SOME INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we show that, if the function \( f : (0, 1) \to \mathbb{R} \) is operator monotone in \((0, 1)\) then there exist \( b \geq 0 \) and a positive measure \( m \) on \((0, 1)\) such that
\[
[f(B) - f(A)](B - A) = b(B - A)^2 \]
\[
+ \int_0^s s^2 \left[ \int_0^t [(1 - t) A + tB + s]^{-1} (B - A)^2 dt \right] dm(s)
\]
for all \( A, B > 0 \). Some necessary and sufficient conditions for the operators \( A, B > 0 \) such that the inequality
\[
f(B) B + f(A) A \geq f(A) B + f(B) A
\]
holds for any operator monotone function \( f \) on \((0, \infty)\) are also given.

1. Introduction

Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle T x, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible. A real valued continuous function \( f(t) \) on \((0, 1)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Theorem 1. A function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) if and only if it has the representation
\[
f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} dm(s)
\]
where \( a \in \mathbb{R} \) and \( b \geq 0 \) and a positive measure \( m \) on \((0, \infty)\) such that
\[
\int_0^\infty \frac{s}{1+s} dm(s) < \infty.
\]

We recall the important fact proved by Löwner and Heinz that states that the power function \( f : (0, \infty) \to \mathbb{R} \), \( f(t) = t^\alpha \) is an operator monotone function for any \( \alpha \in [0, 1] \), [8].

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In [5], T. Furuta observed that for \( \alpha_j \in [0,1], \ j = 1, \ldots, n \) the functions

\[
g(t) := \left( \sum_{j=1}^{n} t^{-\alpha_j} \right)^{-1} \quad \text{and} \quad h(t) = \sum_{j=1}^{n} (1 + t^{-1})^{-\alpha_j}
\]

are operator monotone in \((0, \infty)\).

Let \( f(t) \) be a continuous function \((0, \infty) \to (0, \infty)\). It is known that \( f(t) \) is operator monotone if and only if \( g(t) = t/f(t) =: f^*(t) \) is also operator monotone, see for instance [5] or [10].

Consider the family of functions defined on \((0, 1)\) and \( p \in [-1, 2] \setminus \{0, 1\} \) by

\[
f_p(t) := \frac{p-1}{p} \left( \frac{t^p - 1}{tp^{-1} - 1} \right)
\]

and

\[
f_0(t) := \frac{t}{1-t} \ln t, \quad f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean)}.
\]

We also have the functions of interest

\[
f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean)}, \quad f_{1/2}(t) = \sqrt{t} \text{ (geometric mean)}.
\]

In [4] the authors showed that \( f_p \) is operator monotone for \( 1 \leq p \leq 2 \).

In the same category, we observe that the function

\[
g_p(t) := \frac{t - 1}{tp - 1}
\]

is an operator monotone function for \( p \in (0, 1], [5] \).

It is well known that the logarithmic function \( \ln \) is operator monotone and in [5] the author obtained that the functions

\[
f(t) = t \left( 1 + t \right) \ln \left( 1 + \frac{1}{t} \right), \quad g(t) = \frac{1}{\left( 1 + t \right) \ln \left( 1 + \frac{1}{t} \right)}
\]

are also operator monotone functions on \((0, \infty)\).

For recent operator inequalities related to operator monotone functions, see [1], [11] and [12].

In this paper we show that, if that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) then there exist \( b \geq 0 \) and a positive measure \( m \) on \((0, \infty)\) such that

\[
\left[ f(B) - f(A) \right] (B - A) = b (B - A)^2
\]

\[+ \int_0^\infty s^2 \left[ \int_0^1 \left[ \left( (1-t) A + tB + s \right)^{-1} (B - A) \right]^2 dt \right] dm(s)
\]

for all \( A, B > 0 \). Some necessary and sufficient conditions for the operators \( A, B > 0 \) such that the inequality

\[
f(B) B + f(A) A \geq f(A) B + f(B) A
\]

holds for any operator monotone function \( f \) on \((0, \infty)\) are also given.
2. Main Results

We have the following identities of interest:

