}{w}\right\|_{[0,1],\infty}\left(\int_0^1 w(s)\,ds\right)[f(B)-f(A)]. \tag{3.14}$$

(ii) *If $f \in \mathcal{G}([A,B])$, then*

$$0 \leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \leq \frac{1}{8}[p(1)-p(0)]\left(\int_0^1 w(s)\,ds\right)\sup_{t\in(0,1)}\left(\frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|}{w(t)}\right)1_H. \tag{3.15}$$

(iii) *If p is differentiable on $(0,1)$ and $f \in \mathcal{G}([A,B])$, then*

$$0 \leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \leq \frac{1}{12}\left\|\frac{p'}{w}\right\|_{[0,1],\infty}\sup_{t\in(0,1)}\left(\frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|}{w(t)}\right)\left(\int_a^b w(s)\,ds\right)^2 1_H. \tag{3.16}$$

Proof. Let $x \in H$. If we use the inequality (3.10) for w, $g = p$ and $h = \varphi_{(A,B);x}$, then

$$0 \leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \leq \frac{1}{8}[\langle f(B)x,x\rangle - \langle f(A)x,x\rangle]\left\|\frac{p'}{w}\right\|_{[0,1],\infty}\int_0^1 w(s)\,ds,$$

which is equivalent to the operator inequality (3.14).

If we use the inequality (3.10) for w, $h = p$ and $g = \varphi_{(A,B);x}$ then by Lemmas 1 and 2

(3.17)
$$\begin{aligned} 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \\ &- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \\ &\leq \frac{1}{8}[p(1)-p(0)]\sup_{t\in(0,1)}\left|\frac{\langle \nabla f_{(1-t)A+tB}(B-A)x,x\rangle}{w(t)}\right|, \end{aligned}$$

for any $x \in H$, which is an inequality of interest in itself.

Observe that for all $t \in (0,1)$,

$$\left|\langle \nabla f_{(1-t)A+tB}(B-A)x,x\rangle\right| \leq \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\|x\|^2$$

for any $x \in H$, which implies that

$$\begin{aligned} &\sup_{t\in(0,1)}\left|\frac{\langle \nabla f_{(1-t)A+tB}(B-A)x,x\rangle}{w(t)}\right| \\ &\leq \sup_{t\in(0,1)}\left|\frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|}{w(t)}\right|\|x\|^2 \\ &= \sup_{t\in(0,1)}\left|\frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|}{w(t)}\right|\langle 1_H x,x\rangle \end{aligned}$$

for any $x \in H$.

Therefore, by (3.17) we get

$$\begin{aligned} 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \\ &- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\langle f((1-t)A+tB)x,x\rangle\,dt \\ &\leq \frac{1}{8}[p(1)-p(0)]\sup_{t\in(0,1)}\left|\frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|}{w(t)}\right|\langle 1_H x,x\rangle, \end{aligned}$$

for any $x \in H$, which is equivalent to the operator inequality (3.15).

The inequality (3.16) follows by (3.11) in a similar way and we omit the details. □

In [3] we also obtained the weighted inequality in terms of the Euclidian norms:

Lemma 5. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$. If h, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{h'}{w^{1/2}}$, $\frac{g'}{w^{1/2}} \in L_2[a,b]$, then we have*

$$|C(h,g;w)| \leq \frac{1}{\pi^2}\left\|\frac{h'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds. \tag{3.18}$$

The constant $\frac{1}{\pi^2}$ is best possible.

By making use of Lemma 5 we can also prove the following operator weighted integral inequality:

Theorem 4. *Let* A, $B \in \mathcal{SA}_I(H)$ *with* $A \leq B$ *and* f *be an operator monotonic function on* I *while* $w : [0,1] \to [0,\infty)$ *is an integrable function with* $\int_0^1 w(t)\,dt > 0$. *Assume also that* $p : [0,1] \to \mathbb{R}$ *is monotonic nondecreasing on* $[0,1]$. *Then we have*

$$\begin{aligned}(3.19)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,f((1-t)A+tB)\,dt \\ &\quad - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,f((1-t)A+tB)\,dt \\ &\leq \frac{1}{\pi^2}\left(\int_0^1 \frac{(p'(t))^2}{w(t)}dt\right)^{1/2}\left(\int_0^1 \frac{\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^2}{w(t)}dt\right)^{1/2} 1_H,\end{aligned}$$

provided that the last integrals are finite.

4. Some Examples

The power function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$. If $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$ and $w : [0,1] \to [0,\infty)$ an integrable function with $\int_0^1 w(t)\,dt > 0$, then by (3.4)

$$\begin{aligned}(4.1)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,((1-t)A+tB)^\alpha\,dt \\ &\quad - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,((1-t)A+tB)^\alpha\,dt \\ &\leq \frac{1}{2}\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left|p(t) - \frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds\right| w(t)\,dt \\ &\quad \times (B^\alpha - A^\alpha) \\ &\leq \frac{1}{2}\left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 p^2(t)w(t)\,dt - \left(\frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds\right)^2\right)^{1/2} \\ &\quad \times (B^\alpha - A^\alpha) \\ &\leq \frac{1}{4}[p(1)-p(0)](B^\alpha - A^\alpha),\end{aligned}$$

where $B \geq A \geq 0$.

