SOME INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we show that, if that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ then there exist $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\begin{aligned}&[f(B)-f(A)](B-A)\\&=b(B-A)^2\\&+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned}$$

for all $A,\ B>0$. Some necessary and sufficient conditions for the operators A, $B>0$ such that the inequality

$$f(B)B+f(A)A\geq f(A)B+f(B)A$$

holds for any operator monotone function f on $(0,\infty)$ are also given.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{ts}{t+s}dm(s) \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty\frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [8].

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In [5], T. Furuta observed that for $\alpha_j \in [0,1]$, $j = 1, ..., n$ the functions

$$g(t) := \left(\sum_{j=1}^{n} t^{-\alpha_j} \right)^{-1} \text{ and } h(t) = \sum_{j=1}^{n} \left(1 + t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0, \infty)$.

Let $f(t)$ be a continuous function $(0,\infty) \to (0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [5] or [10].

Consider the family of functions defined on $(0,\infty)$ and $p \in [-1,2] \setminus \{0,1\}$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

and

$$f_0(t) := \frac{t}{1-t} \ln t,$$

$$f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean)}, \; f_{1/2}(t) = \sqrt{t} \text{ (geometric mean).}$$

In [4] the authors showed that f_p is operator monotone for $1 \le p \le 2$.

In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p - 1}$$

is an operator monotone function for $p \in (0,1]$, [5].

It is well known that the logarithmic function ln is operator monotone and in [5] the author obtained that the functions

$$f(t) = t(1+t) \ln\left(1 + \frac{1}{t}\right), \; g(t) = \frac{1}{(1+t)\ln\left(1+\frac{1}{t}\right)}$$

are also operator monotone functions on $(0, \infty)$.

For recent operator inequalities related to operator monotone functions, see [1], [11] and [12].

In this paper we show that, if that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ then there exist $b \ge 0$ and a positive measure m on $(0,\infty)$ such that

$$\begin{aligned} &[f(B) - f(A)](B-A) \\ &= b(B-A)^2 \\ &+ \int_0^\infty s^2 \left[\int_0^1 \left[((1-t)A + tB + s)^{-1} (B-A) \right]^2 dt \right] dm(s) \end{aligned}$$

for all A, $B > 0$. Some necessary and sufficient conditions for the operators A, $B > 0$ such that the inequality

$$f(B)B + f(A)A \ge f(A)B + f(B)A$$

holds for any operator monotone function f on $(0,\infty)$ are also given.

2. Main Results

We have the following identities of interest:

Theorem 2. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.1). Then for all* A, $B>0$ *we have*

$$\begin{aligned}[f(B)-f(A)](B-A) &= b(B-A)^2 \\ &+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}(B-A)[f(B)-f(A)] &= b(B-A)^2 \\ &+\int_0^\infty s^2\left[\int_0^1\left[(B-A)((1-t)A+tB+s)^{-1}\right]^2dt\right]dm(s).\end{aligned}\tag{2.2}$$

Proof. Since the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$, then f can be written as in the equation (1.1) and for A, $B>0$ we have the representation

$$\begin{aligned}f(B)-f(A) &= b(B-A)+\int_0^\infty s\left[B(B+s)^{-1}-A(A+s)^{-1}\right]dm(s).\end{aligned}\tag{2.3}$$

Observe that for $s>0$

$$\begin{aligned}&B(B+s)^{-1}-A(A+s)^{-1}\\ &=(B+s-s)(B+s)^{-1}-(A+s-s)(A+s)^{-1}\\ &=(B+s)(B+s)^{-1}-s(B+s)^{-1}-(A+s)(A+s)^{-1}+s(A+s)^{-1}\\ &=1-s(B+s)^{-1}-1+s(A+s1_H)^{-1}\\ &=s\left[(A+s)^{-1}-(B+s)^{-1}\right].\end{aligned}$$

Therefore, (2.3) becomes, see also [6]

$$f(B)-f(A)=b(B-A)+\int_0^\infty s^2\left[(A+s)^{-1}-(B+s)^{-1}\right]dm(s).\tag{2.4}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}\tag{2.5}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-t)C + tD,\ t \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.6}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.7}$$

Now, if we replace in (2.7) $C = A + s1_H$ and $D = B + s1_H$ for $s > 0$, then

$$\begin{aligned}&(A+s)^{-1} - (B+s)^{-1} \\ &= \int_0^1 ((1-t)A + tB + s)^{-1}(B-A)((1-t)A+tB+s)^{-1}\,dt.\end{aligned} \tag{2.8}$$

By the representation (2.4), we derive the following identity of interest

$$\begin{aligned}f(B) - f(A) &= b(B-A) \\ &\quad + \int_0^\infty s^2 \left[\int_0^1 ((1-t)A + tB + s)^{-1}\right. \\ &\quad \left.\times (B-A)((1-t)A+tB+s)^{-1}\,dt\right] dm(s)\end{aligned} \tag{2.9}$$

for $A,\ B > 0$.