**Theorem 2.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) and has the representation (1.1). Then for all \( A, B > 0 \) we have

\[
[f(B) - f(A)] (B - A) = b(B - A)^2 + \int_0^\infty s^2 \left[ \int_0^1 \left[ ((1-t)A + tB + s)^{-1} (B - A) \right]^2 dt \right] dm(s)
\]

and

\[
(B - A) [f(B) - f(A)] = b(B - A)^2 + \int_0^\infty s^2 \left[ \int_0^1 (B - A) ((1-t)A + tB + s)^{-1} \right]^2 dt \right] dm(s).
\]

**Proof.** Since the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\), then \( f \) can be written as in the equation (1.1) and for \( A, B > 0 \) we have the representation

\[
f(B) - f(A) = b(B - A) + \int_0^\infty s \left[ (B + s)^{-1} - (A + s)^{-1} \right] dm(s).
\]

Observe that for \( s > 0 \)

\[
B(B + s)^{-1} - A(A + s)^{-1} = (B + s - s) (B + s)^{-1} - (A + s - s) (A + s)^{-1}
\]

\[
= (B + s) (B + s)^{-1} - s(B + s)^{-1} - (A + s) (A + s)^{-1} + s(A + s)^{-1}
\]

\[
= 1 - s(B + s)^{-1} - 1 + s(A + s)^{-1}
\]

\[
= s \left[ (A + s)^{-1} - (B + s)^{-1} \right].
\]

Therefore, (2.3) becomes, see also [6]

\[
f(B) - f(A) = b(B - A) + \int_0^\infty s \left[ (A + s)^{-1} - (B + s)^{-1} \right] dm(s).
\]

Let \( T, S > 0 \). The function \( f(t) = -t^{-1} \) is operator monotonic on \((0, \infty)\), operator Gâteaux differentiable and the Gâteaux derivative is given by

\[
\nabla f_T(S) := \lim_{t \to 0} \left[ \frac{f(T + tS) - f(T)}{t} \right] = T^{-1}ST^{-1}
\]

for \( T, S > 0 \).

Consider the continuous function \( f \) defined on an interval \( I \) for which the corresponding operator function is Gâteaux differentiable and for \( C, D \) selfadjoint operators with spectra in \( I \) we consider the auxiliary function defined on \([0, 1]\) by

\[
f_{C,D}(t) = f((1 - t)C + tD), \ t \in [0, 1].
\]
If \( f \) is Gâteaux differentiable on the segment \([C, D] := \{(1 - t) C + tD, \ t \in [0, 1]\}\), then we have, by the properties of the Bochner integral, that

\[
(2.6) \quad f(D) - f(C) = \int_0^1 \frac{dt}{dt} (f_{C,D}(t)) dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C) dt.
\]

If we write this equality for the function \( f(t) = -t^{-1} \) and \( C, D > 0 \), then we get the representation

\[
(2.7) \quad C^{-1} - D^{-1} = \int_0^1 ((1 - t) C + tD)^{-1} (D - C) ((1 - t) C + tD)^{-1} dt.
\]

Now, if we replace in (2.7) \( C = A + s1_H \) and \( D = B + s1_H \) for \( s > 0 \), then

\[
(2.8) \quad (A + s)^{-1} - (B + s)^{-1}
= \int_0^1 ((1 - t) A + tB + s)^{-1} (B - A) ((1 - t) A + tB + s)^{-1} dt.
\]

By the representation (2.4), we derive the following identity of interest

\[
(2.9) \quad f(B) - f(A) = b(B - A)
+ \int_0^\infty s^2 \left[ \int_0^1 ((1 - t) A + tB + s)^{-1}
\times (B - A) ((1 - t) A + tB + s)^{-1} dt \right] dm(s)
\]

for \( A, B > 0 \).

If we multiply this identity at the right with \( B - A \) we get

\[
(2.10) \quad (f(B) - f(A)) (B - A)
= b(B - A)^2
+ \int_0^\infty s^2 \left[ \int_0^1 ((1 - t) A + tB + s)^{-1}
\times (B - A) ((1 - t) A + tB + s)^{-1} (B - A) dt \right] dm(s)
= \int_0^\infty s^2 \left[ \int_0^1 ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt \right] dm(s)
\]

for \( A, B > 0 \) and the equality (2.1) is proved.

