RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR OPERATOR MONOTONIC AND OPERATOR CONVEX INTEGRANDS AND LIPSCHITZIAN INTEGRATORS

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Abstract. Let f be an operator monotonic function on I and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A, B \in \mathcal{SA}_I(H)$ with $A \leq B$ and u is L-Lipschitzian with the constant $L > 0$ on $[0,1]$, then we have the operator inequality

$$-\frac{1}{2}L[f(B)-f(A)]$$
$$\leq \int_0^1 f((1-t)A+tB)\,du(t) - [u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt$$
$$\leq \frac{1}{2}L[f(B)-f(A)].$$

If f is an operator convex function on I and u is as above, then we have the operator inequality

$$-\frac{1}{2}L\left(\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right)$$
$$\leq \int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t)) - [u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt$$
$$\leq \frac{1}{2}L\left(\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right),$$

where $\tilde{u}(t) := \frac{1}{2}[u(t)-u(a+b-t)]$, $t \in [a,b]$.

Some particular examples of interest are also given.

1. Introduction

In order to generalize the Grüss inequality to the Riemann-Stieltjes integral, S.S. Dragomir and I. Fedotov introduced in [9] the following functional

$$D(f;u) := \int_a^b f(t)\,du(t) - [u(b)-u(a)]\frac{1}{b-a}\int_a^b f(t)\,dt$$

provided the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. In the same paper, the authors have shown that

$$|D(f;u)| \leq \frac{1}{2}L(M-m)(b-a), \tag{1.1}$$

provided that u is L-Lipschitzian with the constant $L > 0$, i.e.,

$$|u(t)-u(s)| \leq L|t-s|$$

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for any $t, s \in [a, b]$ and f is Riemann integrable and bounded below by m and above by M, namely

$$m \leq f(t) \leq M \text{ for all } t \in [a, b].$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{1.2}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [11] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Assume that f is operator convex on I. If $p : [0, 1] \to [0, \infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s) = p(s)$ for all $s \in [0, 1]$, then for A, B selfadjoint operators with spectra included in I, we have the operator Fejér's inequality

$$\begin{aligned} \left(\int_0^1 p(s)\,ds\right) f\left(\frac{A+B}{2}\right) &\leq \int_0^1 p(s) f(sA + (1-s)B)\,ds \\ &\leq \left(\int_0^1 p(s)\,ds\right) \frac{f(A) + f(B)}{2}, \end{aligned} \tag{1.3}$$

see [7] where further reverses were obtained.

In the recent paper [8] we obtained the following result:

Let A, $B \in \mathcal{SA}_I(H)$ with $A \leq B$ and f an operator monotonic function on I. If $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{1}{2}\left(\int_0^1 \left|p(t) - \int_0^1 p(s)ds\right| dt\right) [f(B) - f(A)] \\ &\leq \frac{1}{2}\left(\int_0^1 p^2(t)dt - \left(\int_0^1 p(s)ds\right)^2\right)^{1/2} [f(B) - f(A)] \\ &\leq \frac{1}{4}[p(1) - p(0)][f(B) - f(A)]. \end{aligned} \tag{1.4}$$

For other operator inequalities, see [1]-[6] and [12]-[20].

Motivated by the above results, we establish in this paper some lower and upper bounds in operator order for the difference

$$\int_0^1 f\left((1-t)A+tB\right)du(t)-[u(1)-u(0)]\int_0^1 f\left((1-t)A+tB\right)dt,$$

where u is L-Lipschitzian with the constant $L>0$ on $[0,1]$ and f is either operator monotone or operator convex on I. Some examples for the logarithmic function which is both operator monotonic and operator concave are given.

2. Main Results

Let f be a continuous function on I. If $(A,B)\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I and $t\in[0,1]$, then the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f\left((1-t)A+tB\right)$ is a selfadjoint operator in $\mathcal{B}(H)$.

