Some Products of Power of a Nonnegative Sequence Inequalities

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Abstract In [3] , Feng Qi has presented a sharp inequality between the sum of squares and the exponential of the sum of a nonnegative sequence. His result has been extended to more general power sums by H. N. Shi [4, 5], and independently by B. Belaïdi, A. El Farissi and Z. Latreuch [1, 2]. In this note we give some inequalities between the exponential of the sum and product of power of a nonnegative sequence. We also give a simple proof of Arithmetic-Geometric-Harmonic means inequality.

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1 Introduction and main results

For our own convenience, we introduce the following notations:

$$[0,\infty)^n \triangleq \underbrace{[0,\infty)\times[0,\infty)\times\cdots\times[0,\infty)}_{n\ times} \tag{1.1}$$

and

$$(0,\infty)^n \triangleq \underbrace{(0,\infty)\times(0,\infty)\times\cdots\times(0,\infty)}_{n\ times} \tag{1.2}$$

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for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

The following inequalities are due to Feng Qi ([3]).

Theorem A *For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geqslant 2$, inequality*

$$\frac{e^2}{4} \sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.3}$$

is valid. Equality in (1.3) holds if $x_i = 2$ for some given $1 \leqslant i \leqslant n$ and $x_j = 0$ for all $1 \leqslant j \leqslant n$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1.3) is the best possible.

Theorem B *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then*

$$\frac{e^2}{4} \sum_{i=1}^{\infty} x_i^2 \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.4}$$

Equality in (1.4) holds if $x_i = 2$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$.Thus, the constant $\frac{e^2}{4}$ in (1.4) is the best possible.

Recently, Theorems A, B have been generalized to sum of power as follows (see [1], [2], [4], [5]) :

Theorem C *Let $p \geqslant 1$ be a real number. For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geqslant 2$, the inequality*

$$\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.5}$$

is valid. Equality in (1.5) holds if $x_i = p$ for some given $1 \leqslant i \leqslant n$ and $x_j = 0$ for all $1 \leqslant j \leqslant n$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (1.5) is the best possible.

Theorem D *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$ and $p \geqslant 1$ be a real number. Then*

$$\frac{e^p}{p^p} \sum_{i=1}^{\infty} x_i^p \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.6}$$

Equality in (1.6) *holds if* $x_i = p$ *for some given* $i \in \mathbb{N}$ *and* $x_j = 0$ *for all* $j \in \mathbb{N}$ *with* $j \neq i$. *Thus, the constant* $\frac{e^p}{p^p}$ *in* (1.6) *is the best possible.*

In this paper we will establish the following inequalities.

Theorem 1.1 *Let* $(p_1, p_2, ..., p_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *For* $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ *and* $n \geqslant 1$, *the inequality*

$$\exp\left(\sum_{i=1}^{n} x_i\right) \geq \frac{\exp\left(\sum\limits_{i=1}^{n} p_i\right)}{\prod\limits_{i=1}^{n} p_i^{p_i}} \prod_{i=1}^{n} x_{\sigma(i)}^{p_i} \tag{1.7}$$

is valid for all permutations $\sigma(i)$ *of* $\{1, 2, ..., n\}$. *Equality in* (1.7) *holds if* $x_{\sigma(i)} = p_i$ *for all* $1 \leqslant i \leqslant n$. *Thus, the constant* $\frac{\exp\left(\sum\limits_{i=1}^{n} p_i\right)}{\prod\limits_{i=1}^{n} p_i^{p_i}}$ *in* (1.7) *is the best possible.*

Theorem 1.2 *Let* $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *Then, we have*

$$\prod_{i=1}^{n} y_i^{y_i} x_i^{x_i} \geq \prod_{i=1}^{n} y_{\mu(i)}^{x_i} x_{\sigma(i)}^{y_i} \tag{1.8}$$

for all two permutations $\sigma(i)$, $\mu(i)$ *of* $\{1, 2, ..., n\}$.