If we multiply this identity at the right with $B - A$ we get

$$\begin{aligned}&(f(B) - f(A))(B-A) \\ &= b(B-A)^2 \\ &+ \int_0^\infty s^2 \left[\int_0^1 ((1-t)A+tB+s)^{-1}\right. \\ &\left.\times (B-A)((1-t)A+tB+s)^{-1}(B-A)\,dt\right] dm(s) \\ &= \int_0^\infty s^2 \left[\int_0^1 \left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\right] dm(s)\end{aligned} \tag{2.10}$$

for $A,\ B > 0$ and the equality (2.1) is proved.

The equality (2.2) follows by multiplying (2.9) at the left. □

In the following, in order to simplify terminology, when we write $T \geq 0$ we automatically assume that the operator T is selfadjoint.

In the note [3], Fujii and Nakamoto showed that the inequality

$$(f(B) - f(A))(B-A) \geq 0$$

does not hold in general for $A,\ B > 0$.

They also proved the following interesting inequality:

Proposition 1 ([3, Proposition 2])**.** *If $C,\ D > 0$ and $CD^{-1} + DC^{-1}$ is selfadjoint, then*

$$CD^{-1} + DC^{-1} \geq 2. \tag{2.11}$$

Proof. Indeed, as shown in [3], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\mathrm{Sp}(T)$ of T is included in $(0,\infty)$, because $C,\ D > 0$ and $\mathrm{Sp}(T) = \mathrm{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\mathrm{Sp}(V) = \left\{t + \frac{1}{t},\ t \in \mathrm{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$. □

As a consequence, they obtained the following result:

Theorem 3 ([3, Theorem 6])**.** *If*

(i') Operator $A(B+s)^{-1} + B(A+s)^{-1}$ *is selfadjoint for all* $s \geq 0$,

then $(B-A)(f(B)-f(A)) \geq 0$.

Some necessary and sufficient conditions for the operators $A,\ B > 0$ such that the inequality $(f(B)-f(A))(B-A) \geq 0$ holds for any operator monotone function f on $(0,\infty)$ are included in the following theorem.

Theorem 4. *Let* $A,\ B > 0$. *The following statements are equivalent:*

(i) *For all* $s \geq 0$,

$$(A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) \geq 2. \tag{2.12}$$

(ii) *For all* $s \geq 0$,

$$\int_0^1 \left[((1-t)A + tB + s)^{-1}(B-A)\right]^2 dt \geq 0.$$

(iii) *For all* $s \geq 0$,

$$(\ell_s(B) - \ell_s(A))(B-A) \geq 0,$$

where $\ell_s(t) = -(t+s)^{-1}$, $t > 0$.

(iv) *For all operator monotone function* f *on* $(0,\infty)$,

$$(f(B) - f(A))(B-A) \geq 0. \tag{2.13}$$

(v) *For all operator monotone function* f *on* $(0,\infty)$,

$$(B-A)(f(B) - f(A)) \geq 0. \tag{2.14}$$

Proof. From (2.8) we have, by multiplying at right with $B - A$ that

$$\begin{aligned}
&\left[(A+s)^{-1} - (B+s)^{-1}\right](B-A) \\
&= \int_0^1 ((1-t)A + tB + s)^{-1}(B-A)((1-t)A + tB + s)^{-1}(B-A)\,dt \\
&= \int_0^1 \left[((1-t)A + tB + s)^{-1}(B-A)\right]^2 dt
\end{aligned}$$

for all $s \geq 0$.

Also,

$$\begin{aligned}
&\left[(A+s)^{-1} - (B+s)^{-1}\right](B-A) \\
&= \left[(A+s)^{-1} - (B+s)^{-1}\right][B + s - (A+s)] \\
&= (A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) - 2
\end{aligned}$$

for all $s \geq 0$.

Therefore

$$(\ell_s(B)-\ell_s(A))(B-A) = (A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)-2 = \int_0^1 \left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt \tag{2.15}$$

for all $s \geq 0$.