For A, $B\in\mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f\left((1-t)A+tB\right). \tag{2.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\left\langle f\left((1-t)A+tB\right)x,x\right\rangle. \tag{2.2}$$

We have:

Theorem 1. *Let A, $B\in\mathcal{SA}_I(H)$ with $A\leq B$ and f an operator monotonic function on I. If u is L-Lipschitzian with the constant $L>0$ on $[0,1]$, then we have the operator inequality*

$$\begin{aligned}&-\frac{1}{2}L\left[f(B)-f(A)\right]\\&\leq\int_0^1 f\left((1-t)A+tB\right)du(t)-[u(1)-u(0)]\int_0^1 f\left((1-t)A+tB\right)dt\\&\leq\frac{1}{2}L\left[f(B)-f(A)\right].\end{aligned} \tag{2.3}$$

We also have the norm inequality

$$\begin{aligned}&\left\|\int_0^1 f\left((1-t)A+tB\right)du(t)-[u(1)-u(0)]\int_0^1 f\left((1-t)A+tB\right)dt\right\|\\&\leq\frac{1}{2}L\left\|f(B)-f(A)\right\|.\end{aligned} \tag{2.4}$$

Proof. Let $0\leq t_1<t_2\leq 1$ and $A\leq B$. Then

$$(1-t_2)A+t_2B-(1-t_1)A-t_1B=(t_2-t_1)(B-A)\geq 0$$

and by operator monotonicity of f we get

$$f\left((1-t_2)A+t_2B\right)\geq f\left((1-t_1)A+t_1B\right),$$

which is equivalent to

$$\begin{aligned}\varphi_{(A,B);x}(t_2) &= \langle f((1-t_2)A+t_2B)x,x\rangle \\ &\geq \langle f((1-t_1)A+t_1B)x,x\rangle = \varphi_{(A,B);x}(t_1)\end{aligned}$$

that shows that the scalar function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ is monotonic nondecreasing for $A\leq B$ and for any $x\in H$.

We also have that

$$\varphi_{(A,B);x}(0)\leq\varphi_{(A,B);x}(t)\leq\varphi_{(A,B);x}(1),$$

namely

$$\langle f(A)x,x\rangle\leq\varphi_{(A,B);x}(t)\leq\langle f(B)x,x\rangle,$$

for all $t\in[0,1]$ and $x\in H$.

If we use the inequality (1.1) for $\varphi_{(A,B);x}$ and u, then

$$\begin{aligned}&\left|\int_0^1\varphi_{(A,B);x}(t)\,du(t)-[u(1)-u(0)]\int_0^1\varphi_{(A,B);x}(t)\,dt\right| \\ &\leq\frac{1}{2}L(\langle f(B)x,x\rangle-\langle f(A)x,x\rangle),\end{aligned}$$

namely

$$\begin{aligned}&\left|\left\langle\left(\int_0^1 f((1-t)A+tB)\,du(t)\right.\right.\right. \\ &\left.\left.\left.-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right)x,x\right\rangle\right| \\ &\leq\frac{1}{2}L\langle[f(B)-f(A)]x,x\rangle\end{aligned}\tag{2.5}$$

for all $x\in H$.

This inequality can be written as

$$\begin{aligned}-\frac{1}{2}L\langle[f(B)-f(A)]x,x\rangle &\leq\left\langle\left(\int_0^1 f((1-t)A+tB)\,du(t)\right.\right. \\ &\left.\left.-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right)x,x\right\rangle \\ &\leq\frac{1}{2}L\langle[f(B)-f(A)]x,x\rangle\end{aligned}$$

for all $x\in H$, which is equivalent to the operator inequality (2.3).

