RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR OPERATOR MONOTONIC AND OPERATOR CONVEX INTEGRANDS AND (l, L)-LIPSCHITZIAN INTEGRATORS

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Abstract. Let f be an operator monotonic function on I and A, $B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If A, $B \in \mathcal{SA}_I(H)$ with $A \leq B$ and u is (l, L)-Lipschitzian on $[0, 1]$, then we have the operator inequalities

$$\begin{aligned} & -2(L-l)\int_0^1 \left(t-\frac{1}{2}\right) f((1-t)A+tB)\,dt \\ & \leq \int_0^1 f((1-t)A+tB)\,du(t) - [u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt \\ & \leq 2(L-l)\int_0^1 \left(t-\frac{1}{2}\right) f((1-t)A+tB)\,dt. \end{aligned}$$

We also have the norm inequalities

$$\begin{aligned} & \left\| \int_0^1 f((1-t)A+tB)\,du(t) - [u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt \right\| \\ & \leq 2(L-l)\left\| \int_0^1 \left(t-\frac{1}{2}\right) f((1-t)A+tB)\,dt \right\| \\ & \leq \frac{1}{2}(L-l)\max\{\|f(A)\|, \|f(B)\|\}. \end{aligned}$$

Similar inequalities for operator convex functions f are also provided. Some particular examples of interest are given as well.

1. Introduction

In order to generalize the Grüss inequality to the Riemann-Stieltjes integral, S.S. Dragomir and I. Fedotov introduced in [7] the following functional

$$D(f;u) := \int_a^b f(t)\,du(t) - [u(a)-u(b)]\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.1}$$

provided the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. In the same paper, the authors have shown that

$$|D(f;u)| \leq \frac{1}{2}L(M-m)(b-a), \tag{1.2}$$

provided that u is L-Lipschitzian, i.e., $|u(t)-u(s)| \leq L|t-s|$ for any t, $s \in [a, b]$ and f is Riemann integrable and bounded below by m and above by M. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

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In the follow-up paper [8], the same authors established a different result, namely

$$|D(f;u)| \leq \frac{1}{2} K (b-a) \bigvee_a^b (u), \tag{1.3}$$

provided that u is of bounded variation and f is K-Lipschitzian with a constant $K > 0$. Here $\frac{1}{2}$ is also best possible.

We have the following elementary fact, see for instance [2]:

Lemma 1. *Let $u : [a,b] \to \mathbb{R}$ and $l,\ L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzian;*

(ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t-s} \leq L \quad \textit{for each} \quad t,\ s \in [a,b] \quad \textit{with} \quad t \neq s; \tag{1.4}$$

(iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \textit{for each} \quad t,\ s \in [a,b] \quad \textit{with} \quad t > s. \tag{1.5}$$

Following [14] we can introduce the definition of (l,L)-Lipschitzian functions:

Definition 1. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i)–(iii) from Lemma 1 is said to be (l,L)-Lipschitzian on $[a,b]$.*

If $L > 0$ and $l = -L$, then $(-L,L)$-Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l,L)-Lipschitzian functions.

Proposition 1. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l,L)-Lipschitzian on $[a,b]$.*

In [2] we obtained among others the following result for Riemann-Stieltjes integral:

Theorem 1. *Let $f,\ u : [a,b] \to \mathbb{R}$ be such that u is (l,L)-Lipschitzian and f is monotonic nondecreasing, then*

$$|D(f;u)| \leq 2 \cdot \frac{L-l}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt \tag{1.6}$$

$$\leq \begin{cases} \frac{1}{2}(L-l) \max\{|f(a)|, |f(b)|\}(b-a); \\[2mm] \frac{1}{(q+1)^{\frac{1}{q}}} (L-l) \|f\|_p (b-a)^{\frac{1}{q}} & \textit{if} \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2mm] (L-l) \|f\|_1, \end{cases}$$

where $\|f\|_p := \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}}$, $p \geq 1$ are the Lebesgue norms.

