SEVERAL INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS ON FINITE INTERVALS

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Abstract. In this paper we show that, if the function \( f : (-1, 1) \to \mathbb{R} \) is operator monotone in \((-1, 1)\), then there exists a positive measure \( \mu \) on \([-1, 1]\) such that

\[
[f(B) - f(A)](B - A) = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left[ 1 - \lambda ((1 - t) A + tB) \right]^{-1} (B - A) \right)^2 \, d\mu(\lambda)
\]

for all \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \). Some necessary and sufficient conditions for the operators \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \) such that the inequality

\[
f(B)B + f(A)A \geq f(A)B + f(B)A
\]

holds for any operator monotone function \( f \) on \((-1, 1)\) are also given.

1. Introduction

Consider a complex Hilbert space \( (H, \langle \cdot, \cdot \rangle) \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible. A real valued continuous function \( f(t) \) on \((0, \infty)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

**Theorem 1.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) if and only if it has the representation

\[
f(t) = a + bt + \int_{0}^{\infty} \frac{ts}{t+s} \, dm(s)
\]

where \( a \in \mathbb{R} \) and \( b \geq 0 \) and a positive measure \( m \) on \((0, \infty)\) such that

\[
\int_{0}^{\infty} \frac{s}{t+s} \, dm(s) < \infty.
\]

We recall the important fact proved by Löwner and Heinz that states that the power function \( f : (0, \infty) \to \mathbb{R}, f(t) = t^\alpha \) is an operator monotone function for any \( \alpha \in [0, 1], [8] \).

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In [5], T. Furuta observed that for \( \beta_j \in [0, 1], j = 1, ..., n \) the functions
\[
g(t) := \left( \sum_{j=1}^{n} t^{-\beta_j} \right)^{-1} \quad \text{and} \quad h(t) = \sum_{j=1}^{n} (1 + t^{-1})^{-\beta_j}
\]
are operator monotone in \((0, \infty)\).

Let \( f(t) \) be a continuous function \((0, \infty) \to (0, \infty)\). It is known that \( f(t) \) is operator monotone if and only if \( g(t) = t/f(t) =: f^*(t) \) is also operator monotone, see for instance [5] or [10].

Consider the family of functions defined on \((0, \infty)\) and \( p \in [-1, 2] \setminus \{0, 1\} \) by
\[
f_p(t) := \frac{p-1}{p} \left( \frac{t^p - 1}{p^p - 1} \right)
\]
and
\[
f_0(t) := \frac{t}{1-t} \ln t, \quad f_1(t) := \frac{t-1}{\ln t} \quad (\text{logarithmic mean}).
\]
We also have the functions of interest
\[
f_{-1}(t) = \frac{2t}{1+t} \quad (\text{harmonic mean}), \quad f_{1/2}(t) = \sqrt{t} \quad (\text{geometric mean}).
\]

In [4] the authors showed that \( f_p \) is operator monotone for \( 1 \leq p \leq 2 \). In the same category, we observe that the function
\[
g_p(t) := \frac{t-1}{p^p - 1}
\]
is an operator monotone function for \( p \in (0, 1], [5] \).

It is well known that the logarithmic function \( \ln \) is operator monotone and in [5] the author obtained that the functions
\[
f(t) = t(1 + t) \ln \left( 1 + \frac{1}{t} \right), \quad g(t) = \frac{1}{(1 + t) \ln \left( 1 + \frac{1}{t} \right)}
\]
are also operator monotone functions on \((0, \infty)\).

The case of operator monotone functions on finite intervals is as follows [2, p. 134]:

**Theorem 2.** Let \( f \) be a nonconstant operator monotone function on \((-1, 1)\). Then there exists a unique probability measure \( \mu \) on \([-1, 1]\) such that
\[
f(t) = f(0) + f'(0) \int_{-1}^{t} \frac{1}{1-\lambda t} d\mu(\lambda)
\]
for \( t \in (-1, 1) \), where \( f'(0) > 0 \).

It is well known that the function \( f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, f(t) = \tan t \) is operator monotone on \((-\frac{\pi}{2}, \frac{\pi}{2})\) and the function \( g \) is operator monotone on \((0, \infty)\) if and only if the function \( f : (-1, 1) \to \mathbb{R}, f(t) := g \left( \frac{1+i}{1-i} t \right) \) is operator monotone on \((-1, 1), [12]\). Therefore, the functions \( f(t) = \left( \frac{1+i}{1-i} t \right)^r \) and \( f(t) = \ln \left( \frac{1+i}{1-i} t \right) \) are operator monotone on \((-1, 1), [12]\).