Taking the supremum over $x\in H$, $\|x\|=1$ in (2.5) we get (2.4). □

Corollary 1. *Let A, $B\in\mathcal{SA}_I(H)$ with $A\leq B$ and f an operator monotonic function on I. If p is Lebesgue integrable on $[0,1]$ with $\|p\|_\infty:=\operatorname{essup}_{t\in[0,1]}|p(t)|<\infty$ then we have the operator inequality*

$$\begin{aligned}&-\frac{1}{2}\|p\|_\infty[f(B)-f(A)] \\ &\leq\int_0^1 p(t)f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt \\ &\leq\frac{1}{2}\|p\|_\infty[f(B)-f(A)].\end{aligned}\tag{2.6}$$

We also have the norm inequality

$$\left\|\int_0^1 p(t) f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\| \leq \frac{1}{2}\|p\|_\infty \|f(B)-f(A)\|. \tag{2.7}$$

Proof. If we take $u(t) := \int_0^t p(s)\,ds$ and use the properties of Riemann-Stieltjes integral, then we have

$$\int_0^1 f((1-t)A+tB)\,du(t)=\int_0^1 f((1-t)A+tB)\,p(t)\,dt.$$

Also

$$|u(t)-u(s)|=\left|\int_s^t p(s)\,ds\right|\leq \|p\|_\infty |t-s|$$

for all $t,\ s\in[0,1]$, which shows that u is Lipschitzian with the constant $L=\|p\|_\infty$. □

We also have:

Corollary 2. *Let $A,\ B\in \mathcal{SA}_I(H)$ with $A\leq B$ and f an operator monotonic function on I. If q is Lebesgue integrable on $[0,1]$ and there exists the constants m, M with*

$$m\leq q\leq M \ \text{a.e. on}\ [0,1], \tag{2.8}$$

then we have the operator inequality

$$\begin{aligned}&-\frac{1}{4}(M-m)[f(B)-f(A)]\\ &\leq \int_0^1 q(t) f((1-t)A+tB)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\ &\leq \frac{1}{4}(M-m)[f(B)-f(A)].\end{aligned} \tag{2.9}$$

We also have the norm inequality

$$\left\|\int_0^1 q(t) f((1-t)A+tB)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\| \leq \frac{1}{4}(M-m)\|f(B)-f(A)\|. \tag{2.10}$$

Proof. Consider $p=q-\frac{1}{2}(m+M)$. Then

$$|p(t)|=\left|q(t)-\frac{1}{2}(m+M)\right|\leq\frac{1}{2}(M-m)$$

which shows that

$$\|p\|_\infty\leq\frac{1}{2}(M-m).$$

Also

$$\begin{aligned}&\int_0^1 f((1-t)A+tB)\,p(t)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\&=\int_0^1 f((1-t)A+tB)\left[q(t)-\frac{1}{2}(m+M)\right]dt\\&-\int_0^1\left[q(t)-\frac{1}{2}(m+M)\right]dt\int_0^1 f((1-t)A+tB)\,dt\\&=\int_0^1 f((1-t)A+tB)\,q(t)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\end{aligned}$$

and by Corollary 1 we get (2.9). □

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\check{f}_{[a,b]}$ or simply $\check{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\check{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{2.11}$$

The asymmetrical transform of f on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],\ t\in[a,b]. \tag{2.12}$$

Theorem 2. *Let* $A,\ B\in\mathcal{SA}_I(H)$ *and* f *an operator convex function on* I. *If* u *is L-Lipschitzian with the constant* $L>0$ *on* $[0,1]$, *then we have the operator inequality*

$$\begin{aligned}&-\frac{1}{2}L\left(\frac{1}{2}\left[f(A)+f(B)\right]-f\left(\frac{A+B}{2}\right)\right)\\&\le\int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t))-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\\&\le\frac{1}{2}L\left(\frac{1}{2}\left[f(A)+f(B)\right]-f\left(\frac{A+B}{2}\right)\right).\end{aligned}\tag{2.13}$$

We also have the norm inequality

$$\begin{aligned}&\left\|\int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t))-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le\frac{1}{2}L\left\|\frac{1}{2}\left[f(A)+f(B)\right]-f\left(\frac{A+B}{2}\right)\right\|.\end{aligned}\tag{2.14}$$

Proof. Let $A,\ B\in\mathcal{SA}_I(H)$. By the operator convexity of f we have

$$f((1-t)A+tB)\le(1-t)f(A)+tf(B)$$

and

$$f(tA+(1-t)B)\le tf(A)+(1-t)f(B)$$

for all $t\in[0,1]$.

