GENERALIZATIONS OF FEJÉR'S INEQUALITY FOR RIEMANN-STIELTJES INTEGRAL

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Abstract. We say that the function $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$ if the *symmetrical transform* $\check{f}(t):=\frac{1}{2}[f(t)+f(a+b-t)]$, $t\in[a,b]$ is convex on $[a,b]$. Also we say that $g:[a,b]\to\mathbb{R}$ is *asymmetrized monotonic nondecreasing* on $[a,b]$ if the asymmetric transform

$$\tilde{g}(t):=\frac{1}{2}[g(t)-g(a+b-t)],\ t\in[a,b]$$

is monotonic nondecreasing on $[a,b]$. In this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is a symmetrized convex and continuous on the interval $[a,b]$ and $g:[a,b]\to\mathbb{R}$ asymmetrized monotonic nondecreasing on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\leq\int_a^b\check{f}(t)\,dg(t)\leq\frac{f(a)+f(b)}{2}[g(b)-g(a)].$$

If, in addition, we assume that f is differentiable on (a,b), then

$$\int_a^b\check{f}(t)\,dg(t)-\check{f}(s)[g(b)-g(a)]\geq\widetilde{f'}(s)[g(b)-g(a)]\left(\frac{a+b}{2}-s\right)$$

for all $s\in(a,b)$.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\check{f}_{[a,b]}$ or simply $\check{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\check{f}(t):=\frac{1}{2}[f(t)+f(a+b-t)],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}[f(t)-f(a+b-t)],\ t\in[a,b].$$

It is obvious that for any function f we have $\check{f}+\tilde{f}=f$.

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If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\check{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leq\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\check{f}(t_1)+\beta\check{f}(t_2),\end{aligned}$$

which shows that $\check{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [6]

$$\check{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

for any $t\in\mathbb{R}$.

Since the second derivative $\left(\check{f}_0\right)''(t)=3(a+b)$, $t\in\mathbb{R}$, then $\check{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2}>0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2}<0$. Therefore if $a<0<b$ with $\frac{a+b}{2}>0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\check{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [6], see also [9] for an equivalent definition.

Definition 1. *We say that the function $f:[a,b]\to\mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\check{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the closed convex cone of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b]\subsetneq SCon[a,b]. \tag{1.2}$$

Also, if $[c,d]\subset[a,b]$ and $f\in SCon[a,b]$, then this does not imply in general that $f\in SCon[c,d]$.

We have the following result [6], [9] :

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{1.3}$$

We also have [6]:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x\in[a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right)\leq\check{f}(x)\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

In [6, Corollary 1] we also obtained the following generalization of Fejér's inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt, \tag{1.5}$$

provided that $f:[a,b]\to\mathbb{R}$ is symmetrized convex and continuous on the interval $[a,b]$ and $p:[a,b]\to[0,\infty)$ is a symmetric integrable function on $[a,b]$.

For a monograph on Hermite-Hadamard type inequalities see [8]. For related results, see [2]-[5] and [9]-[10].

In a similar way, we can introduce the following concept as well:

Definition 2. *We say that the function $f:[a,b]\to\mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ if the anti-symmetrical transform $\tilde{f}$ is monotonic nondecreasing (nonincreasing) on the interval $[a,b]$.*

If f is monotonic nondecreasing on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$, $t_1<t_2$ we have

$$\begin{aligned}\tilde{f}(t_2)-\tilde{f}(t_1) &= \frac{1}{2}\left[f(t_2)-f(a+b-t_2)\right]-\frac{1}{2}\left[f(t_1)-f(a+b-t_1)\right]\\ &= \frac{1}{2}\left[f(t_2)-f(t_1)\right]+\frac{1}{2}\left[f(a+b-t_1)-f(a+b-t_2)\right]\\ &\geq 0,\end{aligned}$$

which shows that $f:[a,b]\to\mathbb{R}$ is asymmetrized monotonic nondecreasing on the interval $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^2$. We have

$$\tilde{f}_0(t):=\frac{1}{2}\left[t^2-(a+b-t)^2\right]=(a+b)\,t-\frac{1}{2}(a+b)^2$$

and $\left(\tilde{f}_0\right)'(t)=a+b$, therefore $f:[a,b]\to\mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ provided $\frac{a+b}{2}>0\,(<0)$. So, if we take $a<0<b$ with $\frac{a+b}{2}>0$, then f is asymmetrized monotonic nondecreasing on $[a,b]$ but not monotonic nondecreasing on $[a,b]$.

