SEVERAL NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR EXPONENTIAL TYPE CONVEX FUNCTIONS

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Abstract. New Hermite-Hadamard type inequalities will be given in this work for functions whose second derivative in absolute value at certain power is exponential type convex and for functions whose n-th derivative in absolute value at certain power is exponential type convex .

1. **Introduction**

The classical inequality of Hermite-Hadamard has been considered useful in mathematical analysis being extended and generalized in many directions by authors as [8, 7, 12, 1, 17, 23, 14] and the references therein.

Using a recent concept of exponential type convex functions given in [15] some Hermite-Hadamard type inequalities will be presented here for this new kind of functions.

We begin by recalling below the classical definition of the convex functions.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

Definition 2. *(see [15]) A nonnegative function $f : I \to \mathbb{R}$ iscalled exponential type convex function if, for every $m, n \in I$ and $k \in [0,1]$,*

$$f(km + (1-k)n) \leq (e^k - 1)f(m) + (e^{1-k} - 1)f(n). \tag{2}$$

The class of all exponentially type convex functions on interval I is indicated by EXPC(I).

Definition 3. *([28]) Let $h : J \to \mathbf{R}$ be a nonnegative function and $h \neq 0$. We say that $f : I \to \mathbf{R}$ is an h-convex function, or that f belongs to the class SX(h,I), if is nonnegative and for all $m, n \in I,\ k \in [0,1]$ we have*

$$f(km + (1-k)n) \leq h(k)f(m) + h(1-k)f(n).$$

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When previous inequality is reversed then f is said to be a h-concave function, i.e. $f \in SV(h, I)$. It is obvious that when $h(u) = u$ then the h-convexity becomes convexity.

We know from [15] that every nonnegative convex function is exponential type convex function and that every exponential type convex function is an h-convex function with $h(k) = e^k - 1$.

It is necessary to recall below the definition of fractionals integrals, see [10, 13, 12, 24, 25]. For other type of convexity see also [26, 20].

The classical Hermite-Hadamard's inequality for convex functions is given below:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Definition 4. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

We will use also the following result given in [21]:

Lemma 1. *Let $f : I \to \mathbf{R}$, $: I \subset \mathbf{R}$ be a twice differentiable function on I^o, where $a, b \in I,\ a < b$ with $t \in [0, 1]$.*

If $f^{''} \in L[a, b]$, then for all $a < b$ and $\alpha - 1 > 0$, with properties of Gamma function we have

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^+}f(b) + J^{\alpha-1}_{\left(\frac{a+b}{2}\right)^-}f(a)] - f\left(\frac{a+b}{2}\right) =$$

$$= \frac{(b-a)^2}{\alpha 2^{2-\alpha}}[\int_0^{\frac{1}{2}} t^{\alpha}f^{''}(ta+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (1-t)^{\alpha}f^{''}(ta+(1-t)b)\,dt].$$

The following result is a generalization of Lemma 2 from [3], when $\alpha > n-1$ and $n \in \mathbb{N}$.

Lemma 2. *(see [5]) Let $n \in \mathbb{N}^*$, $n \geq 2$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a, b]$ with $a, b \in I^0,\ 0 < a < b,\ x \in [a, b],\ \lambda \in (0, 1)$. Then the following identity holds:*

$$\mathcal{I}(f, x, a, b, \lambda, \alpha, n) = (1-\lambda)(x-a)\int_0^1 t^{\alpha}f^{(n)}(t(\lambda a + (1-\lambda)x) + (1-t)a)dt +$$

$$+\lambda(x-a)\int_0^1 (1-t)^\alpha f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))dt+$$

$$+(1-\lambda)(b-x)\int_0^1 t^\alpha f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)dt+$$

$$+\lambda(b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))dt=$$

$$=\sum_{k=2}^{n}\alpha(\alpha-1)...(\alpha-k+2)[(\frac{(-1)^{k-1}}{(1-\lambda)^{k-1}}-\frac{1}{\lambda^{k-1}})(\frac{f^{(n-k)}(\lambda a+(1-\lambda)x)}{(x-a)^{k-1}}+$$

$$+\frac{f^{(n-k)}(\lambda x+(1-\lambda)b)}{(b-x)^{k-1}})]+\Gamma(\alpha+1)\{\frac{(-1)^n}{(1-\lambda)^\alpha(x-a)^\alpha}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^-}f(a)+$$

$$+\frac{1}{\lambda^\alpha(b-x)^\alpha}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^+}f(b)+\frac{1}{\lambda^\alpha(x-a)^\alpha}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^+}f(x)+$$

$$+\frac{(-1)^n}{(1-\lambda)^\alpha(b-x)^\alpha}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^-}f(x)\},$$

where $\alpha > n-1$.

