Abstract. Let $f$ be an operator monotonic function on $I$ and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in $I$. If $A, B \in \mathcal{SA}_I(H)$ with $A \leq B$ and $u$ is $(l, L)$-Lipschitzian on $[0, 1]$, then we have the operator inequalities

$$-2(L-l)\int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) \, dt$$

$$\leq \int_0^1 f ((1-t)A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 f ((1-t)A + tB) \, dt$$

$$\leq 2(L-l)\int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) \, dt.$$

We also have the norm inequalities

$$\left\| \int_0^1 f ((1-t)A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 f ((1-t)A + tB) \, dt \right\|$$

$$\leq 2(L-l)\left\| \int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) \, dt \right\|$$

$$\leq \frac{1}{2}(L-l)\max\{\|f(A)\|, \|f(B)\|\}.$$ 

Similar inequalities for operator convex functions $f$ are also provided. Some particular examples of interest are given as well.

1. Introduction

In order to generalize the Grüss inequality to the Riemann-Stieltjes integral, S.S. Dragomir and I. Fedotov introduced in [7] the following functional

$$D(f; u) := \int_a^b f(t) \, du(t) - [u(a) - u(b)] \frac{1}{b-a} \int_a^b f(t) \, dt$$

provided the Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ and the Riemann integral $\int_a^b f(t) \, dt$ exist. In the same paper, the authors have shown that

$$|D(f; u)| \leq \frac{1}{2}L(M-m)(b-a),$$

provided that $u$ is $L$-Lipschitzian, i.e., $|u(t) - u(s)| \leq L|t-s|$ for any $t, s \in [a, b]$ and $f$ is Riemann integrable and bounded below by $m$ and above by $M$. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

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In the follow-up paper [8], the same authors established a different result, namely

\begin{equation}
|D (f; u)| \leq \frac{1}{2} K (b - a) \sqrt{b} (u),
\end{equation}

provided that $u$ is of bounded variation and $f$ is $K$-Lipschitzian with a constant $K > 0$. Here $\frac{1}{2}$ is also best possible.

We have the following elementary fact, see for instance [2]:

**Lemma 1.** Let $u : [a, b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:

(i) The function $u - \frac{1 + l}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$ is $\frac{1}{2} (L - l)$-Lipschitzian;

(ii) We have the inequalities

\begin{equation}
l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each} \quad t, s \in [a, b] \quad \text{with} \quad t \neq s;
\end{equation}

(iii) We have the inequalities

\begin{equation}
l (t - s) \leq u(t) - u(s) \leq L (t - s) \quad \text{for each} \quad t, s \in [a, b] \quad \text{with} \quad t > s.
\end{equation}

Following [14] we can introduce the definition of $(l, L)$-Lipschitzian functions:

**Definition 1.** The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i)–(iii) from Lemma 1 is said to be $(l, L)$-Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$-Lipschitzian means $L$-Lipschitzian in the classical sense.

Utilising Lagrange’s mean value theorem, we can state the following result that provides examples of $(l, L)$-Lipschitzian functions.

**Proposition 1.** Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $-\infty < l = \inf_{t \in [a, b]} u'(t)$ and $\sup_{t \in [a, b]} u'(t) = L < \infty$, then $u$ is $(l, L)$-Lipschitzian on $[a, b]$.

In [2] we obtained among others the following result for Riemann-Stieltjes integral:

**Theorem 1.** Let $f, u : [a, b] \to \mathbb{R}$ be such that $u$ is $(l, L)$-Lipschitzian and $f$ is monotonic nondecreasing, then

\begin{equation}
|D (f; u)| \leq 2 \cdot \frac{L - l}{b - a} \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt
\end{equation}

\begin{equation*}
\leq \begin{cases}
\frac{1}{2} (L - l) \max \{|f(a)|, |f(b)|\} (b - a) ; \\
\frac{1}{(a + l)^{\frac{1}{q}}} (L - l) \|f\|_p (b - a)^{\frac{1}{q}} & \text{if} \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\
(L - l) \|f\|_1 ,
\end{cases}
\end{equation*}

where $\|f\|_p := \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$, $p \geq 1$ are the Lebesgue norms.