If we denote by $\mathcal{M}^{\nearrow}[a,b]$ the closed convex cone of monotonic nondecreasing functions defined on $[a,b]$ and by $\mathcal{AM}^{\nearrow}[a,b]$ the closed convex cone of asymmetrized monotonic nondecreasing functions, then from the above remarks we can conclude that

$$\mathcal{M}^{\nearrow}[a,b]\subsetneq\mathcal{AM}^{\nearrow}[a,b]. \tag{1.6}$$

Also, if $[c,d]\subset[a,b]$ and $f\in\mathcal{AM}^{\nearrow}[a,b]$, then this does not imply in general that $f\in\mathcal{AM}^{\nearrow}[c,d]$.

We recall that the pair of functions (f,g) defined on $[a,b]$ are called *synchronous (asynchronous)* on $[a,b]$ if

$$(f(t)-f(s))\,(g(t)-g(s))\geq(\leq)\,0 \tag{1.7}$$

for any $t,\ s\in[a,b]$. It is clear that if both functions (f,g) are monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are synchronous on $[a,b]$. There are also functions that change monotonicity on $[a,b]$, but as a pair they are still synchronous. For instance if $a<0<b$ and $f,g:[a,b]\to\mathbb{R}$, $f(t)=t^2$ and $g(t)=t^4$, then

$$(f(t)-f(s))(g(t)-g(s))=\left(t^2-s^2\right)\left(t^4-s^4\right)=\left(t^2-s^2\right)^2\left(t^2+s^2\right)\geq 0$$

for any t, $s\in[a,b]$, which show that (f,g) is synchronous.

Definition 3. *We say that the pair of functions (f,g) defined on $[a,b]$ is called asymmetrized synchronous (asynchronous) on $[a,b]$ if the pair of transforms $\left(\tilde{f},\tilde{g}\right)$ is synchronous (asynchronous) on $[a,b]$, namely*

$$\left(\tilde{f}(t)-\tilde{f}(s)\right)(\tilde{g}(t)-\tilde{g}(s))\geq(\leq)\,0 \tag{1.8}$$

for any $t,s\in[a,b]$.

It is clear that if f, g are asymmetrized monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are asymmetrized synchronous on $[a,b]$.

We have the following result [7]:

Theorem 3. *Assume that f, g are asymmetrized synchronous (asynchronous) and integrable functions on $[a,b]$. Then*

$$\int_a^b\tilde{f}(t)\,g(t)\,dt\geq(\leq)\,0. \tag{1.9}$$

The case of asymmetrized monotonic functions provides the following reverse inequalities as well [7]:

Theorem 4. *If both f, g are asymmetrized monotonic nondecreasing (nonincreasing) and integrable functions on $[a,b]$, then*

$$\frac{1}{4}\left|f(b)-f(a)\right|\left|g(b)-g(a)\right|\geq\frac{1}{b-a}\int_a^b\tilde{f}(t)\,g(t)\,dt\geq 0, \tag{1.10}$$

and

$$\frac{1}{2}\min\left\{\left|f(b)-f(a)\right|\frac{1}{b-a}\int_a^b|g(t)|\,dt,\left|g(b)-g(a)\right|\frac{1}{b-a}\int_a^b|f(t)|\,dt\right\}\geq\frac{1}{b-a}\int_a^b\tilde{f}(t)\,g(t)\,dt\geq 0. \tag{1.11}$$

Motivated by the above results, in this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is a *symmetrized convex* and continuous on the interval $[a,b]$ and $g:[a,b]\to\mathbb{R}$ *asymmetrized monotonic nondecreasing* on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\leq\int_a^b\check{f}(t)\,dg(t)\leq\frac{f(a)+f(b)}{2}[g(b)-g(a)].$$

If, in addition, we assume that f is differentiable on (a,b), then

$$\int_a^b\check{f}(t)\,dg(t)-\check{f}(s)[g(b)-g(a)]\geq\widetilde{f'}(s)[g(b)-g(a)]\left(\frac{a+b}{2}-s\right)$$

for all $s\in(a,b)$.