The constants 2 and $\frac{1}{2}$ are best possible in (1.6).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq\left(\geq\right)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{1.7}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

Assume that f is operator convex on I. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s)=p(s)$ for all $s\in[0,1]$, then for A, B selfadjoint operators with spectra included in I, we have the operator Fejér's inequality

$$\begin{aligned}\left(\int_0^1 p(s)\,ds\right)f\left(\frac{A+B}{2}\right)&\leq\int_0^1 p(s)\,f\left(sA+(1-s)B\right)ds\\&\leq\left(\int_0^1 p(s)\,ds\right)\frac{f(A)+f(B)}{2},\end{aligned} \tag{1.8}$$

see [5] where further reverses were obtained.

In the recent paper [6] we obtained the following result:

Let A, $B\in\mathcal{SA}_I(H)$ with $A\leq B$ and f an operator monotonic function on I. If $p:[0,1]\to\mathbb{R}$ is monotonic nondecreasing on $[0,1]$, then

$$\begin{aligned}0&\leq\int_0^1 p(t)\,f\left((1-t)A+tB\right)dt-\int_0^1 p(t)\,dt\int_0^1 f\left((1-t)A+tB\right)dt\\&\leq\frac{1}{2}\left(\int_0^1\left|p(t)-\int_0^1 p(s)ds\right|dt\right)\left[f(B)-f(A)\right]\\&\leq\frac{1}{2}\left(\int_0^1 p^2(t)dt-\left(\int_0^1 p(s)ds\right)^2\right)^{1/2}\left[f(B)-f(A)\right]\\&\leq\frac{1}{4}\left[p(1)-p(0)\right]\left[f(B)-f(A)\right].\end{aligned} \tag{1.9}$$

For some related results, see [4]-[6] and [10]-[19].

Motivated by the above results, we establish in this paper some lower and upper bounds in operator order for the difference

$$\int_0^1 f\left((1-t)A+tB\right)du(t)-\left[u(1)-u(0)\right]\int_0^1 f\left((1-t)A+tB\right)dt,$$

where u is (l,L)-Lipschitzian on $[0,1]$ and f is either operator monotone or operator convex on I. Some examples for the logarithmic function which is both operator monotonic and operator concave on $(0,\infty)$ are given.

2. Main Results

Let f be a continuous function on I. If $(A,B)\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I and $t\in[0,1]$, then the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is a convex set in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator in $\mathcal{B}(H)$.

For A, $B\in\mathcal{SA}_I(H)$, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB).\tag{2.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle.\tag{2.2}$$

We have:

Theorem 2. *Let A, $B\in\mathcal{SA}_I(H)$ with $A\le B$ and f an operator monotonic function on I. If u is (l,L)-Lipschitzian on $[0,1]$, then we have the operator inequalities*

$$\begin{aligned}&-2(L-l)\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\\&\le\int_0^1 f((1-t)A+tB)\,du(t)-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\\&\le 2(L-l)\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt.\end{aligned}\tag{2.3}$$

We also have the norm inequality

$$\begin{aligned}&\left\|\int_0^1 f((1-t)A+tB)\,du(t)-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le 2(L-l)\left\|\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\right\|\\&\le\frac{1}{2}(L-l)\max\{\|f(A)\|,\|f(B)\|\}.\end{aligned}\tag{2.4}$$

Proof. Let $0\le t_1<t_2\le 1$ and $A\le B$. Then

$$(1-t_2)A+t_2B-(1-t_1)A-t_1B=(t_2-t_1)(B-A)\ge 0$$

and by operator monotonicity of f we get

$$f((1-t_2)A+t_2B)\ge f((1-t_1)A+t_1B),$$

which is equivalent to

$$\begin{aligned}\varphi_{(A,B);x}(t_2)&=\langle f((1-t_2)A+t_2B)x,x\rangle\\&\ge\langle f((1-t_1)A+t_1B)x,x\rangle=\varphi_{(A,B);x}(t_1)\end{aligned}$$

that shows that the scalar function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ is monotonic nondecreasing for $A \leq B$ and for any $x \in H$.