If we add these inequalities we get for all $t\in[0,1]$

$$\frac{1}{2}\left[f((1-t)A+tB)+f(tA+(1-t)B)\right]\le\frac{1}{2}\left[f(A)+f(B)\right]. \tag{2.15}$$

From the operator convexity of f we have

$$f\left(\frac{C+D}{2}\right) \leq \frac{1}{2}\left[f(C)+f(D)\right]$$

for $A,\, B \in \mathcal{SA}_I(H)$.

If we take in this inequality $C=(1-t)A+tB$ and $D=tA+(1-t)B$, $t\in[0,1]$, then we get

$$f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left[f((1-t)A+tB)+f(tA+(1-t)B)\right] \tag{2.16}$$

for all $t\in[0,1]$.

From (2.15) and (2.16) we get for all $x\in H$ that

$$\begin{aligned}\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle &\leq \frac{1}{2}\left[\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)\right]\\ &= \breve{\varphi}_{(A,B);x}(t) \leq \left\langle \frac{1}{2}\left[f(A)+f(B)\right]x,x\right\rangle\end{aligned}$$

for $t\in[0,1]$.

If we write the inequality for $\breve{\varphi}_{(A,B);x}$ and u, then we get

$$\begin{aligned}&\left|\int_0^1 \breve{\varphi}_{(A,B);x}(t)\,du(t)-[u(1)-u(0)]\int_0^1 \breve{\varphi}_{(A,B);x}(t)\,dt\right|\\ &\leq \frac{1}{2}L\left[\left\langle \frac{1}{2}\left[f(A)+f(B)\right]x,x\right\rangle-\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\right]\end{aligned} \tag{2.17}$$

for all $x\in H$.

Observe that

$$\int_0^1 \breve{\varphi}_{(A,B);x}(t)\,dt=\int_0^1 \varphi_{(A,B);x}(t)\,dt=\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,dt.$$

Also

$$\begin{aligned}&\int_0^1 \breve{\varphi}_{(A,B);x}(t)\,du(t)\\ &=\frac{1}{2}\int_0^1\left[\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)\right]du(t)\\ &=\frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,du(t)+\int_0^1 \varphi_{(A,B);x}(1-t)\,du(t)\right].\end{aligned}$$

Using the change of variable for Riemann-Stieltjes integral, we have for $s=1-t$ that

$$\begin{aligned}\int_0^1 \varphi_{(A,B);x}(1-t)\,du(t)&=\int_1^0 \varphi_{(A,B);x}(s)\,du(1-s)\\ &=\int_0^1 \varphi_{(A,B);x}(s)\,d[-u(1-s)].\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^1 \breve{\varphi}_{(A,B);x}(t)\,du(t)\\&=\frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,du(t)+\int_0^1 \varphi_{(A,B);x}(t)\,d[-u(1-t)]\right]\\&=\int_0^1 \varphi_{(A,B);x}(t)\,d\left[\frac{u(t)-u(1-t)}{2}\right]=\int_0^1 \varphi_{(A,B);x}(t)\,d(\tilde{u}(t))\end{aligned}$$

for all $x\in H$.