The constants 2 and $\frac{1}{2}$ are best possible in (1.6).
A real valued continuous function $f$ on an interval $I$ is said to be \textit{operator convex} (operator concave) on $I$ if

\begin{equation}
(1.7) \quad f ((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f (A) + \lambda f (B)
\end{equation}

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

A real valued continuous function $f$ on an interval $I$ is said to be \textit{operator monotone} if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\text{Sp}(A), \text{Sp}(B) \subset I$ imply $f (A) \leq f (B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Assume that $f$ is operator convex on $I$. If $p : [0, 1] \rightarrow [0, \infty)$ is Lebesgue integrable and symmetric in the sense that $p(1 - s) = p(s)$ for all $s \in [0, 1]$, then for $A, B$ selfadjoint operators with spectra included in $I$, we have the operator Fejér’s inequality

\begin{equation}
(1.8) \quad \left( \int_0^1 p(s) \, ds \right) f \left( \frac{A + B}{2} \right) \leq \int_0^1 p(s) f (sA + (1 - s) B) \, ds \leq \left( \int_0^1 p(s) \, ds \right) \frac{f (A) + f (B)}{2},
\end{equation}

see [5] where further reverses were obtained.

In the recent paper [6] we obtained the following result:

Let $A, B \in \mathcal{SA}_I (H)$ with $A \leq B$ and $f$ an operator monotonic function on $I$. If $p : [0, 1] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then

\begin{equation}
(1.9) \quad 0 \leq \int_0^1 p(t) f ((1 - t) A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 f ((1 - t) A + tB) \, dt
\leq \frac{1}{2} \left( \int_0^1 \left| p(t) \right| dt - \int_0^1 p(s) \, ds \right) \left[ f (B) - f (A) \right]
\leq \frac{1}{2} \left( \int_0^1 p^2 (t) \, dt - \left( \int_0^1 p(s) \, ds \right)^2 \right)^{1/2} \left[ f (B) - f (A) \right]
\leq \frac{1}{4} \left[ p(1) - p(0) \right] \left[ f (B) - f (A) \right].
\end{equation}

For some related results, see [4]-[6] and [10]-[19].

Motivated by the above results, we establish in this paper some lower and upper bounds in operator order for the difference

\[
\int_0^1 f ((1 - t) A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 f ((1 - t) A + tB) \, dt,
\]
where $u$ is $(l, L)$-Lipschitzian on $[0, 1]$ and $f$ is either operator monotone or operator convex on $I$. Some examples for the logarithmic function which is both operator monotonic and operator concave on $(0, \infty)$ are given.

2. Main Results

Let $f$ be a continuous function on $I$. If $(A, B) \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$ and $t \in [0, 1]$, then the convex combination $(1 - t) A + tB$ is a selfadjoint operator with the spectrum in $I$ showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $B(H)$ of all bounded linear operators on $H$. By the continuous functional calculus of selfadjoint operator we also conclude that $f ((1 - t) A + tB)$ is a selfadjoint operator in $B(H)$.

For $A, B \in \mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)} : [0, 1] \to B(H)$ defined by

\begin{equation}
\varphi_{(A,B)}(t) := f ((1 - t) A + tB).
\end{equation}

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B),x} : [0, 1] \to \mathbb{R}$ defined by

\begin{equation}
\varphi_{(A,B),x}(t) := \langle \varphi_{(A,B)}(t) x, x \rangle = \langle f ((1 - t) A + tB) x, x \rangle.
\end{equation}

We have:

**Theorem 2.** Let $A, B \in \mathcal{SA}_I(H)$ with $A \leq B$ and $f$ an operator monotonic function on $I$. If $u$ is $(l, L)$-Lipschitzian on $[0, 1]$, then we have the operator inequalities

\begin{equation}
-2(L-l) \int_0^1 \left(t - \frac{1}{2}\right) f ((1 - t) A + tB) \, dt
\leq \int_0^1 f ((1 - t) A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 f ((1 - t) A + tB) \, dt
\leq 2(L-l) \int_0^1 \left(t - \frac{1}{2}\right) f ((1 - t) A + tB) \, dt.
\end{equation}