If we use the inequality (1.6) for $\varphi_{(A,B);x}$ and u, then

$$\begin{aligned}
&\left|\int_0^1 \varphi_{(A,B);x}(t)\,du(t) - [u(1)-u(0)]\int_0^1 \varphi_{(A,B);x}(t)\,dt\right| \\
&\leq 2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)\varphi_{(A,B);x}(t)\,dt \\
&\leq \frac{1}{2}(L-l)\max\left\{\left|\varphi_{(A,B);x}(0)\right|, \left|\varphi_{(A,B);x}(1)\right|\right\}
\end{aligned}$$

for $A \leq B$ and for any $x \in H$, namely

$$\begin{aligned}
&\left|\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,du(t)\right. \\
&\left.-[u(1)-u(0)]\int_0^1 \langle f((1-t)A+tB)x,x\rangle\,dt\right| \\
&\leq 2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)\langle f((1-t)A+tB)x,x\rangle\,dt \\
&\leq \frac{1}{2}(L-l)\max\{|\langle f(A)x,x\rangle|, |\langle f(B)x,x\rangle|\}
\end{aligned} \tag{2.5}$$

for $A \leq B$ and for any $x \in H$.

The first inequality in (2.5) is equivalent to

$$\begin{aligned}
&\left\langle -2(L-l)\int_0^1 \left(\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\right)x,x\right\rangle \\
&\leq \left\langle\left(\int_0^1 f((1-t)A+tB)\,du(t)\right)x,x\right\rangle \\
&-\left\langle\left([u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right)x,x\right\rangle \\
&\leq \left\langle\left(2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\right)x,x\right\rangle
\end{aligned}$$

for any $x \in H$, which is equivalent to the operator inequality (2.3).

The inequality (2.5) can be written as

$$\begin{aligned}
&\left|\left\langle\left(\int_0^1 f((1-t)A+tB)\,du(t) - [u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt\right)x,x\right\rangle\right| \\
&\leq \left\langle\left(2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\right)x,x\right\rangle \\
&\leq \frac{1}{2}(L-l)\max\{|\langle f(A)x,x\rangle|, |\langle f(B)x,x\rangle|\}
\end{aligned}$$

for any $x \in H$.

By taking the supremum over $x \in H$, $\|x\| = 1$, we derive the norm inequality (2.4). □

We have the following weighted inequalities as well:

Corollary 1. *Let* $A,\ B\in\mathcal{SA}_I(H)$ *with* $A\leq B$ *and* f *an operator monotonic function on* I. *If* p *is continuous on* $[0,1]$ *with* $m:=\min_{t\in[0,1]}p(t)$ *and* $M:=\max_{t\in[0,1]}p(t)$, *then we have the operator inequality*

$$\begin{aligned}
(2.6)\qquad & -2(M-m)\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt \\
&\leq\int_0^1 p(t)\,f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt \\
&\leq 2(M-m)\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt.
\end{aligned}$$

We also have the norm inequality

$$\begin{aligned}
(2.7)\qquad & \left\|\int_0^1 p(t)\,f((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\| \\
&\leq 2(M-m)\left\|\int_0^1\left(t-\frac{1}{2}\right)f((1-t)A+tB)\,dt\right\| \\
&\leq\frac{1}{2}(M-m)\max\{\|f(A)\|,\|f(B)\|\}.
\end{aligned}$$

Proof. If we take $u(t):=\int_0^t p(s)\,ds$ and use the properties of Riemann-Stieltjes integral, then we have

$$\int_0^1 f((1-t)A+tB)\,du(t)=\int_0^1 f((1-t)A+tB)\,p(t)\,dt.$$

Also

$$u'(t)=p(t),\ t\in(0,1),$$

which shows that $u(t)$ is (m,M)-Lipschitzian, and by Theorem 2 we derive the desired inequalities (2.6) and (2.7). □

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\check{f}_{[a,b]}$ or simply $\check{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\check{f}(t):=\frac{1}{2}[f(t)+f(a+b-t)],\ t\in[a,b]. \tag{2.8}$$

The asymmetrical transform of f on the interval $[a,b]$, denoted by $\tilde{f}_{[a,b]}$ or simply $\tilde{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\tilde{f}(t):=\frac{1}{2}[f(t)-f(a+b-t)],\ t\in[a,b]. \tag{2.9}$$

Theorem 3. *Let* $A,\ B\in\mathcal{SA}_I(H)$ *and* f *be an operator convex function on* I. *If* $\tilde{u}$ *is* $\left(\tilde{l},\tilde{L}\right)$*-Lipschitzian on* $[0,1]$, *then we have the operator inequality*

$$\begin{aligned}
(2.10)\qquad & -4\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(t-\frac{1}{4}\right)[f((1-t)A+tB)+f(tA+(1-t)B)]\,dt \\
&\leq\int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t))-[u(1)-u(0)]\int_0^1 f((1-t)A+tB)\,dt \\
&\leq 4\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(t-\frac{1}{4}\right)[f((1-t)A+tB)+f(tA+(1-t)B)]\,dt.
\end{aligned}$$