The equality (2.2) follows by multiplying (2.9) at the left. \( \square \)

In the following, in order to simplify terminology, when we write \( T \geq 0 \) we automatically assume that the operator \( T \) is selfadjoint.

In the note [3], Fujii and Nakamoto showed that the inequality

\[
(f(B) - f(A)) (B - A) \geq 0
\]
does not hold in general for \( A, B > 0 \).

They also proved the following interesting inequality:

**Proposition 1** ([3, Proposition 2]). If \( C, D > 0 \) and \( CD^{-1} + DC^{-1} \) is selfadjoint, then

\[
(2.11) \quad CD^{-1} + DC^{-1} \geq 2.
\]
Proof. Indeed, as shown in [3], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\text{Sp}(T)$ of $T$ is included in $(0, \infty)$, because $C, D > 0$ and $\text{Sp}(T) = \text{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\text{Sp}(V) = \{t + \frac{1}{t}, t \in \text{Sp}(T)\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$. □

As a consequence, they obtained the following result:

**Theorem 3** ([3, Theorem 6]). If

(i') Operator $A(B + s)^{-1} + B(A + s)^{-1}$ is selfadjoint for all $s \geq 0$, then $(B - A)(f(B) - f(A)) \geq 0$.

Some necessary and sufficient conditions for the operators $A, B > 0$ such that the inequality $(f(B) - f(A))(B - A) \geq 0$ holds for any operator monotone function $f$ on $(0, \infty)$ are included in the following theorem.

**Theorem 4.** Let $A, B > 0$. The following statements are equivalent:

(i) For all $s \geq 0$,

\begin{equation}
(2.12) \quad (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) \geq 2.
\end{equation}

(ii) For all $s \geq 0$,

\[
\int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 \, dt \geq 0.
\]

(iii) For all $s \geq 0$,

\[
(\ell_s(B) - \ell_s(A))(B - A) \geq 0,
\]

where $\ell_s(t) = -(t + s)^{-1}, t > 0$.

(iv) For all operator monotone function $f$ on $(0, \infty)$,

\begin{equation}
(2.13) \quad (f(B) - f(A))(B - A) \geq 0.
\end{equation}

(v) For all operator monotone function $f$ on $(0, \infty)$,

\begin{equation}
(2.14) \quad (B - A)(f(B) - f(A)) \geq 0.
\end{equation}

Proof. From (2.8) we have, by multiplying at right with $B - A$ that

\[
\left[ (A + s)^{-1} - (B + s)^{-1} \right] (B - A)
\]

\[
= \int_0^1 ((1 - t) A + tB + s)^{-1} (B - A) \left[ (1 - t) A + tB + s \right] (B - A) \, dt
\]

\[
= \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 \, dt
\]

for all $s \geq 0$.

Also,

\[
\left[ (A + s)^{-1} - (B + s)^{-1} \right] (B - A)
\]

\[
= \left[ (A + s)^{-1} - (B + s)^{-1} \right] [B + s - (A + s)]
\]

\[
= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2
\]

for all $s \geq 0$.
Therefore
\[(\ell_s (B) - \ell_s (A)) (B - A)\]
\[= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2\]
\[= \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt\]
for all \(s \geq 0\).

The identity (2.15) reveals that the statements (i), (ii) and (iii) are equivalent.

Since for fixed \(s \geq 0\), \(\ell_s (t)\) is operator monotone function on \((0, \infty)\), then statement (iv) implies (iii).

Assume that the function \(f : (0, \infty) \to \mathbb{R}\) is operator monotone in \((0, \infty)\). Then for all \(A, B > 0\) we have
\[(2.16)\]
\[f(B) - f(A)] (B - A)\]
\[= b (B - A)^2\]
\[+ \int_0^\infty s^2 \left[ \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt \right] dm (s)\]
\[\geq \int_0^\infty s^2 \left[ \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt \right] dm (s)\]
where \(b \geq 0\) and \(m\) is a positive measure on \((0, \infty)\).

If (ii) is valid, then
\[\int_0^\infty s^2 \left[ \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt \right] dm (s) \geq 0\]
and by (2.16) we obtain (2.13).