The following result is a generalization of Lemma 1 from [4] when $\alpha > n-1$ and $n \in \mathbb{N}$.

Lemma 3. *Let* $n \in \mathbb{N}^*$ *and* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be o function such that* $f^{(n)}$ *exists on the interior* I^0 *of an interval* I *and* $f^{(n)} \in L[a,b]$ *with* $a,b \in I^0$, $0 < a < b$. *Then for any* $x \in [a,b]$, *we have:*

$$I(f,x,a,b,\alpha,n)=(x-a)\int_0^1 t^\alpha f^{(n)}(tx+(1-t)a)dt+(b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)x)dt=$$

$$=\sum_{k=2}^{n}\alpha(\alpha-1)...(\alpha-k+2)f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}}-\frac{1}{(b-x)^{k-1}}\right)+$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha}J^{\alpha-n+1}_{x^-}f(a)+\frac{1}{(b-x)^\alpha}J^{\alpha-n+1}_{x^+}f(b)],$$

where $\alpha > n-1$.

Lemma 4. *([6]) Let* $f : I \to \mathbb{R}$ *be a twice differentiable function on the interior* I^0 *of an interval* I *in* $\mathbb{R}$, *with* $a,b \in I$, $0 < a < b$. *If* $f'' \in L_1[a,b]$ *and* M *is a positive constant so that* $\frac{a}{b} < M$ *for* $l,k,n \in \mathbb{N}$ *with* $n \geq 2$, $l < k < n$ *and* $\frac{l}{k} > M$ *or* ($\frac{a}{b} < \frac{l}{k}$) *then the following identity takes place:*

$$I(f,a,b,n,k,l)=(lb-ka)^3(I_1+I_2)+[\frac{n(b-a)}{2}-(lb-ka)]^3(I_3+I_4)=$$

$$=2n^3\int_a^b f(x)dx+\frac{n^2(b-a)}{4}[4(lb-ka)-n(b-a)][f'(\frac{(n-k)a+lb}{n})-$$

$$-f'(\frac{ka+(n-l)b}{n})]-n^3(b-a)[f(\frac{(n-k)a+lb}{n})+f(\frac{ka+(n-l)b}{n})],$$

where

$$I_1=\int_0^1 t^2 f''\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)dt,$$

$$I_2 = \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{ka+(n-l)b}{n}\right) dt,$$

$$I_3 = \int_0^1 t^2 f^{''}\left(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2}\right) dt,$$

$$I_4 = \int_0^1 t^2 f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{(n-k)a+lb}{n}\right) dt.$$

New Hermite-Hadamard type inequalities will be given in this work in Theorem 1, 2, 3, 4 and 5 for functions whose second derivative in absolute value at certain power is exponential type convex and also for functions whose n-th order derivative in absolute value at certain power is exponential type convex.

2. **On Hermite-Hadamard type inequalities for exponential type convex functions**

The aim of this section is to present new inequalities that refine Hermite-Hadamard inequality for functions whose second derivative in absolute value at certain power is exponential type convex and also for functions whose n-th order derivative in absolute value at certain power is exponential type convex.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval I and $f^{''} \in L[a,b]$ with $a,b \in I^0$, $0 < a < b$. If $|f^{''}|^q$ is an exponential type convex function on $[a,b] \subset I$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality*