We also have the norm inequality

$$\left\| \int_0^1 f((1-t)A+tB)\,d(\tilde{u}(t)) - [u(1)-u(0)] \int_0^1 f((1-t)A+tB)\,dt \right\| \tag{2.11}$$
$$\leq 4\left(\tilde{L}-\tilde{l}\right) \left\| \int_{1/2}^1 \left(t-\frac{1}{4}\right) [f((1-t)A+tB)+f(tA+(1-t)B)]\,dt \right\|$$
$$\leq \frac{1}{2}\left(\tilde{L}-\tilde{l}\right) \max\left\{ \left\| f\left(\frac{A+B}{2}\right)\right\|, \left\| \frac{f(A)+f(B)}{2} \right\| \right\}.$$

Proof. Let $A, B \in \mathcal{SA}_I(H)$. By the operator convexity of f we have that $\varphi_{(A,B);x}$ is convex on $[0,1]$ for all $x \in H$. This implies that the symmetrized transform $\check{\varphi}_{(A,B);x}$ is convex and symmetric on $[0,1]$ and therefore monotonic nonincreasing on $[0,1/2]$ and nondecreasing on $[1/2,1]$.

Observe that

$$\int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t) = \int_0^{1/2} \check{\varphi}_{(A,B);x}(t)\,du(t) + \int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,du(t).$$

Using the change of variable for Riemann-Stieltjes integral, we have for $s=1-t$ that

$$\begin{aligned}\int_0^{1/2} \check{\varphi}_{(A,B);x}(t)\,du(t) &= \int_1^{1/2} \check{\varphi}_{(A,B);x}(1-s)\,d(u(1-s)) \\ &= \int_1^{1/2} \check{\varphi}_{(A,B);x}(s)\,d(u(1-s)) \\ &= \int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d[-u(1-s)],\end{aligned}$$

therefore

$$\begin{aligned}&\int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t) \\ &= \int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d[-u(1-s)] + \int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,du(t) \\ &= \int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d[u(t)-u(1-s)] \\ &= 2\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d(\tilde{u}(t))\end{aligned}$$

for all $x \in H$.

Observe that

$$\begin{aligned}\tilde{u}(1)-\tilde{u}(1/2) &= \frac{u(1)-u(0)}{2} - \frac{u(1/2)-u(1/2)}{2} \\ &= \frac{1}{2}[u(1)-u(0)]\end{aligned}$$

and

$$\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,dt = \frac{1}{2}\int_0^1 \check{\varphi}_{(A,B);x}(t)\,dt = \frac{1}{2}\int_0^1 \varphi_{(A,B);x}(t)\,dt$$

for all $x \in H$.

Therefore

$$\begin{aligned}
(2.12)\quad & \int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t) - [u(1)-u(0)]\int_0^1 \varphi_{(A,B);x}(t)\,dt \\
&= 2\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d(\tilde{u}(t)) \\
&- 2[\tilde{u}(1)-\tilde{u}(1/2)]\,2\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,dt \\
&= 2\left[\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d(\tilde{u}(t)) - [\tilde{u}(1)-\tilde{u}(1/2)]\,2\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,dt\right] \\
&= 2\left[\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,d(\tilde{u}(t)) - [\tilde{u}(1)-\tilde{u}(1/2)]\,\frac{1}{1/2}\int_{1/2}^1 \check{\varphi}_{(A,B);x}(t)\,dt\right] \\
&= 2D_{[1/2,1]}\left(\check{\varphi}_{(A,B);x},\tilde{u}\right),
\end{aligned}$$

where $D_{[1/2,1]}(\cdot,\cdot)$ is defined by (1.1) on the interval $[1/2,1]$.