By utilising (2.17) we derive

$$\begin{aligned}&\left|\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,d(\tilde{u}(t))\right.\\&\left.-[u(1)-u(0)]\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,dt\right|\\&\le\frac{1}{2}L\left[\left\langle\frac{1}{2}[f(A)+f(B)]x,x\right\rangle-\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\right],\end{aligned}$$

namely

$$\begin{aligned}&\left|\left\langle\left(\int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t))\right.\right.\right.\\&\left.\left.\left.-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right)x,x\right\rangle\right|\\&\le\frac{1}{2}L\left[\left\langle\left(\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right)x,x\right\rangle\right.\end{aligned}\tag{2.18}$$

for all $x\in H$, which is equivalent to the operator inequality (2.13).

If we take the supremum over $x\in H$, $\|x\|=1$ in (2.18), then we get the norm inequality (2.14). □

Corollary 3. *Let A, $B\in\mathcal{SA}_I(H)$ and f an operator convex function on I. If p is Lebesgue integrable on $[0,1]$ with $\|p\|_\infty:=\operatorname{essup}_{t\in[0,1]}|p(t)|<\infty$ then we have the operator inequality*

$$\begin{aligned}&-\frac{1}{2}\|p\|_\infty\left[\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right]\\&\le\int_0^1 \breve{p}(t)f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\&\le\frac{1}{2}\|p\|_\infty\left[\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right].\end{aligned}\tag{2.19}$$

We also have the norm inequality

$$\begin{aligned}&\left\|\int_0^1 \breve{p}(t)f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le\frac{1}{2}\|p\|_\infty\left\|\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)\right\|.\end{aligned}\tag{2.20}$$

Proof. For $u(t) := \int_0^t p(s)\,ds$, we have

$$\begin{aligned}
&\int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t)\\
&= \frac{1}{2}\int_0^1 \left[\varphi_{(A,B);x}(t) + \varphi_{(A,B);x}(1-t)\right] du(t)\\
&= \frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,du(t) + \int_0^1 \varphi_{(A,B);x}(1-t)\,du(t)\right]\\
&= \frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,p(t)\,dt + \int_0^1 \varphi_{(A,B);x}(1-t)\,p(t)\,dt\right]\\
&= \frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,p(t)\,dt + \int_0^1 \varphi_{(A,B);x}(s)\,p(1-s)\,ds\right]\\
&= \int_0^1 \varphi_{(A,B);x}(t)\left[\frac{p(t)+p(1-t)}{2}\right] dt = \int_0^1 \varphi_{(A,B);x}(t)\,\check{p}(t)\,dt,
\end{aligned}$$

and by Theorem 2 we get the desired result. □

Corollary 4. *Let A, $B \in \mathcal{SA}_I(H)$ and f an operator convex function on I. If q is Lebesgue integrable on $[0,1]$ and there exists the constants m, M satisfying (2.8), then we have the operator inequality*

$$\begin{aligned}
(2.21)\qquad &-\frac{1}{4}(M-m)\left[\frac{1}{2}[f(A)+f(B)] - f\left(\frac{A+B}{2}\right)\right]\\
&\le \int_0^1 \check{q}(t)\,f((1-t)A+tB)\,dt - \int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\
&\le \frac{1}{4}(M-m)\left[\frac{1}{2}[f(A)+f(B)] - f\left(\frac{A+B}{2}\right)\right].
\end{aligned}$$

We also have the norm inequality

$$\begin{aligned}
(2.22)\qquad &\left\|\int_0^1 \check{q}(t)\,f((1-t)A+tB)\,dt - \int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\|\\
&\le \frac{1}{4}(M-m)\left\|\frac{1}{2}[f(A)+f(B)] - f\left(\frac{A+B}{2}\right)\right\|.
\end{aligned}$$

Remark 1. *If, in addition to the assumptions of Corollary 4, we suppose that q is symmetric on $[0,1]$, namely $q(1-t) = q(t)$ for $t \in [0,1]$, then we have the operator inequality*

$$\begin{aligned}
(2.23)\qquad &-\frac{1}{4}(M-m)\left[\frac{1}{2}[f(A)+f(B)] - f\left(\frac{A+B}{2}\right)\right]\\
&\le \int_0^1 q(t)\,f((1-t)A+tB)\,dt - \int_0^1 q(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\\
&\le \frac{1}{4}(M-m)\left[\frac{1}{2}[f(A)+f(B)] - f\left(\frac{A+B}{2}\right)\right]
\end{aligned}$$

for all A, $B \in \mathcal{SA}_I(H)$.