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+} f(b) + J^{\alpha-1}_{(\frac{a+b}{2})^-} f(a)] - f\left(\frac{a+b}{2}\right)| \leq$$

$$\leq \frac{(b-a)^2}{\alpha(\alpha+1)^{\frac{1}{p}}} \frac{1}{2^{2+\frac{\alpha+1}{p}}} \{[|f^{''}(a)|^q \frac{{}_1F_1(\alpha+1,\alpha+2,\frac{1}{2})-1}{2(\alpha+1)} +$$

$$+|f^{''}(b)|^q \left(e^{\frac{1}{2}}({}_1F_1(1,\alpha+1,\frac{1}{2})-1) - \frac{1}{2(\alpha+1)}\right)]^{\frac{1}{q}} +$$

$$+[|f^{''}(a)|^q \left(e^{\frac{1}{2}}({}_1F_1(1,\alpha+1,\frac{1}{2})-1) - \frac{1}{2(\alpha+1)}\right) + |f^{''}(b)|^q \frac{{}_1F_1(\alpha+1,\alpha+2,\frac{1}{2})-1}{2(\alpha+1)}]^{\frac{1}{q}}\},$$

holds where $\alpha > 1$, and ${}_1F_1(a,b,z) = M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}du$ with $Re\ a > 0$, $Re\ b > 0$ is the confluent hypergeometric function.

Proof. By using Lemma 1 and the properties of the modulus we will find,

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+} f(b) + J^{\alpha-1}_{(\frac{a+b}{2})^-} f(a)] - f\left(\frac{a+b}{2}\right)| \leq$$

$$\leq \frac{(b-a)^2}{\alpha 2^{2-\alpha}}[\int_0^{\frac{1}{2}} t^\alpha |f^{''}(ta+(1-t)b)| dt + \int_{\frac{1}{2}}^1 (1-t)^\alpha |f^{''}(ta+(1-t)b)| dt].$$

Using Holder's inequality, and then the definition of the exponential type convex functions for $|f^{''}|^q$, by calculus, we get:

$$|\frac{2^{\alpha-2}\Gamma(\alpha)}{(b-a)^{\alpha-1}}[J^{\alpha-1}_{(\frac{a+b}{2})^+}f(b)+J^{\alpha-1}_{(\frac{a+b}{2})^-}f(a)]-f\left(\frac{a+b}{2}\right)|\leq$$

$$\leq\frac{(b-a)^2}{\alpha 2^{2-\alpha}}[\left(\int_0^{\frac{1}{2}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^\alpha|f^{''}(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_{\frac{1}{2}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^\alpha|f^{''}(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}]\leq$$

$$\leq\frac{(b-a)^2}{\alpha(\alpha+1)^{\frac{1}{p}}}\frac{1}{2^{2+\frac{1}{p}-\frac{\alpha}{q}}}\{[|f^{''}(a)|^q\int_0^{\frac{1}{2}}t^\alpha(e^t-1)dt+|f^{''}(b)|^q\int_0^{\frac{1}{2}}t^\alpha(e^{1-t}-1)dt]^{\frac{1}{q}}+$$

$$+[|f^{''}(a)|^q\int_{\frac{1}{2}}^1(1-t)^\alpha(e^t-1)dt+|f^{''}(b)|^q\int_{\frac{1}{2}}^1(1-t)^\alpha(e^{1-t}-1)dt]^{\frac{1}{q}}\}.$$

We denote $A_1=\int_0^{\frac{1}{2}}t^\alpha(e^t-1)dt$, $A_2=\int_0^{\frac{1}{2}}t^\alpha(e^{1-t}-1)dt$, $A_3=\int_{\frac{1}{2}}^1(1-t)^\alpha(e^t-1)dt$ and $A_4=\int_{\frac{1}{2}}^1(1-t)^\alpha(e^{1-t}-1)dt$ and by using the substitution $1-t=u$ we see that $A_1=A_4$ and $A_2=A_3$. Taking into account of the definition of the confluent hypergeometric function and their properties, we get for $b=a+1$ and $z=\frac{1}{2}$ that

$$_1F_1(\alpha,\alpha+1,\frac{1}{2})=\alpha\int_0^1 e^{\frac{u}{2}}u^{\alpha-1}du.$$