By making use of the inequality (1.6) we then get

$$\begin{aligned}
(2.13)\quad & \left|D_{[1/2,1]}\left(\check{\varphi}_{(A,B);x},\tilde{u}\right)\right| \\
&\le 2\frac{\tilde{L}-\tilde{l}}{1/2}\int_{1/2}^1\left(t-\frac{1}{4}\right)\check{\varphi}_{(A,B);x}dt \\
&\le \frac{1}{2}1/2\left(\tilde{L}-\tilde{l}\right)\max\left\{\left|\check{\varphi}_{(A,B);x}(1/2)\right|,\left|\check{\varphi}_{(A,B);x}(1)\right|\right\}
\end{aligned}$$

namely

$$\begin{aligned}
(2.14)\quad & \left|D_{[1/2,1]}\left(\check{\varphi}_{(A,B);x},\tilde{u}\right)\right| \\
&\le 2\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(t-\frac{1}{4}\right)\langle[f((1-t)A+tB)+f(tA+(1-t)B)]x,x\rangle\,dt \\
&\le \frac{1}{4}\left(\tilde{L}-\tilde{l}\right)\max\left\{\left|\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\right|,\left|\left\langle\frac{f(A)+f(B)}{2}x,x\right\rangle\right|\right\}
\end{aligned}$$

for all $x\in H$.

Also

$$\begin{aligned}
& \int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t) \\
&= \frac{1}{2}\int_0^1\left[\varphi_{(A,B);x}(t)+\varphi_{(A,B);x}(1-t)\right]du(t) \\
&= \frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,du(t)+\int_0^1 \varphi_{(A,B);x}(1-t)\,du(t)\right].
\end{aligned}$$

Using the change of variable for Riemann-Stieltjes integral, we have for $s=1-t$ that

$$\begin{aligned}\int_0^1 \varphi_{(A,B);x}(1-t)\,du(t) &= \int_1^0 \varphi_{(A,B);x}(s)\,du(1-s)\\ &= \int_0^1 \varphi_{(A,B);x}(s)\,d[-u(1-s)].\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^1 \check{\varphi}_{(A,B);x}(t)\,du(t)\\ &= \frac{1}{2}\left[\int_0^1 \varphi_{(A,B);x}(t)\,du(t) + \int_0^1 \varphi_{(A,B);x}(t)\,d[-u(1-t)]\right]\\ &= \int_0^1 \varphi_{(A,B);x}(t)\,d\left[\frac{u(t)-u(1-t)}{2}\right] = \int_0^1 \varphi_{(A,B);x}(t)\,d(\tilde{u}(t))\end{aligned}\tag{2.15}$$

for all $x\in H$.

Using (2.12), (2.13) and (2.15) we get

$$\begin{aligned}&\left|\int_0^1 \varphi_{(A,B);x}(t)\,d(\tilde{u}(t)) - [u(1)-u(0)]\int_0^1 \varphi_{(A,B);x}(t)\,dt\right|\\ &\le 4\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(t-\frac{1}{4}\right)\langle[f((1-t)A+tB)+f(tA+(1-t)B)]x,x\rangle\,dt\\ &\le \frac{1}{2}\left(\tilde{L}-\tilde{l}\right)\max\left\{\left|\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\right|,\left|\left\langle\frac{f(A)+f(B)}{2}x,x\right\rangle\right|\right\}\end{aligned}\tag{2.16}$$

for all $x\in H$.

The first inequality in (2.16) is equivalent to (2.10) and the whole (2.16) implies, by taking the supremum over $x\in H$, $\|x\|=1$, the norm inequality (2.11). □

Corollary 2. *Let A, $B\in\mathcal{SA}_I(H)$ and f be an operator convex function on I. If p is continuous on $[0,1]$ with $m:=\min_{t\in[0,1]}p(t)$ and $M:=\max_{t\in[0,1]}p(t)$, then we have the operator inequality*

$$\begin{aligned}&-4(M-m)\int_{1/2}^1\left(t-\frac{1}{4}\right)[f((1-t)A+tB)+f(tA+(1-t)B)]\,dt\\ &\le \int_0^1 \check{p}(t)f((1-t)A+tB)\,dt - \int_0^1 p(s)\,ds\int_0^1 f((1-t)A+tB)\,dt\\ &\le 4(M-m)\int_{1/2}^1\left(t-\frac{1}{4}\right)[f((1-t)A+tB)+f(tA+(1-t)B)]\,dt.\end{aligned}\tag{2.17}$$