We also have the norm inequality

$$\left\| \int_0^1 q(t) f((1-t)A + tB)\, dt - \int_0^1 q(t)\, dt \int_0^1 f((1-t)A + tB)\, dt \right\| \leq \frac{1}{4}(M-m) \left\| \frac{1}{2}[f(A) + f(B)] - f\left(\frac{A+B}{2}\right) \right\| \tag{2.24}$$

for all $A,\ B \in \mathcal{SA}_I(H)$.

3. Some Examples

The logarithmic function $f(t) = \ln t$ is operator monotone on $(0,\infty)$. If u is L-Lipschitzian with the constant $L > 0$ on $[0,1]$, then by Theorem 1 we have the operator inequality

$$\begin{aligned} & -\frac{1}{2}L(\ln B - \ln A) \\ & \leq \int_0^1 \ln((1-t)A + tB)\, du(t) - [u(1) - u(0)] \int_0^1 \ln((1-t)A + tB)\, dt \\ & \leq \frac{1}{2}L(\ln B - \ln A) \end{aligned} \tag{3.1}$$

for $B \geq A > 0$.

We also have the norm inequality

$$\left\| \int_0^1 \ln((1-t)A + tB)\, du(t) - [u(1) - u(0)] \int_0^1 \ln((1-t)A + tB)\, dt \right\| \leq \frac{1}{2}L\|\ln B - \ln A\|. \tag{3.2}$$

If q is Lebesgue integrable on $[0,1]$ and there exists the constants $m,\ M$ with $m \leq q \leq M$ a.e. on $[0,1]$, then by Corollary 2 we have the operator inequality

$$\begin{aligned} & -\frac{1}{4}(M-m)(\ln B - \ln A) \\ & \leq \int_0^1 q(t) \ln((1-t)A + tB)\, dt - \int_0^1 q(t)\, dt \int_0^1 \ln((1-t)A + tB)\, dt \\ & \leq \frac{1}{4}(M-m)(\ln B - \ln A), \end{aligned} \tag{3.3}$$

for $B \geq A > 0$.

We also have the norm inequality

$$\left\| \int_0^1 q(t) \ln((1-t)A + tB)\, dt - \int_0^1 q(t)\, dt \int_0^1 \ln((1-t)A + tB)\, dt \right\| \leq \frac{1}{4}(M-m)\|\ln B - \ln A\|. \tag{3.4}$$

The logarithmic function $f(t) = \ln t$ is operator concave on $(0,\infty)$. If u is L-Lipschitzian with the constant $L > 0$ on $[0,1]$, then by Theorem 2 we have the operator inequality

$$
\begin{aligned}
(3.5)\quad & -\frac{1}{2}L\left(\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right)\\
&\leq [u(1)-u(0)]\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \ln((1-t)A+tB)\,d(\tilde{u}(t))\\
&\leq \frac{1}{2}L\left(\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right)
\end{aligned}
$$

for all $A,\ B>0$.

We also have the norm inequality

$$
\begin{aligned}
(3.6)\quad & \left\|[u(1)-u(0)]\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \ln((1-t)A+tB)\,d(\tilde{u}(t))\right\|\\
&\leq \frac{1}{2}L\left\|\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right\|,
\end{aligned}
$$

for all $A,\ B>0$.

If q is Lebesgue integrable on $[0,1]$ and there exists the constants $m,\ M$ satisfying condition (2.8), then by Corollary 4 we have the operator inequality

$$
\begin{aligned}
(3.7)\quad & -\frac{1}{4}(M-m)\left[\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right]\\
&\leq \int_0^1 q(t)\,dt\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \check{q}(t)\ln((1-t)A+tB)\,dt\\
&\leq \frac{1}{4}(M-m)\left[\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right]
\end{aligned}
$$

for all $A,\ B>0$.