If p is differentiable on $(0,1)$, then by (3.14)

$$\begin{aligned}(4.2)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,((1-t)A+tB)^\alpha\,dt \\ &\quad - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,((1-t)A+tB)^\alpha\,dt \\ &\leq \frac{1}{8}\left\|\frac{p'}{w}\right\|_{[0,1],\infty}\left(\int_0^1 w(s)\,ds\right)(B^\alpha - A^\alpha),\end{aligned}$$

where $B \geq A \geq 0$.

The function $f(t) = -t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S) = T^{-1} S T^{-1}$$

for T, $S > 0$.

If $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$ and $w : [0,1] \to [0,\infty)$ an integrable function with $\int_0^1 w(t)\,dt > 0$, then by (3.15) we get

$$\begin{aligned}(4.3)\quad 0 &\le \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\left((1-t)A+tB\right)^{-1} dt \\ &- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\left((1-t)A+tB\right)^{-1} dt \\ &\le \frac{1}{8}\left[p(1)-p(0)\right]\left(\int_0^1 w(s)\,ds\right) \\ &\times \sup_{t\in(0,1)}\left(\frac{1}{w(t)}\left\|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right\|\right)1_H,\end{aligned}$$

where $B \ge A > 0$.

Moreover, if p is differentiable on $(0,1)$, then by (3.16)

$$\begin{aligned}(4.4)\quad 0 &\le \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\left((1-t)A+tB\right)^{-1} dt \\ &- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\left((1-t)A+tB\right)^{-1} dt \\ &\le \frac{1}{12}\left(\int_a^b w(s)\,ds\right)^2\left\|\frac{p'}{w}\right\|_{[0,1],\infty} \\ &\times \sup_{t\in(0,1)}\left(\frac{1}{w(t)}\left\|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right\|\right)1_H,\end{aligned}$$

where $B \ge A > 0$.

From (3.19) we also have for $B \ge A > 0$ that

$$\begin{aligned}(4.5)\quad 0 &\le \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\left((1-t)A+tB\right)^{-1} dt \\ &- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\left((1-t)A+tB\right)^{-1} dt \\ &\le \frac{1}{\pi^2}\left(\int_0^1 \frac{\left(p'(t)\right)^2}{w(t)}dt\right)^{1/2} \\ &\times \left(\int_0^1 \frac{1}{w(t)}\left\|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right\|^2 dt\right)^{1/2} \acute{1}_H,\end{aligned}$$

provided that the last integrals are finite.

We note that the function $f(t) = \ln t$ is operator monotonic on $(0,\infty)$.

If $0 < A \leq B$, $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$ and $w : [0,1] \to [0,\infty)$ an integrable function with $\int_0^1 w(t)\,dt > 0$, then by (3.4)

$$
\begin{aligned}
(4.6)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\ln((1-t)A+tB)\,dt \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\ln((1-t)A+tB)\,dt \\
&\leq \frac{1}{2}\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left| p(t) - \frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right| w(t)\,dt \\
&\times (\ln B - \ln A) \\
&\leq \frac{1}{2}\left(\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 p^2(t)w(t)\,dt - \left(\frac{1}{\int_0^1 w(s)\,ds}\int_0^1 p(s)w(s)\,ds \right)^2 \right)^{1/2} \\
&\times (\ln B - \ln A) \\
&\leq \frac{1}{4}[p(1) - p(0)](\ln B - \ln A).
\end{aligned}
$$

If p is differentiable on $(0,1)$, then by (3.14)

$$
\begin{aligned}
(4.7)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\ln((1-t)A+tB)\,dt \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\ln((1-t)A+tB)\,dt \\
&\leq \frac{1}{8}\left\| \frac{p'}{w} \right\|_{[0,1],\infty} \left(\int_0^1 w(s)\,ds \right)(\ln B - \ln A)
\end{aligned}
$$

for $0 < A \leq B$.

The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [12, p. 155]):

$$
\nabla \ln_T(S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1}\,ds
$$

for T, $S > 0$.

If $p : [0,1] \to \mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then by (3.15) we get

$$
\begin{aligned}
(4.8)\quad 0 &\leq \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\ln((1-t)A+tB)\,dt \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\ln((1-t)A+tB)\,dt \\
&\leq \frac{1}{8}[p(1) - p(0)]\left(\int_0^1 w(s)\,ds \right) \\
\times \sup_{t\in(0,1)} &\left(\frac{1}{w(t)} \left\| \int_0^\infty (s1_H + (1-t)A + tB)^{-1}(B-A)(s1_H + (1-t)A + tB)^{-1}\,ds \right\| \right) 1_H
\end{aligned}
$$

for $0 < A \leq B$.

Moreover, if p is differentiable on $(0,1)$, then by (3.16)

$$0\leq\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\ln((1-t)A+tB)\,dt - \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,p(t)\,dt\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\ln((1-t)A+tB)\,dt \leq \frac{1}{12}\left(\int_a^b w(s)\,ds\right)^2\left\|\frac{p'}{w}\right\|_{[0,1],\infty} \times\sup_{t\in(0,1)}\left(\frac{1}{w(t)}\left\|\int_0^\infty (s1_H+(1-t)A+tB)^{-1}(B-A)(s1_H+(1-t)A+tB)^{-1}ds\right\|\right)1_H \tag{4.9}$$

for $0<A\leq B$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.