We also have the norm inequality

$$\begin{aligned}&\left\|\int_0^1 \check{p}(t)f((1-t)A+tB)\,dt - \int_0^1 p(s)\,ds\int_0^1 f((1-t)A+tB)\,dt\right\|\\ &\le 4(M-m)\left\|\int_{1/2}^1\left(t-\frac{1}{4}\right)[f((1-t)A+tB)+f(tA+(1-t)B)]\,dt\right\|\\ &\le \frac{1}{2}(M-m)\max\left\{\left\|f\left(\frac{A+B}{2}\right)\right\|,\left\|\frac{f(A)+f(B)}{2}\right\|\right\}.\end{aligned}\tag{2.18}$$

Proof. For $u(t) := \int_0^t p(s)\,ds$, we have

$$\tilde{u}(t) = \frac{1}{2}[u(t) - u(1-t)] = \frac{1}{2}\left[\int_0^t p(s)\,ds - \int_0^{1-t} p(s)\,ds\right].$$

Then $\tilde{u}(t)$ is differentiable and

$$(\tilde{u}(t))' = \frac{1}{2}[p(t) + p(1-t)] \in [m, M],\ t \in [0,1],$$

which shows that $\tilde{u}(t)$ is (m, M)-Lipschitzian on $[0,1]$.

Since, by the properties of Riemann-Stieltjes integral

$$\int_0^1 f((1-t)A + tB)\,d(\tilde{u}(t)) = \int_0^1 \check{p}(t) f((1-t)A + tB)\,dt$$

hence by Theorem 3 we get the desired results. □

3. Some Examples

The logarithmic function $f(t) = \ln t$ is operator monotone on $(0, \infty)$. If u is (l, L)-Lipschitzian on $[0,1]$, then by Theorem 2 we have the operator inequalities

$$\begin{aligned} &-2(L-l)\int_0^1 \left(t - \frac{1}{2}\right)\ln((1-t)A + tB)\,dt \\ &\le \int_0^1 \ln((1-t)A + tB)\,du(t) - [u(1) - u(0)]\int_0^1 \ln((1-t)A + tB)\,dt \\ &\le 2(L-l)\int_0^1 \left(t - \frac{1}{2}\right)\ln((1-t)A + tB)\,dt \end{aligned} \tag{3.1}$$

for $B \ge A > 0$.

We also have the norm inequalities

$$\begin{aligned} &\left\|\int_0^1 \ln((1-t)A + tB)\,du(t) - [u(1) - u(0)]\int_0^1 \ln((1-t)A + tB)\,dt\right\| \\ &\le 2(L-l)\left\|\int_0^1 \left(t - \frac{1}{2}\right)\ln((1-t)A + tB)\,dt\right\| \\ &\le \frac{1}{2}(L-l)\max\{\|\ln A\|, \|\ln B\|\} \end{aligned} \tag{3.2}$$

for $B \ge A > 0$.

If p is continuous on $[0,1]$ with $m := \min_{t\in[0,1]} p(t)$ and $M := \max_{t\in[0,1]} p(t)$, then by Corollary 1 we have the operator inequality

$$\begin{aligned} &-2(M-m)\int_0^1 \left(t - \frac{1}{2}\right)\ln((1-t)A + tB)\,dt \\ &\le \int_0^1 p(t)\ln((1-t)A + tB)\,dt - \int_0^1 p(t)\,dt\int_0^1 \ln((1-t)A + tB)\,dt \\ &\le 2(M-m)\int_0^1 \left(t - \frac{1}{2}\right)\ln((1-t)A + tB)\,dt \end{aligned} \tag{3.3}$$

for $B \ge A > 0$.

We also have the norm inequality

$$\begin{aligned}(3.4)\quad &\left\|\int_0^1 p(t)\ln((1-t)A+tB)\,dt-\int_0^1 p(t)\,dt\int_0^1 \ln((1-t)A+tB)\,dt\right\|\\ &\leq 2(M-m)\left\|\int_0^1\left(t-\frac{1}{2}\right)\ln((1-t)A+tB)\,dt\right\|\\ &\leq \frac{1}{2}(M-m)\max\{\|\ln A\|,\|\ln B\|\}\end{aligned}$$

for $B\geq A>0$.