We also have the norm inequality

$$
\begin{aligned}
(3.8)\quad & \left\|\int_0^1 q(t)\,dt\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \check{q}(t)\ln((1-t)A+tB)\,dt\right\|\\
&\leq \frac{1}{4}(M-m)\left\|\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right\|.
\end{aligned}
$$

In addition, if q *is symmetric*, then from (3.7) and (3.8) we get

$$
\begin{aligned}
(3.9)\quad & -\frac{1}{4}(M-m)\left[\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right]\\
&\leq \int_0^1 q(t)\,dt\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 q(t)\ln((1-t)A+tB)\,dt\\
&\leq \frac{1}{4}(M-m)\left[\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.10)\quad & \left\|\int_0^1 q(t)\,dt\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 q(t)\ln((1-t)A+tB)\,dt\right\|\\
&\leq \frac{1}{4}(M-m)\left\|\ln\left(\frac{A+B}{2}\right)-\frac{1}{2}(\ln A+\ln B)\right\|
\end{aligned}
$$

for all $A,\ B>0$.

The interested reader may obtain other similar inequalities by choosing the operator monotonic functions $f(t)=t^r$ for $r\in(0,1)$ and operator convex functions $f(t)=t^p$ if either $1\le p\le 2$ or $-1\le p\le 0$.

Indeed, if u is L-Lipschitzian with the constant $L>0$ on $[0,1]$, then by Theorem 1 we have the power operator inequality for $r\in(0,1)$,

$$\begin{aligned}(3.11)\qquad & -\frac{1}{2}L\left(B^r-A^r\right)\\ &\le \int_0^1 \left((1-t)A+tB\right)^r du(t)-\left[u(1)-u(0)\right]\int_0^1 \left((1-t)A+tB\right)^r dt\\ &\le \frac{1}{2}L\left(B^r-A^r\right),\end{aligned}$$

where $B\ge A>0$.

We also have the norm inequality

$$\begin{aligned}(3.12)\qquad & \left\|\int_0^1 \left((1-t)A+tB\right)^r du(t)-\left[u(1)-u(0)\right]\int_0^1 \left((1-t)A+tB\right)^r dt\right\|\\ &\le \frac{1}{2}L\left\|B^r-A^r\right\|,\end{aligned}$$

where $B\ge A>0$.

Since $f(t)=t^r$ for $r\in(0,1)$ is also operator concave, then by Theorem 2 we have the operator inequality

$$\begin{aligned}(3.13)\qquad & -\frac{1}{2}L\left(\left(\frac{A+B}{2}\right)^r-\frac{1}{2}\left(A^r+B^r\right)\right)\\ &\le \left[u(1)-u(0)\right]\int_0^1 \left((1-t)A+tB\right)^r dt-\int_0^1 \left((1-t)A+tB\right)^r d\left(\tilde{u}(t)\right)\\ &\le \frac{1}{2}L\left(\left(\frac{A+B}{2}\right)^r-\frac{1}{2}\left(A^r+B^r\right)\right),\end{aligned}$$

where $B\ge A>0$.

We also have the norm inequality

$$\begin{aligned}(3.14)\qquad & \left\|\int_0^1 \left((1-t)A+tB\right)^r d\left(\tilde{u}(t)\right)-\left[u(1)-u(0)\right]\int_0^1 f\left((1-t)A+tB\right)^r dt\right\|\\ &\le \frac{1}{2}L\left\|\left(\frac{A+B}{2}\right)^r-\frac{1}{2}\left(A^r+B^r\right)\right\|,\end{aligned}$$

where $B\ge A>0$.

Similar weighted inequalities may be stated, however we do not present them here.

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