Theorem 1.3 *Let* $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$ *and* $(p_1, p_2, ..., p_n) \in (0, \infty)^n$ $(n \geqslant 1)$ *such that* $\sum\limits_{i=1}^{n} p_i = 1$. *Then, we have*

$$\sum_{i=1}^{n} p_i x_i \geq \prod_{i=1}^{n} x_i^{p_i}. \tag{1.9}$$

Setting $p_i = p$ $(i = 1, ..., n)$ in Theorem 1.1, we obtain:

Corollary 1.1 *Let* $p > 0$ *and* $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ $(n \geqslant 1)$. *Then, the inequality*

$$\exp\left(\sum_{i=1}^{n} x_i\right) \geq \frac{e^{np}}{p^{np}} \prod_{i=1}^{n} x_{\sigma(i)}^{p} \tag{1.10}$$

is valid for all permutations $\sigma(i)$ *of* $\{1, 2, ..., n\}$. *Equality in* (1.10) *holds if* $x_i = p$ *for all* $1 \leqslant i \leqslant n$. *Thus, the constant* $\frac{e^{np}}{p^{np}}$ *in* (1.10) *is the best possible.*

Corollary 1.2 (Arithmetic-Geometric-Harmonic means inequality). *Let* $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *Then, we have*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \geq \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{x_i}}. \tag{1.11}$$

Corollary 1.3 *Let* $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *If* $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$, *then we have*

$$\prod_{i=1}^{n} x_i^{x_i} \geq \prod_{i=1}^{n} y_{\sigma(i)}^{x_i} \tag{1.12}$$

for all permutations $\sigma(i)$ *of* $\{1, 2, ..., n\}$.

Corollary 1.4 *Let* $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *Then, we have*

$$\prod_{i=1}^{n} x_i^{x_i} \geq \prod_{i=1}^{n} x_{\sigma(i)}^{x_i} \tag{1.13}$$

for all permutations $\sigma(i)$ *of* $\{1, 2, ..., n\}$.

Corollary 1.5 *Let* $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. *Then, we have*

$$\prod_{i=1}^{n} x_i^{x_i} \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{\sum\limits_{i=1}^{n} x_i} \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}\sum\limits_{i=1}^{n} x_i}. \tag{1.14}$$

2 Lemma

Lemma 2.1 *Let* $p > 0$ *be a real number and* $x \geqslant 0$. *Then, the inequality*

$$e^x \geq \frac{e^p}{p^p} x^p \tag{2.1}$$

is valid. Equality in (2.1) *holds if* $x = p$*. Thus, the constant* $\frac{e^p}{p^p}$ *in* (2.1) *is the best possible.*

Proof. It is clearly that for all $t \geq -1$, we have $e^t \geq t + 1$. So, if we put $t = \frac{x}{p} - 1$, then we get (2.1).

3 Proof of Theorems

Proof of Theorem 1.1 Let $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. By Lemma 2.1, for all permutations $\sigma(i)$ of $\{1, 2, ..., n\}$ and $p_i > 0$ $(1 \leqslant i \leqslant n)$, we have

$$e^{x_{\sigma(i)}} \geq \frac{e^{p_i}}{p_i^{p_i}} x_{\sigma(i)}^{p_i},$$

which implies that

$$\prod_{i=1}^{n} e^{x_{\sigma(i)}} \geq \prod_{i=1}^{n} \frac{e^{p_i}}{p_i^{p_i}} x_{\sigma(i)}^{p_i}.$$

Then, we can write

$$\exp\left(\sum_{i=1}^{n} x_i\right) \geq \frac{\exp\left(\sum\limits_{i=1}^{n} p_i\right)}{\prod\limits_{i=1}^{n} p_i^{p_i}} \prod_{i=1}^{n} x_{\sigma(i)}^{p_i}. \tag{3.1}$$