The identity (2.15) reveals that the statements (i), (ii) and (iii) are equivalent.

Since for fixed $s \geq 0$, $\ell_s(t)$ is operator monotone function on $(0,\infty)$, then statement (iv) implies (iii).

Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. Then for all A, $B>0$ we have

$$\begin{aligned}&[f(B)-f(A)](B-A)\\&= b(B-A)^2\\&+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\right]dm(s)\\&\geq \int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\right]dm(s)\end{aligned} \tag{2.16}$$

where $b\geq 0$ and m is a positive measure on $(0,\infty)$.

If (ii) is valid, then

$$\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\right]dm(s)\geq 0$$

and by (2.16) we obtain (2.13).

Define the operator $K:=(f(B)-f(A))(B-A)$. Since

$$\begin{aligned}K^* &= [(f(B)-f(A))(B-A)]^* = (B-A)^*(f(B)-f(A))^*\\&= (B-A)(f(B)-f(A))\end{aligned}$$

then the fact that K is selfadjoint is equivalent to

$$(f(B)-f(A))(B-A)=(B-A)(f(B)-f(A)),$$

which is also equivalent to the fact that

$$f(A)B+f(B)A=Bf(A)+Af(B).$$

These prove the equivalence between (iv) and (v). □

Remark 1. *The identity*

$$(B-A)(\ell_s(B)-\ell_s(A))=(B+s)(A+s)^{-1}+(A+s)(B+s)^{-1}-2$$

for $s\geq 0$ was the main tool in the proof of Theorem 6, [3].

We can state:

Corollary 1. *Let A, $B>0$. The statement (i) is equivalent to the inequality*

$$f(B)B+f(A)A\geq f(A)B+f(B)A, \tag{2.17}$$

for all f an operator monotone function on $(0,\infty)$.

Observe that, in fact we have:

Proposition 2. *Let A, $B>0$, then the statements (i) and (i') are equivalent.*

Proof. Notice that for all $s\geq 0$,

$$\begin{aligned}(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s) \\ =(A+s)^{-1}B+(B+s)^{-1}A+s(A+s)^{-1}+s(B+s)^{-1}.\end{aligned} \tag{2.18}$$

Also, the operator $s(A+s)^{-1}+s(B+s)^{-1}$ is selfadjoint for $s\geq 0$.

If the statement *(i)* holds, then $(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$ is selfadjoint and by (2.18) we must have that $(A+s)^{-1}B+(B+s)^{-1}A$ is selfadjoint, which shows that

$$\left((A+s)^{-1}B+(B+s)^{-1}A\right)^{*}=B(A+s)^{-1}+A(B+s)^{-1}$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (2.18) we get

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$$

is selfadjoint and by (2.11) for $C=(A+s)^{-1}$, $D=(B+s)^{-1}$ we obtain the inequality (2.12), namely *(i)* is true. □

We define the class of operators

$$\mathfrak{Cl}_{(0,\infty)}(H):=\{(A,B)\mid A,\ B>0 \text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$ then $(B,A)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB=BA$, A, $B>0$, then $U_s:=(A+s)^{-1}(B+s)$ and $U_s^{-1}=(B+s)^{-1}(A+s)$ are selfadjoint and since $U_s+U_s^{-1}\geq 2$, $s\geq 0$ we derive that $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators A, $B>0$, then we have

$$\emptyset\neq\mathfrak{Co}_{(0,\infty)}(H)\subset\mathfrak{Cl}_{(0,\infty)}(H). \tag{2.19}$$

Corollary 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $g:I\to(0,\infty)$ is continuous, then for all selfadjoint operators A, B with spectra in I for which $(g(A),g(B))\in\mathfrak{Cl}_{(0,\infty)}(H)$ we have*

$$(f\circ g)(B)g(B)+(f\circ g)(A)g(A)\geq(f\circ g)(A)g(B)+(f\circ g)(B)g(A). \tag{2.20}$$

Follows by Theorem 2 by replacing A with $g(A)$, B with $g(B)$ and using the composition rule for continuous functions of selfadjoint operators which gives that $f(g(A))=(f\circ g)(A)$ and $f(g(B))=(f\circ g)(B)$, see for instance [7, p. 49].

Corollary 3. *Assume that the function $f:(0,\infty)\to(0,\infty)$ is operator monotone in $(0,\infty)$. Then*

$$B^2(f(B))^{-1}+A^2(f(A))^{-1}\geq(f(A))^{-1}AB+(f(B))^{-1}BA, \tag{2.21}$$

for all $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Follows by Corollary 1 and the fact that $t/f(t)$ is also operator monotone on $(0,\infty)$.