By substitution $\frac{u}{2}=v$ we have $_1F_1(\alpha,\alpha+1,\frac{1}{2})=\alpha 2^\alpha\int_0^{\frac{1}{2}}v^{\alpha-1}e^v dv$. From here we obtain,

$$A_1=\frac{_1F_1(\alpha+1,\alpha+2,\frac{1}{2})-1}{(\alpha+1)2^{\alpha+1}}.$$

We find the expression of the A_2, by using the incomplete gamma function, $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$, where s is a complex parameter with $Re\ s>0$. It is known, see [29], that $\gamma(s+1,x)=s\gamma(s,x)-x^s e^{-x}$ and then we find in our case

$$\gamma(\alpha+1,\frac{1}{2})=\alpha\gamma(\alpha,\frac{1}{2})-\frac{1}{2^\alpha}e^{-\frac{1}{2}}.$$

In order to find $\gamma(\alpha,\frac{1}{2})$ we use the following property, see [29]: if $Re\ z>0$ then $\gamma(s,z)=s^{-1}z^s e^{-z}M(1,s+1,z)$. Therefore we have,

$$\gamma(\alpha,\frac{1}{2})=\frac{1}{\alpha 2^\alpha e^{\frac{1}{2}}}M(1,\alpha+1,\frac{1}{2})=\frac{1}{\alpha 2^\alpha e^{\frac{1}{2}}}{}_1F_1(1,\alpha+1,\frac{1}{2})$$

and from here,

$$A_2=\frac{1}{2^\alpha}[e^{\frac{1}{2}}({}_1F_1(1,\alpha+1,\frac{1}{2})-1)-\frac{1}{(\alpha+1)2^{\alpha+1}}].$$

From here we find the inequality of the theorem.

∎

Consequence 1. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$. If $|f^{(n)}|$ is an exponential type convex function on $[a,b]$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then we have*

$$|I(f,x,a,b,\alpha,n)| \leq$$

$$\leq \frac{2}{\alpha+1}\{(x-a)A\left(({}_1F_1(\alpha+1,\alpha+2,1)-1)|f^{(n)}(x)|,({}_1F_1(1,\alpha+2,1)-1)|f^{(n)}(a)|\right)+$$

$$+(b-x)A\left(({}_1F_1(1,\alpha+2,1)-1)|f^{(n)}(b)|,({}_1F_1(\alpha+1,\alpha+2,1)-1)|f^{(n)}(x)|\right)\}$$

holds, where $\alpha > n-1$, and ${}_1F_1(a,b,z) = M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}du$ with Re $a > 0$, Re $b > 0$ is the confluent hypergeometric function and where $A(u,v)$ is the arithmetic mean of u and v.

Proof. This time we use Holder's inequality. We start like in previous demonstration, by using Lemma 3 and then the modulus properties and we get:

$$|I(f,x,a,b,\alpha,n)| \leq$$

$$\leq (x-a)\int_0^1 t^\alpha|f^{(n)}(tx+(1-t)a)|dt + (b-x)\int_0^1(1-t)^\alpha|f^{(n)}(tb+(1-t)x)|dt \leq$$

By using the definition of the exponential type convex functions for the function $|f^{(n)}|$ and calculus we obtain,

$$|I(f,x,a,b,\alpha,n)| \leq (x-a)[|f^{(n)}(x)|\int_0^1 t^\alpha(e^t-1)dt + |f^{(n)}(a)|\int_0^1 t^\alpha(e^{1-t}-1)dt]+$$

$$+(b-x)[|f^{(n)}(b)|\int_0^1(1-t)^\alpha(e^t-1)dt + |f^{(n)}(x)|\int_0^1(1-t)^\alpha(e^{1-t}-1)dt].$$

We take into account integrals $E_1 = \int_0^1 t^\alpha(e^t-1)dt$, $E_2 = \int_0^1 t^\alpha(e^{1-t}-1)dt$, $E_3 = \int_0^1(1-t)^\alpha(e^t-1)dt$ and $E_4 = \int_0^1(1-t)^\alpha(e^{1-t}-1)dt$. We see that $E_1 = E_4$ and $E_2 = E_3$ if we use the substitution $1-t = u$ and then, using the confluent hyperbolic functions and their properties, see [29], we get $E_1 = \frac{1}{\alpha+1}({}_1F_1(\alpha+1,\alpha+2,1)-1)$ and $E_2 = \frac{1}{\alpha+1}({}_1F_1(1,\alpha+2,1)-1)$.