We also have the norm inequality

\begin{equation}
\left\| \int_0^1 f ((1 - t) A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 f ((1 - t) A + tB) \, dt \right\|
\leq 2(L-l) \left\| \int_0^1 \left(t - \frac{1}{2}\right) f ((1 - t) A + tB) \, dt \right\|
\leq \frac{1}{2} (L-l) \max \{ \|f(A)\|, \|f(B)\| \}.
\end{equation}

**Proof.** Let $0 \leq t_1 < t_2 \leq 1$ and $A \leq B$. Then

\[(1 - t_2) A + t_2B - (1 - t_1) A - t_1B = (t_2 - t_1) (B - A) \geq 0\]

and by operator monotonicity of $f$ we get

\[f ((1 - t_2) A + t_2B) \geq f ((1 - t_1) A + t_1B),\]

which is equivalent to

\[\varphi_{(A,B),x}(t_2) = \langle f ((1 - t_2) A + t_2B) x, x \rangle \geq \langle f ((1 - t_1) A + t_1B) x, x \rangle = \varphi_{(A,B),x}(t_1)\]
that shows that the scalar function \( \varphi_{(A,B);x} : [0, 1] \to \mathbb{R} \) is monotonic nondecreasing for \( A \leq B \) and for any \( x \in H \).

If we use the inequality (1.6) for \( \varphi_{(A,B);x} \) and \( u \), then

\[
\left| \int_0^1 \varphi_{(A,B);x}(t) \, du(t) - [u(1) - u(0)] \int_0^1 \varphi_{(A,B);x}(t) \, dt \right|
\leq 2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) \varphi_{(A,B);x}(t) \, dt
\]

\[
\leq \frac{1}{2} (L - l) \max \{ |\varphi_{(A,B);x}(0)|, |\varphi_{(A,B);x}(1)| \}
\]

for \( A \leq B \) and for any \( x \in H \), namely

\[
(2.5) \quad \left| \int_0^1 (f((1 - t) A + tB) x, x) \, du(t)ight|
- [u(1) - u(0)] \int_0^1 (f((1 - t) A + tB) x, x) \, dt\]
\leq 2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) (f((1 - t) A + tB) x, x) \, dt
\]

\[
\leq \frac{1}{2} (L - l) \max \{|f(A), x||, |f(B), x||\}
\]

for \( A \leq B \) and for any \( x \in H \).

The first inequality in (2.5) is equivalent to

\[
\left\langle -2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) f((1 - t) A + tB) dt \right), x, x \right\rangle
\leq \left\langle \left( \int_0^1 f((1 - t) A + tB) du(t) \right), x, x \right\rangle
- \left\langle \left[ u(1) - u(0) \right] \int_0^1 f((1 - t) A + tB) dt \right), x, x \right\rangle
\]

\[
\leq \left\langle \left( 2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) f((1 - t) A + tB) dt \right), x, x \right\rangle
\]

for any \( x \in H \), which is equivalent to the operator inequality (2.3).

The inequality (2.5) can be written as

\[
\left| \left\langle \left( \int_0^1 f((1 - t) A + tB) du(t) - [u(1) - u(0)] \int_0^1 f((1 - t) A + tB) dt \right), x, x \right\rangle \right|
\leq \left\langle \left( 2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) f((1 - t) A + tB) dt \right), x, x \right\rangle
\]

\[
\leq \frac{1}{2} (L - l) \max \{|f(A), x||, |f(B), x||\}
\]

for any \( x \in H \).