For recent operator inequalities related to operator monotone functions, see [1], [11] and [13].
In this paper we show that, if the function \( f : (-1, 1) \to \mathbb{R} \) is operator monotone in \((-1, 1)\), then there exists a positive measure \( \mu \) on \([-1, 1]\) such that
\[
[f(B) - f(A)](B - A) = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} (1 - \lambda ((1 - t) A + tB))^{-1} (B - A) \, dt \right) d\mu(\lambda)
\]
for all \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \). Some necessary and sufficient conditions for the operators \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \) such that the inequality
\[
f(B)B + f(A)A \geq f(A)B + f(B)A
\]
holds for any operator monotone function \( f \) on \((-1, 1)\) are also given.

2. Main Results

We have the following representation results that is of interest in itself as well:

**Theorem 3.** Let \( f \) be a nonconstant operator monotone function on \((-1, 1)\) that satisfies the representation (1.2). Then,
\[
(2.1) \quad f(B) - f(A) = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} (1 - \lambda ((1 - t) A + tB))^{-1} \times (B - A) (1 - \lambda ((1 - t) A + tB))^{-1} \, dt \right) d\mu(\lambda).
\]
for all \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \).

**Proof.** For \( C \) a selfadjoint operator with \( \text{Sp}(C) \subset (-1, 1) \), then by (1.2) we have
\[
f(C) = f(0) + f'(0) \int_{-1}^{1} C (1 - \lambda C)^{-1} \, d\mu(\lambda).
\]
Therefore, for \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \),
\[
f(B) - f(A) = f'(0) \int_{-1}^{1} \left[ B (1 - \lambda B)^{-1} - A (1 - \lambda A)^{-1} \right] d\mu(\lambda).
\]
For \( \lambda \in [-1, 1], \lambda \neq 0 \) we put
\[
U_{\lambda} := B (1 - \lambda B)^{-1} - A (1 - \lambda A)^{-1}.
\]
Then
\[
\lambda U_{\lambda} = \lambda B (1 - \lambda B)^{-1} - \lambda A (1 - \lambda A)^{-1}
\]
\[
= (1 - 1 + \lambda B) (1 - \lambda B)^{-1} - (1 - 1 + \lambda A) (1 - \lambda A)^{-1}
\]
\[
= (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1}
\]
\[
= (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1},
\]
namely
\[
U_{\lambda} = \frac{(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1}}{\lambda}
\]
for \( \lambda \in [-1, 1], \lambda \neq 0 \). Using the identity
\[
X^{-1} - Y^{-1} = X^{-1}(Y - X)Y^{-1},
\]
we get
\[
\begin{align*}
(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} &= (1 - \lambda B)^{-1} (1 - \lambda A - 1 + \lambda B) (1 - \lambda A)^{-1} \\
&= \lambda (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}
\end{align*}
\]
and then
\[
\lambda U_\lambda = \lambda (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}
\]
for \( \lambda \in [-1, 1] \), \( \lambda \neq 0 \). Therefore
\[
U_\lambda = (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}, \quad \lambda \in [-1, 1]
\]
and we get the identity of interest
\[
f(B) - f(A) = f'(0) \int_{-1}^{1} (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1} d\mu(\lambda).
\]
For small \( t \) around 0 we have
\[
f(A + tC) - f(A) = f'(0) t \int_{-1}^{1} [ (1 - \lambda (A + tC))^{-1} C (1 - \lambda A)^{-1} ] d\mu(\lambda),
\]
for \( C \in B(H), \) \( C \) selfadjoint and \( A \) with \( \text{Sp}(A) \subset (-1, 1) \), which implies that the Gâteaux derivative in \( A \) of \( f \) exists and
\[
\nabla_A f(C) = f'(0) \int_{-1}^{1} (1 - \lambda A)^{-1} C (1 - \lambda A)^{-1} d\mu(\lambda).
\]
Let \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \). Consider the auxiliary function \( \varphi_{A,B} : [0,1] \to B(H), \)
\[
\varphi_{A,B}(t) := f((1-t)A + tB).
\]
Then
\[
\frac{d\varphi_{A,B}(t)}{dt} = \nabla_{(1-t)A+tB} f(B - A), \quad t \in (-1,1)
\]
and by (2.3) we get
\[
\frac{d\varphi_{A,B}(t)}{dt} = f'(0) \int_{-1}^{1} (1 - \lambda ((1-t)A + tB))^{-1} \\
\times (B - A) (1 - \lambda ((1-t)A + tB))^{-1} d\mu(\lambda)
\]
for \( t \in (-1,1) \).
Since
\[
f(B) - f(A) = \varphi_{A,B}(1) - \varphi_{A,B}(0) = \int_{0}^{1} \frac{d\varphi_{A,B}(t)}{dt} dt,
\]
hence by (2.4) we get
\[
f(B) - f(A) = f'(0) \int_{0}^{1} \left( \int_{-1}^{1} (1 - \lambda ((1-t)A + tB))^{-1} \\
\times (B - A) (1 - \lambda ((1-t)A + tB))^{-1} d\mu(\lambda) \right) dt
\]
and by Fubini’s theorem we get (2.1). \( \square \)
Corollary 1. With the assumptions of Theorem 3, we have the norm inequality
\[ \| f(B) - f(A) \| \leq f'(0) \| B - A \| \]
\[ \times \int_{-1}^{1} \left( \int_{0}^{1} \left\| (1 - \lambda ((1 - t) A + tB))^{-1} \right\|^2 dt \right) d\mu(\lambda). \]