If $\tilde{u}$ is $\left(\tilde{l},\tilde{L}\right)$-Lipschitzian on $[0,1]$, then by Theorem 3 we have the operator inequality

$$\begin{aligned}(3.5)\quad &4\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(t-\frac{1}{4}\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt\\ &\leq [u(1)-u(0)]\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \ln((1-t)A+tB)\,d(\tilde{u}(t))\\ &\leq 4\left(\tilde{L}-\tilde{l}\right)\int_{1/2}^1\left(\frac{1}{4}-t\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt.\end{aligned}$$

for $B\geq A>0$.

We also have the norm inequality

$$\begin{aligned}(3.6)\quad &\left\|[u(1)-u(0)]\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \ln((1-t)A+tB)\,d(\tilde{u}(t))\right\|\\ &\leq 4\left(\tilde{L}-\tilde{l}\right)\left\|\int_{1/2}^1\left(t-\frac{1}{4}\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt\right\|\\ &\leq \frac{1}{2}\left(\tilde{L}-\tilde{l}\right)\max\left\{\left\|\ln\left(\frac{A+B}{2}\right)\right\|,\left\|\frac{\ln A+\ln B}{2}\right\|\right\}.\end{aligned}$$

If p is continuous on $[0,1]$ with $m:=\min_{t\in[0,1]}p(t)$ and $M:=\max_{t\in[0,1]}p(t)$, then by Corollary 2 we have the operator inequality

$$\begin{aligned}(3.7)\quad &4(M-m)\int_{1/2}^1\left(t-\frac{1}{4}\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt\\ &\leq \int_0^1 p(s)\,ds\int_0^1 \ln((1-t)A+tB)\,dt-\int_0^1 \check{p}(t)\ln((1-t)A+tB)\,dt\\ &\leq 4(M-m)\int_{1/2}^1\left(\frac{1}{4}-t\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt\end{aligned}$$

for $B\geq A>0$.

We also have the norm inequality

$$\begin{aligned}(3.8)\quad &\left\|\int_0^1 \check{p}(t)\ln((1-t)A+tB)\,dt-\int_0^1 p(s)\,ds\int_0^1 \ln((1-t)A+tB)\,dt\right\|\\ &\leq 4(M-m)\left\|\int_{1/2}^1\left(t-\frac{1}{4}\right)[\ln((1-t)A+tB)+\ln(tA+(1-t)B)]\,dt\right\|\\ &\leq \frac{1}{2}(M-m)\max\left\{\left\|\ln\left(\frac{A+B}{2}\right)\right\|,\left\|\frac{\ln A+\ln B}{2}\right\|\right\}\end{aligned}$$

for $B \geq A > 0$.

The interested reader may obtain other similar inequalities by choosing the operator monotonic functions $f(t) = t^r$ for $r \in (0,1)$ and operator convex functions $f(t) = t^p$ if either $1 \leq p \leq 2$ or $-1 \leq p \leq 0$.

Indeed, since $f(t) = t^r$ is operator monotonic for $r \in (0,1)$, then by (2.3),

$$
\begin{aligned}
&-2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)((1-t)A+tB)^r dt \\
&\leq \int_0^1 ((1-t)A+tB)^r du(t) - [u(1)-u(0)]\int_0^1 ((1-t)A+tB)^r dt \\
&\leq 2(L-l)\int_0^1 \left(t-\frac{1}{2}\right)((1-t)A+tB)^r dt.
\end{aligned}
\tag{3.9}
$$

where u is (l,L)-Lipschitzian on $[0,1]$ and $B \geq A \geq 0$.

We also have the norm inequality

$$
\begin{aligned}
&\left\|\int_0^1 ((1-t)A+tB)^r du(t) - [u(1)-u(0)]\int_0^1 ((1-t)A+tB)^r dt\right\| \\
&\leq 2(L-l)\left\|\int_0^1 \left(t-\frac{1}{2}\right)((1-t)A+tB)^r dt\right\| \\
&\leq \frac{1}{2}(L-l)\|B^r\|,
\end{aligned}
\tag{3.10}
$$

for $B \geq A \geq 0$.