Define the operator \(K := (f(B) - f(A)) (B - A)\). Since
\[K^* = [(f(B) - f(A)) (B - A)]^* = (B - A)^* (f(B) - f(A))^*\]
\[= (B - A) (f(B) - f(A))\]
then the fact that \(K\) is selfadjoint is equivalent to
\[(f(B) - f(A)) (B - A) = (B - A) (f(B) - f(A)),\]
which is also equivalent to the fact that
\[f(A)B + f(B)A = Bf(A) + Af(B)\].
These prove the equivalence between (iv) and (v).

\[\square\]

**Remark 1.** The identity
\[(B - A) (\ell_s (B) - \ell_s (A)) = (B + s) (A + s)^{-1} + (A + s) (B + s)^{-1} - 2\]
for \(s \geq 0\) was the main tool in the proof of Theorem 6, [3].

We can state:

**Corollary 1.** Let \(A, B > 0\). The statement (i) is equivalent to the inequality
\[(2.17)\]
\[f(B)B + f(A)A \geq f(A)B + f(B)A,\]
for all \(f\) an operator monotone function on \((0, \infty)\).

Observe that, in fact we have:
Proposition 2. Let $A, B > 0$, then the statements (i) and (i') are equivalent.

Proof. Notice that for all $s \geq 0$,
\begin{equation}
(2.18) \quad (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) = (A + s)^{-1} B + (B + s)^{-1} A + s (A + s)^{-1} + s (B + s)^{-1}.
\end{equation}
Also, the operator $s (A + s)^{-1} + s (B + s)^{-1}$ is selfadjoint for $s \geq 0$.

If the statement (i) holds, then $(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s)$ is selfadjoint and by (2.18) we must have that $(A + s)^{-1} B + (B + s)^{-1} A$ is selfadjoint, which shows that
\[
\left( (A + s)^{-1} B + (B + s)^{-1} A \right)^* = B (A + s)^{-1} + A (B + s)^{-1}
\]
is selfadjoint, namely (i') is true.

If the statement (i') holds, then by (2.18) we get
\[
(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s)
\]
is selfadjoint and by (2.11) for $C = (A + s)^{-1}, D = (B + s)^{-1}$ we obtain the inequality (2.12), namely (i) is true. \hfill \square

We define the class of operators
\[
\mathcal{C}_{(0,\infty)}(H) := \{ (A, B) \mid A, B > 0 \text{ and satisfy condition (i')} \}.
\]

We observe that if $(A, B) \in \mathcal{C}_{(0,\infty)}(H)$ then $(B, A) \in \mathcal{C}_{(0,\infty)}(H)$.

Also if $AB = BA, A, B > 0$, then $U_s := (A + s)^{-1} (B + s)$ and $U_s^{-1} = (B + s)^{-1} (A + s)$ are selfadjoint and since $U_s + U_s^{-1} \geq 2, s \geq 0$ we derive that $(A, B) \in \mathcal{C}_{(0,\infty)}(H)$. Therefore, if $\mathcal{C}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators $A, B > 0$, then we have
\begin{equation}
(2.19) \quad \emptyset \neq \mathcal{C}_{(0,\infty)}(H) \subset \mathcal{C}_{(0,\infty)}(H).
\end{equation}

Corollary 2. Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$. If $g : I \to (0,\infty)$ is continuous, then for all selfadjoint operators $A, B$ with spectra in $I$ for which $(g(A), g(B)) \in \mathcal{C}_{(0,\infty)}(H)$ we have
\begin{equation}
(2.20) \quad (f \circ g)(B) g(B) + (f \circ g)(A) g(A) \geq (f \circ g)(A) g(B) + (f \circ g)(B) g(A).
\end{equation}

Follows by Theorem 2 by replacing $A$ with $g(A), B$ with $g(B)$ and using the composition rule for continuous functions of selfadjoint operators which gives that $f(g(A)) = (f \circ g)(A)$ and $f(g(B)) = (f \circ g)(B)$, see for instance [7, p. 49].

Corollary 3. Assume that the function $f : (0,\infty) \to (0,\infty)$ is operator monotone in $(0,\infty)$. Then
\begin{equation}
(2.21) \quad B^2 f(B)^{-1} + A^2 f(A)^{-1} \geq f(A)^{-1} AB + f(B)^{-1} BA,
\end{equation}
for all $(A, B) \in \mathcal{C}_{(0,\infty)}(H)$.