∎

Theorem 2. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$. If $|f^{(n)}|^q$ is an exponential type convex function on $[a,b]$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \leq$$

$$\leq \frac{2^{\frac{1}{q}}}{\alpha+1}\{(1-\lambda)(x-a)A^{\frac{1}{q}}\left(E_1^*|f^{(n)}(\lambda a+(1-\lambda)x)|^q, E_2^*|f^{(n)}(a)|^q\right)+$$

$$+\lambda(x-a)A^{\frac{1}{q}}\left(E_2^*|f^{(n)}(x)|^q, E_1^*|f^{(n)}(\lambda a+(1-\lambda)x)|^q\right)+$$

$$+(1-\lambda)(b-x)A^{\frac{1}{q}}\left(E_1^*|f^{(n)}(\lambda x+(1-\lambda)b)|^q, E_2^*|f^{(n)}(x)|^q\right)+$$

$$+\lambda(b-x)A^{\frac{1}{q}}\left(E_2^*|f^{(n)}(b)|^q, E_1^*|f^{(n)}(\lambda x+(1-\lambda)b)|^q\right)\}$$

holds, where $A(u,v)$ is the arithmetic mean of u and v and

$$E_1^* = {}_1F_1(\alpha+1,\alpha+2,1)-1$$

$$E_2^* = {}_1F_1(1,\alpha+2,1)-1.$$

Proof. By Lemma 2, the modulus properties and then Holder;s inequality we get:

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \le (1-\lambda)(x-a)\int_0^1 t^\alpha |f^{(n)}\left(t(\lambda a+(1-\lambda)x)+(1-t)a\right)|dt+$$

$$+\lambda(x-a)\int_0^1 (1-t)^\alpha |f^{(n)}\left(tx+(1-t)(\lambda a+(1-\lambda)x)\right)|dt+$$

$$+(1-\lambda)(b-x)\int_0^1 t^\alpha |f^{(n)}\left(t(\lambda x+(1-\lambda)b)+(1-t)x\right)|dt+$$

$$+\lambda(b-x)\int_0^1 (1-t)^\alpha |f^{(n)}\left(tb+(1-t)(\lambda x+(1-\lambda)b)\right)|dt \le$$

$$\le (1-\lambda)(x-a)\left(\int_0^1 t^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 t^\alpha |f^{(n)}\left(t(\lambda a+(1-\lambda)x)+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\lambda(x-a)\left(\int_0^1 (1-t)^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^\alpha |f^{(n)}\left(tx+(1-t)(\lambda a+(1-\lambda)x)\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+(1-\lambda)(b-x)\left(\int_0^1 t^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 t^\alpha |f^{(n)}\left(t(\lambda x+(1-\lambda)b)+(1-t)x\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\lambda(b-x)\left(\int_0^1 (1-t)^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^\alpha |f^{(n)}\left(tb+(1-t)(\lambda x+(1-\lambda)b)\right)|^q dt\right)^{\frac{1}{q}}.$$

Because $|f^{(n)}|^q$ is an exponential type convex function, we have,

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \le$$

$$\le \frac{1}{(\alpha+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)[E_1|f^{(n)}(\lambda a+(1-\lambda)x)|^q+E_2|f^{(n)}(a)|^q]^{\frac{1}{q}}+$$

$$+\lambda(x-a)[E_2|f^{(n)}(x)|^q+E_1|f^{(n)}(\lambda a+(1-\lambda)x)|^q]^{\frac{1}{q}}+$$

$$+(1-\lambda)(b-x)[E_1|f^{(n)}(\lambda x+(1-\lambda)b)|^q+E_2|f^{(n)}(x)|^q]^{\frac{1}{q}}+$$

$$+\lambda(b-x)[E_2|f^{(n)}(b)|^q+E_1|f^{(n)}(\lambda x+(1-\lambda)b)|^q]^{\frac{1}{q}}\},$$

where $E_1,\ E_2,\ E_3,\ E_4$ are as in Consequence 1. From here we obtain the desired inequality.