If p is continuous on $[0,1]$ with $m := \min_{t\in[0,1]} p(t)$ and $M := \max_{t\in[0,1]} p(t)$ and since $f(t) = t^r$ is operator concave for $r \in (0,1)$, then we have the operator inequality

$$
\begin{aligned}
&4(M-m)\int_{1/2}^1 \left(t-\frac{1}{4}\right)[((1-t)A+tB)^r + (tA+(1-t)B)^r] dt \\
&\leq \int_0^1 p(s)ds\int_0^1 ((1-t)A+tB)^r dt - \int_0^1 \check{p}(t)((1-t)A+tB)^r dt \\
&\leq 4(M-m)\int_{1/2}^1 \left(\frac{1}{4}-t\right)[((1-t)A+tB)^r + (tA+(1-t)B)^r] dt
\end{aligned}
\tag{3.11}
$$

for any A, $B \geq 0$.

We also have the norm inequality

$$
\begin{aligned}
&\left\|\int_0^1 p(s)ds\int_0^1 ((1-t)A+tB)^r dt - \int_0^1 \check{p}(t)((1-t)A+tB)^r dt\right\| \\
&\leq 4(M-m)\left\|\int_{1/2}^1 \left(t-\frac{1}{4}\right)[((1-t)A+tB)^r + (tA+(1-t)B)^r] dt\right\| \\
&\leq \frac{1}{2}(M-m)\left\|\left(\frac{A+B}{2}\right)^r\right\|.
\end{aligned}
\tag{3.12}
$$

References

[1] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.*, **20** (2007), 123–130.

[2] S. S. Dragomir, Accurate approximations of the Riemann-Stieltjes integral with (l, L)-Lipschitzian integrators, *AIP Conf. Proc. 939, Numerical Anal. & Appl. Math.*, Ed. T.H. Simos et al., pp. 686–690. Preprint *RGMIA Res. Rep. Coll.* **10** (2007), No. 3, Article 5. [Online https://rgmia.org/papers/v10n3/AppMathComp.pdf].

[3] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218** (2011), no. 3, 766–772.

[4] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps. *Spec. Matrices* **7** (2019), 38–51. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 80. [Online http://rgmia.org/papers/v19/v19a80.pdf].

[5] S. S. Dragomir, Reverses of operator Féjer's inequalities, to appear in *Tokyo Math. J.*, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 91, 14 pp. [Online https://rgmia.org/papers/v22/v22a91.pdf].

[6] S. S. Dragomir, Some weighted integral inequalities for operator monotonic functions on Hilbert Spaces, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 50, 14 pp. [Online https://rgmia.org/papers/v23/v23a50.pdf].

[7] S. S. Dragomir and I. Fedotov, An inequality of Grüss type for Riemann-Stieltjes integral and application for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.

[8] S. S. Dragomir and I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, *Nonlinear Funct. Anal. Appl.* (Korea), **6**(3) (2001), 415-433.

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] A. G. Ghazanfari, Hermite-Hadamard type inequalities for functions whose derivatives are operator convex. *Complex Anal. Oper. Theory* **10** (2016), no. 8, 1695–1703.

[11] A. G. Ghazanfari, The Hermite-Hadamard type inequalities for operator s-convex functions. *J. Adv. Res. Pure Math.* **6** (2014), no. 3, 52–61.

[12] J. Han and J. Shi, Refinements of Hermite-Hadamard inequality for operator convex function. *J. Nonlinear Sci. Appl.* **10** (2017), no. 11, 6035–6041.

[13] B. Li, Refinements of Hermite-Hadamard's type inequalities for operator convex functions. *Int. J. Contemp. Math. Sci.* **8** (2013), no. 9-12, 463–467.

[14] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.

[15] A. Taghavi, V. Darvish, H. M. Nazari and S. S. Dragomir, Hermite-Hadamard type inequalities for operator geometrically convex functions. *Monatsh. Math.* **181** (2016), no. 1, 187–203.

[16] M. Vivas Cortez and E. J. Hernández-Hernández, Refinements for Hermite-Hadamard type inequalities for operator h-convex function. *Appl. Math. Inf. Sci.* **11** (2017), no. 5, 1299–1307.

[17] M. Vivas Cortez and E. J. Hernández-Hernández, On some new generalized Hermite-Hadamard-Fejér inequalities for product of two operator h-convex functions. A*ppl. Math. Inf. Sci.* **11** (2017), no. 4, 983–992.

[18] S.-H. Wang, Hermite-Hadamard type inequalities for operator convex functions on the coordinates. *J. Nonlinear Sci. Appl.* **10** (2017), no. 3, 1116–1125

[19] S.-H. Wang, New integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 4, 744–753.

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