By taking the supremum over \( x \in H, \|x\| = 1 \), we derive the norm inequality (2.4).

\[ \square \]

We have the following weighted inequalities as well:
Corollary 1. Let $A, B \in \mathcal{S}A_1(H)$ with $A \leq B$ and $f$ an operator monotonic function on $I$. If $p$ is continuous on $[0,1]$ with $m := \min_{t \in [0,1]} p(t)$ and $M := \max_{t \in [0,1]} p(t)$, then we have the operator inequality

\begin{equation}
-2(M-m) \int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) dt \\
\leq \int_0^1 p(t) f ((1-t)A + tB) dt - \int_0^1 p(t) dt \int_0^1 f ((1-t)A + tB) dt \\
\leq 2(M-m) \int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) dt.
\end{equation}

We also have the norm inequality

\begin{equation}
\left\| \int_0^1 p(t) f ((1-t)A + tB) dt - \int_0^1 p(t) dt \int_0^1 f ((1-t)A + tB) dt \right\| \\
\leq 2(M-m) \left\| \int_0^1 \left( t - \frac{1}{2} \right) f ((1-t)A + tB) dt \right\| \\
\leq \frac{1}{2} (M-m) \max \{ \| f(A) \|, \| f(B) \| \}.
\end{equation}

Proof. If we take $u(t) := \int_0^t p(s) ds$ and use the properties of Riemann-Stieltjes integral, then we have

\[ \int_0^1 f ((1-t)A + tB) du(t) = \int_0^1 f ((1-t)A + tB) p(t) dt. \]

Also

\[ u'(t) = p(t), \quad t \in (0,1), \]

which shows that $u(t)$ is $(m,M)$-Lipschitzian, and by Theorem 2 we derive the desired inequalities (2.6) and (2.7). \qed

For a function $f : [a,b] \to \mathbb{C}$ we consider the symmetrical transform of $f$ on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

\begin{equation}
\tilde{f}(t) := \frac{1}{2} [f(t) + f(a + b - t)], \quad t \in [a,b].
\end{equation}

The asymmetrical transform of $f$ on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

\begin{equation}
\tilde{f}(t) := \frac{1}{2} [f(t) - f(a + b - t)], \quad t \in [a,b].
\end{equation}

Theorem 3. Let $A, B \in \mathcal{S}A_1(H)$ and $f$ be an operator convex function on $I$. If $\tilde{u}$ is $(\tilde{I}, \tilde{L})$-Lipschitzian on $[0,1]$, then we have the operator inequality

\begin{equation}
-4(\tilde{L} - \tilde{l}) \int_{1/2}^1 \left( t - \frac{1}{4} \right) [f ((1-t)A + tB) + f (tA + (1-t)B)] dt \\
\leq \int_0^1 f ((1-t)A + tB) d(\tilde{u}(t)) - [u(1) - u(0)] \int_0^1 f ((1-t)A + tB) dt \\
\leq 4(\tilde{L} - \tilde{l}) \int_{1/2}^1 \left( t - \frac{1}{4} \right) [f ((1-t)A + tB) + f (tA + (1-t)B)] dt.
\end{equation}
We also have the norm inequality

\[(2.11) \quad \left\| \int_0^1 f ((1 - t) A + tB) d (\tilde{u} (t)) - [u (1) - u (0)] \right\| \leq 4 \left( L - I \right) \left\| \int_{1/2}^1 \left( t - \frac{1}{4} \right) [f ((1 - t) A + tB) + f (tA + (1 - t) B)] dt \right\| \leq \frac{1}{2} \left( L - I \right) \max \left\{ \left\| f \left( \frac{A + B}{2} \right) \right\|, \left\| f (A) + f (B) \right\| \right\} .\]

Proof. Let \( A, B \in \mathcal{S}_A (H) \). By the operator convexity of \( f \) we have that \( \varphi_{(A,B);x} \) is convex on \([0, 1]\) for all \( x \in H \). This implies that the symmetrized transform \( \hat{\varphi}_{(A,B);x} \) is convex and symmetric on \([0, 1]\) and therefore monotonic nonincreasing on \([0, 1/2]\) and nondecreasing on \([1/2, 1]\).