Remark 2. *Since $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^{\alpha}$ is an operator monotone function for any $\alpha\in[0,1]$, then by (2.17) we get*

$$B^{\alpha+1}+A^{\alpha+1}\geq A^{\alpha}B+B^{\alpha}A, \tag{2.22}$$

for all $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

Since $\ln$ *is operator monotone on* $(0,\infty)$, *then by (2.17) we get*

$$B \ln B + A \ln A \geq (\ln A) B + (\ln B) A \tag{2.23}$$

for all $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

Assume that A, B *are selfadjoint operators such that* $(\exp A, \exp B) \in \mathfrak{Cl}_{(0,\infty)}(H)$. *If we take* $f = \ln$, *which is operator monotone on* $(0,\infty)$ *and* $g = \exp$, *then by Corollary 2 we get*

$$B \exp B + A \exp A \geq \exp(A) B + \exp(B) A, \tag{2.24}$$

Suppose that $p > 1$. *Then by taking* $f(t) = t^{1/p}$ *and* $g(t) = t^p$, *for* $t \in (0,\infty)$, *in Corollary 2, we get*

$$B^{p+1} + A^{p+1} \geq AB^p + BA^p,$$

for all A, $B > 0$ *such that* $(A^p, B^p) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

3. Some Integral and Discrete Inequalities

We have the following integral inequality:

Proposition 3. *Assume that the function* $f : (0,\infty) \to \mathbb{R}$ *is operator monotone in* $(0,\infty)$. *Then for all* A, $B > 0$ *with* $\left((1-t)A + tB, \frac{A+B}{2}\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ *for all* $t \in [0,1]$,

$$\begin{aligned} &\int_0^1 ((1-t)A + tB) f((1-t)A + tB)\, dt \\ &\geq \left(\int_0^1 f((1-t)A + tB)\, dt\right) \frac{A+B}{2}. \end{aligned} \tag{3.1}$$

Proof. From (2.17) we get

$$\begin{aligned} &((1-t)A + tB) f((1-t)A + tB) + \frac{A+B}{2} f\left(\frac{A+B}{2}\right) \\ &\geq f\left(\frac{A+B}{2}\right)((1-t)A + tB) + f((1-t)A + tB)\frac{A+B}{2} \end{aligned} \tag{3.2}$$

for all $t \in [0,1]$.

By taking the integral in (3.2) we get

$$\begin{aligned} &\int_0^1 ((1-t)A + tB) f((1-t)A + tB)\, dt + \frac{A+B}{2} f\left(\frac{A+B}{2}\right) \\ &\geq f\left(\frac{A+B}{2}\right) \int_0^1 ((1-t)A + tB)\, dt \\ &+ \left(\int_0^1 f((1-t)A + tB)\, dt\right) \frac{A+B}{2} \end{aligned} \tag{3.3}$$

and since

$$\int_0^1 ((1-t)A + tB)\, dt = \frac{A+B}{2},$$

hence by (3.3) we derive

$$\begin{aligned}&\int_0^1 ((1-t)A+tB) f((1-t)A+tB)\,dt + \frac{A+B}{2} f\left(\frac{A+B}{2}\right)\\ &\geq f\left(\frac{A+B}{2}\right)\frac{A+B}{2} + \left(\int_0^1 f((1-t)A+tB)\,dt\right)\frac{A+B}{2},\end{aligned}$$

which is equivalent to the first inequality in (3.1).

The second inequality in (3.1) follows by the second part of (2.17). □

If we take $f(t)=t^r$, $r\in(0,1)$ in (3.1), then we get

$$\int_0^1 ((1-t)A+tB)^{r+1}\,dt \geq \left(\int_0^1 ((1-t)A+tB)^r\,dt\right)\frac{A+B}{2}, \tag{3.4}$$

for all A, $B>0$ with $\left((1-t)A+tB, \frac{A+B}{2}\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $t\in[0,1]$.

Also, with the same assumptions for A and B, by choosing $f(t)=\ln t$, $t>0$ in (3.1), then we get

$$\begin{aligned}&\int_0^1 ((1-t)A+tB)\ln((1-t)A+tB)\,dt\\ &\geq \left(\int_0^1 \ln((1-t)A+tB)\,dt\right)\frac{A+B}{2},\end{aligned} \tag{3.5}$$

for all A, $B>0$.