∎

Theorem 3. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I$, $0 < a < b$. If $f^{''} \in L[a,b]$ for $l,k,n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$. If $|f^{''}|$ is an exponential convex function on $[a,b]$, M is a positive constant so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following inequality*

$$|I(f,a,b,n,k,l)| \leq$$
$$\leq (lb-ka)^3\{2(e-\frac{8}{3})[|f^{''}(a)|+|f^{''}(b)|]+(e-\frac{7}{3})[|f^{''}(\frac{(n-k)a+lb}{n})|+|f^{''}(\frac{ka+(n-l)b}{n})]\}$$
$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[4(e-\frac{8}{3})|f^{''}(\frac{a+b}{2})|+(e-\frac{7}{3})[|f^{''}(\frac{ka+(n-l)b}{n})]+|f^{''}(\frac{(n-k)a+lb}{n})|]$$

takes place.

Proof. First we use the modulus properties in Lemma 4, having:

$$|I(f,a,b,n,k,l)| \leq (lb-ka)^3(|I_1|+|I_2|) + [\frac{n(b-a)}{2} - (lb-ka)]^3|(I_3|+|I_4|) \leq$$
$$\leq (lb-ka)^3[\int_0^1 t^2|f^{''}\left(t\frac{(n-k)a+lb}{n} + (1-t)a\right)|dt+$$
$$+\int_0^1 (t-1)^2 f^{''}|\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|dt]+$$
$$+[\frac{n(b-a)}{2} - (lb-ka)]^3[\int_0^1 t^2|f^{''}\left(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2}\right)|dt+$$
$$+\int_0^1 (t-1)^2|f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{(n-k)a+lb}{n}\right)|dt]$$

where

$$I_1 = \int_0^1 t^2 f^{''}\left(t\frac{(n-k)a+lb}{n} + (1-t)a\right)dt,$$
$$I_2 = \int_0^1 (t-1)^2 f^{''}\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)dt,$$
$$I_3 = \int_0^1 t^2 f^{''}\left(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2}\right)dt,$$
$$I_4 = \int_0^1 (t-1)^2 f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{(n-k)a+lb}{n}\right)dt.$$

Now by definition of the exponential type convexity of $|f^{''}|$, we get,

$$|I(f,a,b,n,k,l)| \leq (lb-ka)^3\{\int_0^1 t^2(e^t-1)dt \cdot |f^{''}\left(\frac{(n-k)a+lb}{n}\right)|+$$
$$+\int_0^1 t^2(e^{1-t}-1)dt \cdot |f^{''}(a)| + \int_0^1 (1-t)^2(e^t-1)dt \cdot |f^{''}(b)|+$$
$$+\int_0^1 (1-t)^2(e^{1-t}-1)dt \cdot |f^{''}\left(\frac{ka+(n-l)b}{n}\right)|\}+$$
$$+[\frac{n(b-a)}{2} - (lb-ka)]^3\{\int_0^1 t^2(e^t-1)dt \cdot |f^{''}\left(\frac{(ka+(n-l)b}{n}\right)|+$$
$$+\int_0^1 t^2(e^{1-t}-1)dt \cdot |f^{''}\left(\frac{a+b}{2}\right)| + \int_0^1 (1-t)^2(e^t-1)dt \cdot |f^{''}\left(\frac{a+b}{2}\right)|+$$

$$+\int_0^1 (1-t)^2(e^{1-t}-1)dt\cdot|f''\left(\frac{(n-k)a+lb}{2}\right)|\},$$

and by taking into account that these four integrals are computed before we obtain the desired result.

∎

Theorem 4. *Let $f: I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a,b \in I$, $0<a<b$. If $f'' \in L[a,b]$ for $l,k,n \in \mathbb{N}$ with $n \geq 2$ and $l<k<n$. If $|f''|^q$ is an exponential convex function on $[a,b]$, M is a positive constant so that $\frac{a}{b}<M$ and $\frac{l}{k}>M$ or $(\frac{a}{b}<\frac{l}{k})$ then the following inequality takes place*

$$|I(f,a,b,n,k,l)| \leq$$

$$\leq 2^{\frac{1}{q}}\frac{(e-2)^{\frac{1}{q}}}{(2p+1)^{\frac{1}{p}}}\{(lb-ka)^3[A^{\frac{1}{q}}(|f''(\frac{(n-k)a+lb}{n})|^q,|f''(a)|^q)+$$

$$+A^{\frac{1}{q}}(|f''(\frac{ka+(n-l)b}{n})|^q,|f''(b)|^q)]+$$

$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[A^{\frac{1}{q}}(|f''(\frac{ka+(n-l)b}{n})|^q,|f''(\frac{a+b}{2})|)+$$

$$+A^{\frac{1}{q}}(|f''(\frac{(n-k)a+lb}{n})|^q,|f''(\frac{a+b}{2})|^q)]\},$$

where $A(u,v)=\frac{u+v}{2}$ is the arithmetic mean of u and v.