Proof of Theorem 1.2 Let $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. By using Theorem 1.1, we have

$$\exp\left(\sum_{i=1}^{n} x_i\right) \geq \frac{\exp\left(\sum\limits_{i=1}^{n} y_i\right)}{\prod\limits_{i=1}^{n} y_i^{y_i}} \prod_{i=1}^{n} x_{\sigma(i)}^{y_i}, \tag{3.2}$$

for all permutations $\sigma(i)$ of $\{1, 2, ..., n\}$ and

$$\exp\left(\sum_{i=1}^{n} y_i\right) \geq \frac{\exp\left(\sum\limits_{i=1}^{n} x_i\right)}{\prod\limits_{i=1}^{n} x_i^{x_i}} \prod_{i=1}^{n} y_{\mu(i)}^{x_i} \tag{3.3}$$

for all permutations $\mu(i)$ of $\{1,2,...,n\}$. By using (3.2) into (3.3), we get (1.8).

Proof of Theorem 1.3 Let $(x_1,x_2,...,x_n)\in(0,\infty)^n$ $(n\geqslant 1)$ and set $\alpha=\prod_{i=1}^{n}x_i^{p_i}$. By Lemma 2.1, we have

$$e^{p_i x_i}\geq e^{p_i}x^{p_i}.$$

It follows that

$$e^{\sum_{i=1}^{n}p_i x_i}\geq e^{\sum_{i=1}^{n}p_i}\prod_{i=1}^{n}x_i^{p_i}=e\prod_{i=1}^{n}x_i^{p_i}. \tag{3.4}$$

Replacing x_i with $\frac{x_i}{\alpha}$ in (3.4), we obtain

$$e^{\sum_{i=1}^{n}\frac{p_i x_i}{\alpha}}\geq e\prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{p_i}=\frac{e\prod_{i=1}^{n}x_i^{p_i}}{\alpha^{\sum_{i=1}^{n}p_i}}=e.$$

Hence

$$\sum_{i=1}^{n}p_i x_i\geq\alpha=\prod_{i=1}^{n}x_i^{p_i}. \tag{3.5}$$

Proof of Corollary 1.2 Let $(x_1,x_2,...,x_n)\in(0,\infty)^n$ $(n\geqslant 1)$. By Theorem 1.3 for $p_i=\frac{1}{n}$ $(i=1,...,n)$, we have

$$\frac{1}{n}\sum_{i=1}^{n}x_i\geq\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}. \tag{3.6}$$

Replacing x_i with $\dfrac{1}{x_i}$ in (3.6), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\geq\frac{1}{\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}}.$$

It follows that

$$\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}\geq\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}. \tag{3.7}$$

Now using (3.6) and (3.7), we get (1.11).

Proof of Corollary 1.3 Let $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, $(x_1, x_2, ..., x_n) \in (0, \infty)^n$ $(n \geqslant 1)$. By using Theorem 1.1, we have

$$\exp\left(\sum_{i=1}^{n} y_i\right) \geq \frac{\exp\left(\sum\limits_{i=1}^{n} x_i\right)}{\prod\limits_{i=1}^{n} x_i^{x_i}} \prod_{i=1}^{n} y_{\sigma(i)}^{x_i}$$

for all permutations $\sigma(i)$ of $\{1, 2, ..., n\}$. Using the fact that $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$, then we get (1.12).

Proof of Corollary 1.4 This can be concluded by letting $x_i = y_i$ in Corollary 1.3.

Proof of Corollary 1.5 By letting $y_i = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$ $(i = 1, ..., n)$ in Corollary 1.3, we obtain

$$\prod_{i=1}^{n} x_i^{x_i} \geq \left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)^{\sum\limits_{i=1}^{n} x_i}. \tag{3.8}$$

Now by Corollary 1.2, we have

$$\frac{1}{n} \sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

It follows that

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)^{\sum\limits_{i=1}^{n} x_i} \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n} \sum\limits_{i=1}^{n} x_i}. \tag{3.9}$$

By (3.8) and (3.9), we get (1.14).

References

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