Follows by Corollary 1 and the fact that $t/f(t)$ is also operator monotone on $(0,\infty)$.

Remark 2. Since $f : (0,\infty) \to \mathbb{R}, f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$, then by (2.17) we get
\begin{equation}
(2.22) \quad B^{\alpha+1} + A^{\alpha+1} \geq A^\alpha B + B^\alpha A,
\end{equation}
for all \((A, B) \in \mathcal{C}_I(0, \infty) (H)\).

Since \(\ln\) is operator monotone on \((0, \infty)\), then by (2.17) we get

\[
\ln B + \ln A \geq (\ln A) B + (\ln B) A
\]

for all \((A, B) \in \mathcal{C}_I(0, \infty) (H)\).

Assume that \(A, B\) are selfadjoint operators such that \((\exp A, \exp B) \in \mathcal{C}_I(0, \infty) (H)\).

If we take \(f = \ln\), which is operator monotone on \((0, \infty)\) and \(g = \exp\), then by Corollary 2 we get

\[
B \ln B + A \ln A \geq (\ln A) B + (\ln B) A
\]

for all \((A, B) \in \mathcal{C}_I(0, \infty) (H)\).

Assume that \(A, B\) are selfadjoint operators such that \((\exp A, \exp B) \in \mathcal{C}_I(0, \infty) (H)\).

If we take \(f = \ln\), which is operator monotone on \((0, \infty)\) and \(g = \exp\), then by Corollary 2 we get

\[
B \exp B + A \exp A \geq \exp (A) B + \exp (B) A,
\]

Suppose that \(p > 1\). Then by taking \(f(t) = t^{1/p}\) and \(g(t) = t^{p}\), for \(t \in (0, \infty)\), in Corollary 2, we get

\[
B^{p+1} + A^{p+1} \geq AB^p + BA^p,
\]

for all \(A, B > 0\) such that \((A^p, B^p) \in \mathcal{C}_I(0, \infty) (H)\).

### 3. Some Integral and Discrete Inequalities

We have the following integral inequality:

**Proposition 3.** Assume that the function \(f : (0, \infty) \rightarrow \mathbb{R}\) is operator monotone in \((0, \infty)\). Then for all \(A, B > 0\) with \(((1-t)A + tB, \frac{A+B}{2}) \in \mathcal{C}_I(0, \infty) (H)\) for all \(t \in [0, 1]\),

\[
\int_0^1 ((1-t)A + tB) f ((1-t)A + tB) dt \geq \left( \int_0^1 f ((1-t)A + tB) dt \right) \frac{A+B}{2}
\]

**Proof.** From (2.17) we get

\[
((1-t)A + tB) f ((1-t)A + tB) + \frac{A+B}{2} f \left( \frac{A+B}{2} \right)
\]

\[
\geq f \left( \frac{A+B}{2} \right) ((1-t)A + tB) + f ((1-t)A + tB) \frac{A+B}{2}
\]

for all \(t \in [0, 1]\).

By taking the integral in (3.2) we get

\[
\int_0^1 ((1-t)A + tB) f ((1-t)A + tB) dt + \frac{A+B}{2} f \left( \frac{A+B}{2} \right)
\]

\[
\geq f \left( \frac{A+B}{2} \right) \int_0^1 ((1-t)A + tB) dt
\]

\[
+ \left( \int_0^1 f ((1-t)A + tB) dt \right) \frac{A+B}{2}
\]

and since

\[
\int_0^1 ((1-t)A + tB) dt = \frac{A+B}{2},
\]
hence by (3.3) we derive
\[
\int_0^1 ((1-t)A + tB) f ((1-t)A + tB) \, dt + \frac{A + B}{2} f \left( \frac{A + B}{2} \right) \\
\geq f \left( \frac{A + B}{2} \right) \frac{A + B}{2} + \left( \int_0^1 f ((1-t)A + tB) \, dt \right) \frac{A + B}{2},
\]
which is equivalent to the first inequality in (3.1).