Observe that

\[
\begin{align*}
\int_0^1 \hat{\varphi}_{(A,B);x} (t) \, du (t) &= \int_0^{1/2} \hat{\varphi}_{(A,B);x} (t) \, du (t) + \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, du (t).
\end{align*}
\]

Using the change of variable for Riemann-Stieltjes integral, we have for \( s = 1 - t \) that

\[
\begin{align*}
\int_0^{1/2} \hat{\varphi}_{(A,B);x} (t) \, du (t) &= \int_1^{1/2} \hat{\varphi}_{(A,B);x} (1 - s) \, d (u (1 - s)) \\
&= \int_1^{1/2} \hat{\varphi}_{(A,B);x} (s) \, d (u (1 - s)) \\
&= \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, d [-u (1 - s)],
\end{align*}
\]

therefore

\[
\begin{align*}
\int_0^1 \hat{\varphi}_{(A,B);x} (t) \, du (t) &= \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, d [-u (1 - s)] + \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, du (t) \\
&= \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, d [u (t) - u (1 - s)] \\
&= 2 \int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, d (\tilde{u} (t))
\end{align*}
\]

for all \( x \in H \).

Observe that

\[
\begin{align*}
\tilde{u} (1) - \tilde{u} (1/2) &= \frac{u (1) - u (0)}{2} - \frac{u (1/2) - u (0)}{2} \\
&= \frac{1}{2} [u (1) - u (0)]
\end{align*}
\]

and

\[
\int_{1/2}^1 \hat{\varphi}_{(A,B);x} (t) \, dt = \frac{1}{2} \int_0^1 \hat{\varphi}_{(A,B);x} (t) \, dt = \frac{1}{2} \int_0^1 \hat{\varphi}_{(A,B);x} (t) \, dt
\]

for all \( x \in H \).
Therefore

\[ (2.12) \quad \int_{0}^{1} \varphi_{(A,B):x}(t) \, du(t) - [u(1) - u(0)] \int_{0}^{1} \varphi_{(A,B):x}(t) \, dt \]

\[ = 2 \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, d(\bar{u}(t)) \]

\[ - 2 [\bar{u}(1) - \bar{u}(1/2)] 2 \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, dt \]

\[ = 2 \left[ \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, d(\bar{u}(t)) - [\bar{u}(1) - \bar{u}(1/2)] 2 \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, dt \right] \]

\[ = 2 \left[ \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, d(\bar{u}(t)) - [\bar{u}(1) - \bar{u}(1/2)] \frac{1}{1/2} \int_{1/2}^{1} \varphi_{(A,B):x}(t) \, dt \right] \]

\[ = 2 D_{[1/2,1]} \left( \varphi_{(A,B):x}, \bar{u} \right), \]

where \( D_{[1/2,1]}(\cdot, \cdot) \) is defined by (1.1) on the interval \([1/2,1]\).

By making use of the inequality (1.6) we then get

\[ (2.13) \quad \left| D_{[1/2,1]} \left( \varphi_{(A,B):x}, \bar{u} \right) \right| \]

\[ \leq 2 \left( \frac{\tilde{L} - \tilde{l}}{1/2} \right) \int_{1/2}^{1} \left( t - \frac{1}{4} \right) \varphi_{(A,B):x} \, dt \]

\[ \leq \frac{1}{2} \left( \frac{\tilde{L} - \tilde{l}}{1/2} \right) \max \left\{ \left| \varphi_{(A,B):x}(1/2) \right|, \left| \varphi_{(A,B):x}(1) \right| \right\} \]

namely

\[ (2.14) \quad \left| D_{[1/2,1]} \left( \varphi_{(A,B):x}, \bar{u} \right) \right| \]

\[ \leq 2 \left( \frac{\tilde{L} - \tilde{l}}{1/2} \right) \int_{1/2}^{1} \left( t - \frac{1}{4} \right) \left( \| f((1-t)A + tB) + f(tA + (1-t)B) \| x, x \right) \, dt \]

\[ \leq \frac{1}{4} \left( \frac{\tilde{L} - \tilde{l}}{1/2} \right) \max \left\{ \left\langle f \left( \frac{A + B}{2} \right) x, x \right\rangle, \left\langle f(A) + f(B) \frac{1}{2}, x, x \right\rangle \right\} \]

for all \( x \in H \).