2. Main Results

We have the following result for Riemann-Stieltjes integral:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a symmetrized convex and continuous on the interval $[a,b]$ and $g:[a,b]\to\mathbb{R}$ asymmetrized monotonic nondecreasing on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\leq\int_a^b\check{f}(t)\,dg(t)\leq\frac{f(a)+f(b)}{2}[g(b)-g(a)]. \tag{2.1}$$

In particular, if $f:[a,b]\to\mathbb{R}$ symmetric, convex and continuous, then

$$f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\leq\int_a^b f(t)\,dg(t)\leq\frac{f(a)+f(b)}{2}[g(b)-g(a)]. \tag{2.2}$$

Proof. Since $\check{f}$ is continuous convex on $[a,b]$ and $\tilde{g}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b\check{f}(t)\,d\tilde{g}(t)$ exists, see for instance [2, p. 159].

The Riemann-Stieltjes integral with monotonic nondecreasing integrator is an isotonic linear functional, see for instance [2, p. 155]. By Theorem 2 we then have

$$f\left(\frac{a+b}{2}\right)\int_a^b d\tilde{g}(t)\leq\int_a^b\check{f}(t)\,d\tilde{g}(t)\leq\frac{f(a)+f(b)}{2}\int_a^b d\tilde{g}(t). \tag{2.3}$$

Observe that

$$\begin{aligned}\int_a^b d\tilde{g}(t) &= \tilde{g}(b)-\tilde{g}(a)=\frac{1}{2}[g(b)-g(a)]-\frac{1}{2}[g(a)-g(b)]\\ &= g(b)-g(a).\end{aligned}$$

Also

$$\begin{aligned}\int_a^b\check{f}(t)\,d\tilde{g}(t) &= \frac{1}{2}\int_a^b(f(t)+f(a+b-t))\,d\tilde{g}(t)\\ &= \frac{1}{2}\left[\int_a^b f(t)\,d\tilde{g}(t)+\int_a^b f(a+b-t)\,d\tilde{g}(t)\right].\end{aligned} \tag{2.4}$$

Furthermore,

$$\begin{aligned}\int_a^b f(t)\,d\tilde{g}(t) &= \int_a^b f(t)\,d\left[\frac{g(t)-g(a+b-t)}{2}\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dg(t)-\int_a^b f(t)\,dg(a+b-t)\right].\end{aligned} \tag{2.5}$$

Using the change of variable for the Riemann-Stieltjes integral, see for instance [2, 144-145], then by putting $s=a+b-t$, $t\in[a,b]$ we have

$$\int_a^b f(t)\,dg(a+b-t)=\int_b^a f(a+b-s)\,dg(s)=-\int_a^b f(a+b-s)\,dg(s)$$

and by (2.5) we derive

$$\int_a^b f(t)\,d\tilde{g}(t) = \frac{1}{2}\left[\int_a^b f(t)\,dg(t) + \int_a^b f(a+b-t)\,dg(t)\right] \tag{2.6}$$
$$= \int_a^b \check{f}(t)\,dg(t).$$

Also, we have

$$\int_a^b f(a+b-t)\,d\tilde{g}(t) \tag{2.7}$$
$$= \int_a^b f(a+b-t)\,d\left[\frac{g(t)-g(a+b-t)}{2}\right]$$
$$= \frac{1}{2}\left[\int_a^b f(a+b-t)\,dg(t) - \int_a^b f(a+b-t)\,dg(a+b-t)\right].$$

Using the change of variable for the Riemann-Stieltjes integral, then by putting $s = a+b-t$, $t \in [a,b]$ we have

$$\int_a^b f(a+b-t)\,dg(a+b-t) = \int_b^a f(s)\,dg(s) = -\int_a^b f(s)\,dg(s)$$

and by (2.7) we derive

$$\int_a^b f(a+b-t)\,d\tilde{g}(t) = \frac{1}{2}\left[\int_a^b f(a+b-t)\,dg(t) + \int_a^b f(t)\,dg(t)\right] \tag{2.8}$$
$$= \int_a^b \check{f}(t)\,dg(t).$$

By (2.4), (2.6) and (2.8) we then obtain

$$\int_a^b \check{f}(t)\,d\tilde{g}(t) = \int_a^b \check{f}(t)\,dg(t)$$

and by (2.3) we derive (2.1). □

Remark 1. *We observe that, if g is monotonic nondecreasing on $[a,b]$, then the inequalities (2.1) and (2.2) are valid.*

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a symmetrized convex and continuous on the interval $[a,b]$ and $p : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \le \int_a^b \check{f}(t)\,p(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{2.9}$$

In particular, if $f : [a,b] \to \mathbb{R}$ is symmetric, convex and continuous, then

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \le \int_a^b f(t)\,p(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{2.10}$$