The second inequality in (3.1) follows by the second part of (2.17).

\[\square\]

If we take \( f(t) = t^r \), \( r \in (0, 1) \) in (3.1), then we get
\[
\int_0^1 ((1-t)A + tB)^{r+1} \, dt \geq \left( \int_0^1 ((1-t)A + tB)^r \, dt \right) \frac{A + B}{2},
\]
for all \( A, B > 0 \) with \((1-t)A + tB, \frac{A + B}{2}) \in \mathcal{C}_{(0, \infty)}(H)\) for all \( t \in [0, 1] \).

Also, with the same assumptions for \( A \) and \( B \), by choosing \( f(t) = \ln t \), \( t > 0 \) in (3.1), then we get
\[
\int_0^1 ((1-t)A + tB) \ln ((1-t)A + tB) \, dt \\
\geq \left( \int_0^1 \ln ((1-t)A + tB) \, dt \right) \frac{A + B}{2},
\]
for all \( A, B > 0 \).

We have the following Chebychev type operator inequality:

**Proposition 4.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\). If \( g : I \to (0, \infty) \) is continuous, then for all selfadjoint operators \( A_k \), \( k = 1, \ldots, n \) with spectra in \( I \) and such that \((g(A_k), g(A_j)) \in \mathcal{C}_{(0, \infty)}(H)\) for \( j, k = 1, \ldots, n \) and \( p_k \geq 0 \), \( k = 1, \ldots, n \) with \( \sum_{k=1}^n p_k = 1 \),

\[
\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) \geq \sum_{k=1}^n p_k (f \circ g)(A_k) \sum_{k=1}^n p_k g(A_k).
\]

**Proof.** From (2.20) we get
\[
(f \circ g)(A_k) g(A_k) + (f \circ g)(A_j) g(A_j) \\
\geq (f \circ g)(A_j) g(A_k) + (f \circ g)(A_k) g(A_j)
\]
for all \( k, j \in \{1, \ldots, n\} \).

If we multiply (3.7) by \( p_k p_j \geq 0 \) and sum over \( k \) and \( j \) from 1 to \( n \), we get
\[
\sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_j) \\
\geq \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_j),
\]

(3.6)
which is equivalent to

\[
\sum_{k=1}^{n} p_k \left( f \circ g \right)(A_k) g(A_k) + \sum_{j=1}^{n} p_j \left( f \circ g \right)(A_j) g(A_j)
\]

\[
\geq \sum_{j=1}^{n} p_j \left( f \circ g \right)(A_j) \sum_{k=1}^{n} p_k g(A_k) + \sum_{k=1}^{n} p_k \left( f \circ g \right)(A_k) \sum_{j=1}^{n} p_j g(A_j)
\]

that is equivalent to the first part of (3.6).

\[\square\]

**Remark 3.** If the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) and \( A_k > 0, k = 1, \ldots, n \), with \((A_k, A_j) \in \mathcal{C}_{(0, \infty)}(H)\) for \( j, k = 1, \ldots, n \) and \( p_k \geq 0, k = 1, \ldots, n \) with \( \sum_{k=1}^{n} p_k = 1 \), then by (3.6),

\[
\sum_{k=1}^{n} p_k A_k f(A_k) \geq \sum_{k=1}^{n} p_k f(A_k) \sum_{k=1}^{n} p_k A_k.
\]

In particular, for \( f(t) = t^r, r \in (0, 1) \) we get

\[
\sum_{k=1}^{n} p_k A_k^{r+1} \geq \sum_{k=1}^{n} p_k A_k^r \sum_{k=1}^{n} p_k A_k.
\]

For \( f(t) = -t^{-1} \) we get

\[
1_H \leq \sum_{k=1}^{n} p_k A_k^{-1} \sum_{k=1}^{n} p_k A_k.
\]

We also have the logarithmic inequalities

\[
\sum_{k=1}^{n} p_k A_k \ln A_k \geq \sum_{k=1}^{n} p_k \ln A_k \sum_{k=1}^{n} p_k A_k.
\]

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**References**


[6] T. Furuta, Precise lower bound of \( f(A) - f(B) \) for \( A > B > 0 \) and non-constant operator monotone function \( f \) on \([0, \infty)\). *J. Math. Inequal.* 9 (2015), no. 1, 47–52.


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