Also

\[ \int_{0}^{1} \varphi_{(A,B):x}(t) \, du(t) \]

\[ = \frac{1}{2} \int_{0}^{1} \left[ \varphi_{(A,B):x}(t) + \varphi_{(A,B):x}(1-t) \right] \, du(t) \]

\[ = \frac{1}{2} \left[ \int_{0}^{1} \varphi_{(A,B):x}(t) \, du(t) + \int_{0}^{1} \varphi_{(A,B):x}(1-t) \, du(t) \right]. \]
Using the change of variable for Riemann-Stieltjes integral, we have for $s = 1 - t$
that
\[
\int_0^1 \varphi_{(A,B);x} (1 - t) \, du (t) = \int_0^1 \varphi_{(A,B);x} (s) \, du (1 - s)
= \int_0^1 \varphi_{(A,B);x} (s) \, d [u (1 - s)].
\]
Therefore
\[
(2.15) \quad \int_0^1 \varphi_{(A,B);x} (t) \, du (t)
= \frac{1}{2} \left[ \int_0^1 \varphi_{(A,B);x} (t) \, du (t) + \int_0^1 \varphi_{(A,B);x} (t) \, d [u (1 - t)] \right]
= \int_0^1 \varphi_{(A,B);x} (t) \, d \left[ \frac{u (t) - u (1 - t)}{2} \right] = \int_0^1 \varphi_{(A,B);x} (t) \, d (\bar{u} (t))
\]
for all $x \in H$.

Using (2.12), (2.13) and (2.15) we get
\[
(2.16) \quad \left| \int_0^1 \varphi_{(A,B);x} (t) \, d (\bar{u} (t)) - [u (1) - u (0)] \int_0^1 \varphi_{(A,B);x} (t) \, dt \right|
\leq 4 \left( \bar{L} - \bar{l} \right) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left| \left[ f ((1 - t) A + t B) + f (t A + (1 - t) B) \right] x, x \right| \, dt
\leq \frac{1}{2} \left( \bar{L} - \bar{l} \right) \max \left\{ \left| \left( \frac{A + B}{2} \right) x, x \right|, \left| \frac{f (A) + f (B)}{2} x, x \right| \right\}
\]
for all $x \in H$.

The first inequality in (2.16) is equivalent to (2.10) and the whole (2.16) implies, by taking the supremum over $x \in H$, $\|x\| = 1$, the norm inequality (2.11).

**Corollary 2.** Let $A, B \in \mathcal{S_A} (H)$ and $f$ be an operator convex function on $I$. If $p$ is continuous on $[0,1]$ with $m := \min_{t \in [0,1]} p (t)$ and $M := \max_{t \in [0,1]} p (t)$, then we have the operator inequality
\[
(2.17) \quad - 4 (M - m) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ f ((1 - t) A + t B) + f (t A + (1 - t) B) \right] \, dt
\leq \int_0^1 \tilde{p} (t) f ((1 - t) A + t B) \, dt - \int_0^1 p (s) \, ds \int_0^1 f ((1 - t) A + t B) \, dt
\leq 4 (M - m) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ f ((1 - t) A + t B) + f (t A + (1 - t) B) \right] \, dt.
\]

We also have the norm inequality
\[
(2.18) \quad \left\| \int_0^1 \tilde{p} (t) f ((1 - t) A + t B) \, dt - \int_0^1 p (s) \, ds \int_0^1 f ((1 - t) A + t B) \, dt \right\|
\leq 4 (M - m) \left\| \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ f ((1 - t) A + t B) + f (t A + (1 - t) B) \right] \, dt \right\|
\leq \frac{1}{2} (M - m) \max \left\{ \left\| f \left( \frac{A + B}{2} \right) \right\|, \left\| \frac{f (A) + f (B)}{2} \right\| \right\}.
\]
Proof. For $u(t) := \int_0^t p(s) \, ds$, we have
\[
\tilde{u}(t) = \frac{1}{2} [u(t) - u(1 - t)] = \frac{1}{2} \left[ \int_0^t p(s) \, ds - \int_0^{1-t} p(s) \, ds \right].
\]
Then $\tilde{u}(t)$ is differentiable and
\[
(\tilde{u}(t))' = \frac{1}{2} [p(t) + p(1 - t)] \in [m, M], \quad t \in [0, 1],
\]
which shows that $\tilde{u}(t)$ is $(m, M)$-Lipschitzian on $[0, 1]$.