We have the following Chebychev type operator inequality:

Proposition 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $g: I\to(0,\infty)$ is continuous, then for all selfadjoint operators A_k, $k=1,...,n$ with spectra in I and such that $(g(A_k), g(A_j)) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for $j,k=1,...,n$ and $p_k\geq 0$, $k=1,...,n$ with $\sum_{k=1}^n p_k=1$,*

$$\sum_{k=1}^n p_k (f\circ g)(A_k)\, g(A_k) \geq \sum_{k=1}^n p_k (f\circ g)(A_k) \sum_{k=1}^n p_k g(A_k). \tag{3.6}$$

Proof. From (2.20) we get

$$\begin{aligned}&(f\circ g)(A_k)\,g(A_k) + (f\circ g)(A_j)\,g(A_j)\\ &\geq (f\circ g)(A_j)\,g(A_k) + (f\circ g)(A_k)\,g(A_j)\end{aligned} \tag{3.7}$$

for all k, $j\in\{1,...,n\}$.

If we multiply (3.7) by $p_k p_j\geq 0$ and sum over k and j from 1 to n, we get

$$\begin{aligned}&\sum_{k=1}^n\sum_{j=1}^n p_k p_j (f\circ g)(A_k)\,g(A_k) + \sum_{k=1}^n\sum_{j=1}^n p_k p_j (f\circ g)(A_j)\,g(A_j)\\ &\geq \sum_{k=1}^n\sum_{j=1}^n p_k p_j (f\circ g)(A_j)\,g(A_k) + \sum_{k=1}^n\sum_{j=1}^n p_k p_j (f\circ g)(A_k)\,g(A_j),\end{aligned}$$

which is equivalent to

$$\sum_{k=1}^{n} p_k (f \circ g)(A_k)\, g(A_k) + \sum_{j=1}^{n} p_j (f \circ g)(A_j)\, g(A_j)$$
$$\geq \sum_{j=1}^{n} p_j (f \circ g)(A_j) \sum_{k=1}^{n} p_k g(A_k) + \sum_{k=1}^{n} p_k (f \circ g)(A_k) \sum_{j=1}^{n} p_j g(A_j)$$

that is equivalent to the first part of (3.6). □

Remark 3. *If the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and $A_k > 0$, $k = 1,...,n$, with $(A_k, A_j) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for j, $k = 1,...,n$ and $p_k \geq 0$, $k = 1,...,n$ with $\sum_{k=1}^{n} p_k = 1$, then by (3.6),*

$$\sum_{k=1}^{n} p_k A_k f(A_k) \geq \sum_{k=1}^{n} p_k f(A_k) \sum_{k=1}^{n} p_k A_k. \tag{3.8}$$

In particular, for $f(t) = t^r$, $r \in (0,1)$ we get

$$\sum_{k=1}^{n} p_k A_k^{r+1} \geq \sum_{k=1}^{n} p_k A_k^r \sum_{k=1}^{n} p_k A_k. \tag{3.9}$$

For $f(t) = -t^{-1}$ we get

$$1_H \leq \sum_{k=1}^{n} p_k A_k^{-1} \sum_{k=1}^{n} p_k A_k. \tag{3.10}$$

We also have the logarithmic inequalities

$$\sum_{k=1}^{n} p_k A_k \ln A_k \geq \sum_{k=1}^{n} p_k \ln A_k \sum_{k=1}^{n} p_k A_k. \tag{3.11}$$

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References

[1] M. Bagher Ghaemi, V. Kaleibary, Some inequalities involving operator monotone functions and operator means. *Math. Inequal. Appl.* **19** (2016), no. 2, 757–764.

[2] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[3] M. Fujii and R. Nakamoto, Note on Dragomir's theorems, *Private note*, 5 Aug. 2020. Preprint *RGMIA Res. Rep. Coll* **23** (2020), Art. 113. 4 pp.

[4] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[5] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra and its Applications* **429** (2008) 972–980.

[6] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[7] M. Haase, *Lectures on Functional Calculus*, 21st International Internet Seminar, March 19, 2018, Kiel University, 2018. https://www.mathematik.tu-darmstadt.de/media/analysis/lehrmaterial_anapde/hallerd/ISem21complete.pdf

[8] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[9] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[10] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.* **246** (1980) 205–224.

[11] J. Mićić, H. R. Moradi, Some inequalities involving operator means and monotone convex functions. *J. Math. Inequal.* **14** (2020), no. 1, 135–145.

[12] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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