Proof. Observe that the function $g(t) := \int_a^t p(s)\,ds$ is monotonic nondecreasing on $[a,b]$, and by the properties of Riemann-Stieltjes integral, see for instance [2, p. 158]

$$\int_a^b \check{f}(t)\,d\left(\int_a^t p(s)\,ds\right) = \int_a^b \check{f}(t)\,p(t)\,dt.$$

By using (2.1) and (2.2) we derive the desired results (2.9) and (2.10). □

Remark 2. *We observe that, if $p:[a,b]\to[0,\infty)$ is a symmetric integrable function on $[a,b]$, then*

$$\begin{aligned}\int_a^b \check{f}(t)\,p(t)\,dt &= \frac{1}{2}\int_a^b [f(t)+f(a+b-t)]\,p(t)\,dt\\ &= \frac{1}{2}\left[\int_a^b f(t)\,p(t)\,dt+\int_a^b f(a+b-t)\,p(t)\,dt\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,p(t)\,dt+\int_a^b f(a+b-t)\,p(a+b-t)\,dt\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,p(t)\,dt+\int_a^b f(s)\,p(s)\,dt\right]=\int_a^b f(t)\,p(t)\,dt\end{aligned}$$

and by (2.9) we get (1.5).

Definition 4. *We say that the function $f:[a,b]\to\mathbb{R}$ is asymmetric on $[a,b]$ if*

$$f(a+b-t)=-f(t)\ \textit{ for all } t\in[a,b]\,.$$

We also have:

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a symmetrized convex and continuous function on the interval $[a,b]$ and $g:[a,b]\to\mathbb{R}$ monotonic nondecreasing and asymmetric on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)g(b)\le\frac{1}{2}\int_a^b f(t)\,dg(t)\le\frac{f(a)+f(b)}{2}g(b)\,. \tag{2.11}$$

In particular, if $f:[a,b]\to\mathbb{R}$ is convex and continuous on the interval $[a,b]$, then (2.11) is valid as well.

Proof. Since $\check{f}$ is convex and g monotonic nondecreasing, then by (1.4) we get

$$f\left(\frac{a+b}{2}\right)\int_a^b dg(t)\le\int_a^b \check{f}(t)\,dg(t)\le\frac{f(a)+f(b)}{2}\int_a^b dg(t)\,. \tag{2.12}$$

Observe that

$$\int_a^b \check{f}(t)\,dg(t)=\frac{1}{2}\left[\int_a^b f(t)\,dg(t)+\int_a^b f(a+b-t)\,dg(t)\right] \tag{2.13}$$

If we use the change of variable $s=a+b-t$ in the Riemann-Stieltjes integral, we have

$$\int_a^b f(a+b-t)\,dg(t)=\int_b^a f(s)\,dg(a+b-s)=\int_a^b f(s)\,d[-g(a+b-s)]\,.$$

Since g is antisymmetric on $[a,b]$, hence $-g(a+b-s)=g(s)$, $s\in[a,b]$ and we have

$$\int_a^b f(s)\,d[-g(a+b-s)]=\int_a^b f(s)\,dg(s)\,.$$

Therefore

$$\int_a^b \check{f}(t)\,dg(t)=\int_a^b f(t)\,dg(t)\,.$$

Also, by the asymmetry of g we have

$$\int_a^b dg(t) = g(b) - g(a) = 2g(b).$$

By making use of (2.12) we derive (2.11). □

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be a symmetrized convex and continuous on the interval $[a,b]$ and $p : [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and symmetric nonnegative, i.e., $\breve{p}(t) \geq 0$, $t \in [a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,\breve{p}(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{2.14}$$

In particular, if $f : [a,b] \to \mathbb{R}$ is convex and continuous on the interval $[a,b]$, then (2.11) is valid as well.

Proof. Consider the function

$$g(t) := \int_{a+b-t}^{t} p(s)\,ds.$$

Observe that

$$g(a+b-t) = \int_t^{a+b-t} p(s)\,ds = -\int_{a+b-t}^{t} p(s)\,ds = -g(t),$$

which shows that g is asymmetric on $[a,b]$.