Proof. The proof will be as before, by using Lemma 4, but we apply in addition the Holder's inequality, like below:

$$|I(f,a,b,n,k,l)| \leq$$

$$\leq (lb-ka)^3[\int_0^1 t^2|f''\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|dt+$$

$$+\int_0^1 (t-1)^2 f''|\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|dt]+$$

$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[\int_0^1 t^2|f''\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|dt+$$

$$+\int_0^1 (t-1)^2|f''\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|dt] \leq$$

$$\leq (lb-ka)^3[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|^q dt\right)^{\frac{1}{q}}]+$$

$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|^q dt\right)^{\frac{1}{q}}].$$

From definition of the exponential type convex functions, for $|f^{''}|^q$ and using that $\int_0^1 (e^t-1)dt = \int_0^1 (e^{1-t}-1)dt = e-2$ we get the desired inequality.
∎

Theorem 5. *Under conditions of previous theorem, we have the following inequality:*

$$|I(f,a,b,n,k,l)| \leq$$
$$\leq \frac{2^{\frac{1}{q}}}{3^{\frac{1}{p}}}\{(lb-ka)^3[A^{\frac{1}{q}}((e-\frac{7}{3})|f^{''}(\frac{(n-k)a+lb}{n})|^q, 2(e-\frac{8}{3})|f^{''}(a)|^q)+$$
$$+A^{\frac{1}{q}}((e-\frac{7}{3})|f^{''}(\frac{ka+(n-l)b}{n})|^q, 2(e-\frac{8}{3})|f^{''}(b)|^q)]+$$
$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[A^{\frac{1}{q}}((e-\frac{7}{3})|f^{''}(\frac{ka+(n-l)b}{n})|^q, 2(e-\frac{8}{3})|f^{''}(\frac{a+b}{2})|)+$$
$$+A^{\frac{1}{q}}((e-\frac{7}{3})|f^{''}(\frac{(n-k)a+lb}{n})|^q, 2(e-\frac{8}{3})|f^{''}(\frac{a+b}{2})|^q)]\},$$

where $A(u,v)=\frac{u+v}{2}$ is the arithmetic mean of u and v.

Proof. Applying Lemma 4 and the modulus properties, we have,

$$|I(f,a,b,n,k,l)| \leq$$
$$\leq (lb-ka)^3[\int_0^1 t^2|f^{''}\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|dt+$$
$$+\int_0^1 (t-1)^2 f^{''}|\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|dt]+$$
$$+[\frac{n(b-a)}{2}-(lb-ka)]^3[\int_0^1 t^2|f^{''}\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|dt+$$
$$+\int_0^1 (t-1)^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|dt].$$

Now we use Holder's inequality like in Theorem 1 and 2 and obtain,

$$|I(f,a,b,n,k,l)| \leq$$
$$\leq (lb-ka)^3[\left(\int_0^1 t^2 dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2|f^{''}\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$
$$+\left(\int_0^1 (t-1)^2 dt\right)^{\frac{1}{p}}\left(\int_0^1 (t-1)^2|f^{''}\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|^q dt\right)^{\frac{1}{q}}]+$$
$$+[\frac{n(b-a)}{2}-(lb-ka)]^3\left([\int_0^1 t^2 dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2|f^{''}\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|^q dt\right))^{\frac{1}{q}}+$$
$$+\left(\int_0^1 (t-1)^2 dt\right))^{\frac{1}{p}}\left(\int_0^1 (t-1)^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|^q dt\right))^{\frac{1}{q}}].$$

Then by using the definition of exponential type convexity for the function $|f^{''}|^q$ we find the desired inequality after calculus.
∎

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