Since, by the properties of Riemann-Stieltjes integral
\[
\int_0^1 f((1-t)A + tB) \, d(\tilde{u}(t)) = \int_0^1 \tilde{p}(t) f((1-t)A + tB) \, dt
\]
hence by Theorem 3 we get the desired results. \hfill \square

3. SOME EXAMPLES

The logarithmic function $f(t) = \ln t$ is operator monotone on $(0, \infty)$. If $u$ is $(l, L)$-Lipschitzian on $[0, 1]$, then by Theorem 2 we have the operator inequalities

\begin{equation}
-2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t)A + tB) \, dt \\
\leq \int_0^1 \ln ((1-t)A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 \ln ((1-t)A + tB) \, dt \\
\leq 2 (L - l) \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t)A + tB) \, dt
\end{equation}

for $B \preceq A > 0$.

We also have the norm inequalities

\begin{equation}
\left\| \int_0^1 \ln ((1-t)A + tB) \, du(t) - [u(1) - u(0)] \int_0^1 \ln ((1-t)A + tB) \, dt \right\| \\
\leq 2 (L - l) \left\| \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t)A + tB) \, dt \right\| \\
\leq \frac{1}{2} (L - l) \max \{ \|\ln A\|, \|\ln B\| \}
\end{equation}

for $B \preceq A > 0$.

If $p$ is continuous on $[0, 1]$ with $m := \min_{t \in [0, 1]} p(t)$ and $M := \max_{t \in [0, 1]} p(t)$, then by Corollary 1 we have the operator inequality

\begin{equation}
-2 (M - m) \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t)A + tB) \, dt \\
\leq \int_0^1 p(t) \ln ((1-t)A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln ((1-t)A + tB) \, dt \\
\leq 2 (M - m) \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t)A + tB) \, dt
\end{equation}

for $B \preceq A > 0$. 
We also have the norm inequality

\[(3.4) \quad \left\| \int_0^1 p(t) \ln ((1-t) A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln ((1-t) A + tB) \, dt \right\|
\leq 2 (M - m) \left\| \int_0^1 \left( t - \frac{1}{2} \right) \ln ((1-t) A + tB) \, dt \right\|
\leq \frac{1}{2} (M - m) \max \left\{ \| \ln A \|, \| \ln B \| \right\}
\]

for \( B \geq A > 0 \).

If \( \tilde{u} \) is \((\tilde{l}, \tilde{L})\)-Lipschitzian on \([0, 1] \), then by Theorem 3 we have the operator inequality

\[(3.5) \quad 4 \left( \tilde{L} - \tilde{l} \right) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \, dt
\leq \left| u (1) - u (0) \right| \int_0^1 \ln ((1-t) A + tB) \, dt - \int_0^1 \ln ((1-t) A + tB) d (\tilde{u} (t))
\leq 4 \left( \tilde{L} - \tilde{l} \right) \int_{1/2}^1 \left( \frac{1}{4} - t \right) \left| \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \right| \, dt.
\]

for \( B \geq A > 0 \).

We also have the norm inequality

\[(3.6) \quad \left\| \left[ u (1) - u (0) \right] \int_0^1 \ln ((1-t) A + tB) \, dt - \int_0^1 \ln ((1-t) A + tB) d (\tilde{u} (t)) \right\|
\leq 4 \left( \tilde{L} - \tilde{l} \right) \left\| \int_{1/2}^1 \left( t - \frac{1}{4} \right) \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \, dt \right\|
\leq \frac{1}{2} \left( \tilde{L} - \tilde{l} \right) \max \left\{ \left\| \ln \left( \frac{A + B}{2} \right) \right\|, \left\| \ln A + \ln B \right\| \right\}.
\]

If \( p \) is continuous on \([0, 1] \) with \( m := \min_{t \in [0, 1]} p (t) \) and \( M := \max_{t \in [0, 1]} p (t) \), then by Corollary 2 we have the operator inequality

\[(3.7) \quad 4 (M - m) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \, dt
\leq \int_0^1 p (s) \, ds \int_0^1 \ln ((1-t) A + tB) \, dt - \int_0^1 \tilde{p} (t) \ln ((1-t) A + tB) \, dt
\leq 4 (M - m) \int_{1/2}^1 \left( \frac{1}{4} - t \right) \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \, dt.
\]

for \( B \geq A > 0 \).