The function g is also differentiable on (a,b). Using Leibniz's integral rule we have

$$\begin{aligned} g'(t) &:= \frac{d}{dt}\left(\int_{a+b-t}^{t} p(s)\,ds\right) = p(t) - p(a+b-t)(a+b-t)' \\ &= p(t) + p(a+b-t) = 2\breve{p}(t) \geq 0. \end{aligned}$$

Since $g(t) = \int_a^b p(t)\,dt$ and, see for instance [2, p. 163],

$$\frac{1}{2}\int_a^b f(t)\,dg(t) = \frac{1}{2}\int_a^b f(t)\,g'(t)\,dt = \int_a^b f(t)\,\breve{p}(t)\,dt,$$

hence by (2.11) we get (2.14). □

Remark 3. *We observe that, if p is nonnegative, then $\breve{p}$ is also nonnegative. The converse is not true in general. Indeed, if we consider the function $p : [a,b] \to \mathbb{R}$, then $\breve{p}(t) = \frac{1}{2}(t+a+b-t) = \frac{a+b}{2}$. If we choose $a < 0 < b$ with $\frac{a+b}{2} > 0$, then $\breve{p}$ is nonnegative on $[a,b]$ but p is not.*

3. Related Results

We can prove the following result as well:

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a symmetrized convex and differentiable function on the interval (a,b) and $g : [a,b] \to \mathbb{R}$ asymmetrized monotonic nondecreasing on $[a,b]$. Then for all $s \in (a,b)$,*

$$\int_a^b \breve{f}(t)\,dg(t) - \breve{f}(s)[g(b)-g(a)] \geq \widetilde{f}'(s)[g(b)-g(a)]\left(\frac{a+b}{2} - s\right). \tag{3.1}$$

Proof. By the convexity of $\check{f}$ and the gradient inequality, we have

$$\check{f}(t)-\check{f}(s)\geq\left(\check{f}\right)'(s)(t-s)=\widetilde{f'}(s)(t-s) \tag{3.2}$$

for all $t\in[a,b]$ and $s\in(a,b)$.

Since $g:[a,b]\to\mathbb{R}$ is asymmetrized monotonic nondecreasing on $[a,b]$, hence

$$\int_a^b\check{f}(t)\,d\tilde{g}(t)-\check{f}(s)\int_a^b d\tilde{g}(t)\geq\widetilde{f'}(s)\int_a^b(t-s)\,d\tilde{g}(t). \tag{3.3}$$

Observe that

$$\int_a^b\check{f}(t)\,d\tilde{g}(t)=\int_a^b\check{f}(t)\,dg(t),\quad\int_a^b d\tilde{g}(t)=g(b)-g(a).$$

Using the integration by parts formula for Riemann-Stieltjes integral [2, p. 144], we have

$$\begin{aligned}\int_a^b(t-s)\,d\tilde{g}(t)&=\tilde{g}(t)(t-s)|_a^b-\int_a^b\tilde{g}(t)\,dt\\&=\tilde{g}(b)(b-s)-\tilde{g}(a)(a-s)-\frac{1}{2}\int_a^b[g(t)-g(a+b-t)]\,dt\\&=\frac{1}{2}[g(b)-g(a)](b-s)-\frac{1}{2}[g(a)-g(b)](a-s)\\&=\frac{1}{2}[g(b)-g(a)](a+b-2s)=[g(b)-g(a)]\left(\frac{a+b}{2}-s\right)\end{aligned}$$

since

$$\int_a^b g(t)\,dt=\int_a^b g(a+b-t)\,dt.$$

By utilising the inequality (3.3) we then obtain (3.1). □

Remark 4. *We observe that, if we take* $s=\frac{a+b}{2}$ *in (3.1), then we obtain the first inequality in (2.1).*

Corollary 3. *With the assumptions of Theorem 7 we have*

$$\frac{1}{g(b)-g(a)}\int_a^b\check{f}(t)\,dg(t)\geq\frac{2}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}. \tag{3.4}$$

Proof. By taking the integral mean in (3.1) we get

$$\begin{aligned}&\int_a^b\check{f}(t)\,dg(t)-[g(b)-g(a)]\frac{1}{b-a}\int_a^b\check{f}(s)\,ds\\&\geq[g(b)-g(a)]\frac{1}{b-a}\int_a^b\widetilde{f'}(s)\left(\frac{a+b}{2}-s\right)ds.\end{aligned} \tag{3.5}$$