We also have the norm inequality

\[(3.8) \quad \left\| \int_0^1 \tilde{p} (t) \ln ((1-t) A + tB) \, dt - \int_0^1 p (s) \, ds \int_0^1 \ln ((1-t) A + tB) \, dt \right\|
\leq 4 (M - m) \left\| \int_{1/2}^1 \left( t - \frac{1}{4} \right) \ln ((1-t) A + tB) + \ln (tA + (1-t) B) \, dt \right\|
\leq \frac{1}{2} (M - m) \max \left\{ \left\| \ln \left( \frac{A + B}{2} \right) \right\|, \left\| \ln A + \ln B \right\| \right\}.
\]
for \( B \geq A > 0 \).

The interested reader may obtain other similar inequalities by choosing the operator monotonic functions \( f(t) = t^r \) for \( r \in (0, 1) \) and operator convex functions \( f(t) = t^r \) if either \( 1 \leq p \leq 2 \) or \(-1 \leq p \leq 0\).

Indeed, since \( f(t) = t^r \) is operator monotonic for \( r \in (0, 1) \), then by (2.3),

\[
(3.9) \quad -2(L - l) \int_0^1 \left( t - \frac{1}{2} \right) ((1 - t) A + tB)^r dt \\
\leq \int_0^1 ((1 - t) A + tB)^r du(t) - [u(1) - u(0)] \int_0^1 ((1 - t) A + tB)^r dt \\
\leq 2(L - l) \int_0^1 \left( t - \frac{1}{2} \right) ((1 - t) A + tB)^r dt.
\]

where \( u \) is \((l, L)\)-Lipschitzian on \([0, 1]\) and \( B \geq A \geq 0 \).

We also have the norm inequality

\[
(3.10) \quad \left\| \int_0^1 ((1 - t) A + tB)^r du(t) - [u(1) - u(0)] \int_0^1 ((1 - t) A + tB)^r dt \right\| \\
\leq 2(L - l) \left\| \int_0^1 \left( t - \frac{1}{2} \right) ((1 - t) A + tB)^r dt \right\| \\
\leq \frac{1}{2} (L - l) \| B^r \|
\]

for \( B \geq A \geq 0 \).

If \( p \) is continuous on \([0, 1]\) with \( m := \min_{t \in [0, 1]} p(t) \) and \( M := \max_{t \in [0, 1]} p(t) \) and since \( f(t) = t^r \) is operator concave for \( r \in (0, 1) \), then we have the operator inequality

\[
(3.11) \quad 4(M - m) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ ((1 - t) A + tB)^r + (tA + (1 - t) B)^r \right] dt \\
\leq \int_0^1 p(s) ds \int_0^1 ((1 - t) A + tB)^r dt - \int_0^1 \tilde{p}(t) ((1 - t) A + tB)^r dt \\
\leq 4(M - m) \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ ((1 - t) A + tB)^r + (tA + (1 - t) B)^r \right] dt
\]

for any \( A, B \geq 0 \).

We also have the norm inequality

\[
(3.12) \quad \left\| \int_0^1 p(s) ds \int_0^1 ((1 - t) A + tB)^r dt - \int_0^1 \tilde{p}(t) ((1 - t) A + tB)^r dt \right\| \\
\leq 4(M - m) \left\| \int_{1/2}^1 \left( t - \frac{1}{4} \right) \left[ ((1 - t) A + tB)^r + (tA + (1 - t) B)^r \right] dt \right\| \\
\leq \frac{1}{2} (M - m) \left\| \left( \frac{A + B}{2} \right)^r \right\|.
\]

**References**

RIEMANN-STIELTJES INTEGRAL INEQUALITIES


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