Observe that

$$\frac{1}{b-a}\int_a^b\check{f}(s)\,ds=\frac{1}{b-a}\int_a^b f(t)\,dt.$$

Integrating by parts, we have

$$\begin{aligned}\int_a^b \widetilde{f}'(s)\left(\frac{a+b}{2}-s\right)ds &= \int_a^b \left(\check{f}\right)'(s)\left(\frac{a+b}{2}-s\right)ds\\ &= \left(\frac{a+b}{2}-s\right)\check{f}(s)\Big|_a^b + \int_a^b \check{f}(s)\,ds\\ &= \left(\frac{a+b}{2}-b\right)\check{f}(b) - \left(\frac{a+b}{2}-a\right)\check{f}(a) + \int_a^b f(t)\,dt\\ &= \int_a^b f(t)\,dt - (b-a)\frac{\check{f}(a)+\check{f}(b)}{2}\\ &= \int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2}.\end{aligned}$$

By making use of (3.5), we derive

$$\begin{aligned}&\int_a^b \check{f}(t)\,dg(t) - [g(b)-g(a)]\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\geq [g(b)-g(a)]\left[\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right],\end{aligned}$$

namely

$$\begin{aligned}\int_a^b \check{f}(t)\,dg(t) &\geq [g(b)-g(a)]\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\quad + [g(b)-g(a)]\left[\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right]\\ &= 2[g(b)-g(a)]\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}[g(b)-g(a)],\end{aligned}$$

which is equivalent to (3.4). □

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be a symmetrized convex and differentiable function on the interval (a,b) and $p:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, then*

$$\int_a^b \check{f}(t)\,p(t)\,dt - \check{f}(s)\int_a^b p(t)\,dt \geq \widetilde{f}'(s)\left(\frac{a+b}{2}-s\right)\int_a^b p(t)\,dt \tag{3.6}$$

for $s\in(a,b)$ and

$$\frac{1}{\int_a^b p(t)\,dt}\int_a^b \check{f}(t)\,p(t)\,dt \geq \frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}. \tag{3.7}$$

The case of monotonic nondecreasing and asymmetric integrators is as follows:

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a symmetrized convex and differentiable function on the interval (a,b) and $g:[a,b]\to\mathbb{R}$ monotonic nondecreasing and asymmetric on $[a,b]$, then for all $s\in(a,b)$,*

$$\frac{1}{2}\int_a^b f(t)\,dg(t) - \check{f}(s)\,g(b) \geq \widetilde{f}'(s)\left(\frac{b+a}{2}-s\right)g(b). \tag{3.8}$$

We also have

$$\frac{1}{2g(b)}\int_a^b f(t)\,dg(t) \geq \frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}. \tag{3.9}$$

Proof. By the gradient inequality and since $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, hence

$$\int_a^b \check{f}(t)\,dg(t) - \check{f}(s)\int_a^b dg(t) \geq \widetilde{f'}(s)\int_a^b (t-s)\,dg(t). \tag{3.10}$$

Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b (t-s)\,dg(t) &= (t-s)\,g(t)|_a^b - \int_a^b g(t)\,dt \\ &= (b-s)\,g(b) - (a-s)\,g(a) - \int_a^b g(t)\,dt \\ &= (b-s)\,g(b) + (a-s)\,g(b) - \int_a^b g(t)\,dt \\ &= (b+a-2s)\,g(b) - \int_a^b g(t)\,dt.\end{aligned}$$

Since

$$\int_a^b \check{f}(t)\,dg(t) = \int_a^b f(t)\,dg(t),\quad \int_a^b dg(t) = g(b) - g(a) = 2g(b),$$

hence by (3.10) we obtain

$$\begin{aligned}\int_a^b f(t)\,dg(t) - 2\check{f}(s)\,g(b) &\geq \widetilde{f'}(s)\left[(b+a-2s)\,g(b) - \int_a^b g(t)\,dt\right] \\ &= \widetilde{f'}(s)\,(b+a-2s)\,g(b)\end{aligned}$$

due to the fact that, by asymmetry of g on $[a,b]$, $\int_a^b g(t)\,dt = 0$.

This inequality is equivalent to (3.10).

By taking the integral mean and performing the required calculations, we obtain

□

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be a symmetrized convex and differentiable function on the interval (a,b) and $p:[a,b]\to\mathbb{R}$ continuous on $[a,b]$ and symmetric nonnegative, then for all $s\in(a,b)$,*

$$\int_a^b f(t)\,\check{p}(t)\,dt - \check{f}(s)\int_a^b p(t)\,dt \geq \widetilde{f'}(s)\left(\frac{b+a}{2} - s\right)\int_a^b p(t)\,dt. \tag{3.11}$$

We also have

$$\frac{1}{\int_a^b p(t)\,dt}\int_a^b f(t)\,\check{p}(t)\,dt \geq \frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}. \tag{3.12}$$

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