Theorem 4. Let \( f \) be a nonconstant operator monotone function on \((-1,1)\) that satisfies the representation (1.2), then
\[ \left[ f(B) - f(A) \right] (B - A) \]
\[ = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left[ (1 - \lambda ((1 - t) A + tB))^{-1} (B - A) \right] \right)^2 dt \right) d\mu(\lambda) \]
and
\[ \left[ f(B) - f(A) \right] (B - A) \]
\[ = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left[ (1 - \lambda ((1 - t) A + tB))^{-1} (B - A) \right] \right)^2 dt \right) d\mu(\lambda), \]
for all \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1,1) \).

Proof. If we multiply the equality (2.1) at the right with \( B - A \) we get
\[ \left( f(B) - f(A) \right) (B - A) \]
\[ = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left(1 - \lambda ((1 - t) A + tB))^{-1} \right) (B - A) dt \right) d\mu(\lambda) \]
\[ = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left[ (1 - \lambda ((1 - t) A + tB))^{-1} (B - A) \right] \right)^2 dt \right) d\mu(\lambda), \]
which proves (2.6).

The proof of (2.7) follows in a similar way. \( \Box \)

We also have:

Theorem 5. Let \( f \) be a nonconstant operator monotone function on \((-1,1)\) that satisfies the representation (1.2), then for \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1,1) \) and
\[ M \geq B - A \geq m, \]
for some real numbers \( M > m \), we have
\[ M f'(0) \int_{-1}^{1} \left[ \int_{0}^{1} \left(1 - \lambda ((1 - t) A + tB))^{-2} \right) dt \right] d\mu(\lambda) \]
\[ \geq f(B) - f(A) \]
\[ \geq m f'(0) \int_{-1}^{1} \left[ \int_{0}^{1} \left(1 - \lambda ((1 - t) A + tB))^{-2} \right) dt \right] d\mu(\lambda). \]

Proof. If \( B - A \geq m \), then
\[ \left(1 - \lambda ((1 - t) A + tB))^{-1} (B - A) (1 - \lambda ((1 - t) A + tB))^{-1} \right) dt \]
\[ \geq m \left(1 - \lambda ((1 - t) A + tB))^{-2} \right) \]
for \( t \in [0, 1] \) and \( \lambda \in [-1, 1] \), which implies that

\[
f(B) - f(A) \geq m f'(0) \int_0^1 \left( \int_0^t (1 - \lambda ((1 - t) A + tB))^{-2} \, dt \right) d\mu(\lambda).
\]

If \( M \geq B - A \), then in a similar way

\[
M f'(0) \int_{-1}^1 \left[ \int_0^1 (1 - \lambda ((1 - t) A + tB))^{-2} \, dt \right] d\mu(\lambda) \geq f(B) - f(A)
\]

and the theorem is proved.

**Corollary 2.** With the assumptions of Theorem 5 and if \( B - A \geq m > 0 \), then

\[
(2.10) \quad f(B) - f(A) \geq m f'(0) \left[ 1 + \left( \int_{-1}^1 \lambda \mu(\lambda) \right) (A + B) \right.
\]

\[
+ \frac{1}{3} \left( \int_{-1}^1 \lambda^2 \mu(\lambda) \right) \left( A^2 + B^2 + \frac{AB + BA}{2} \right)
\]

\[
\geq 0.
\]

**Proof.** Since for \( x < 1 \), then

\[
\frac{1}{1 - x} \geq 1 + x,
\]

which implies that

\[
(1 - x)^{-2} \geq (1 + x)^2 = 1 + 2x + x^2 \geq 0.
\]

We then get

\[
(1 - \lambda((1 - t) A + tB))^{-2}
\]

\[
\geq 1 + 2\lambda((1 - t) A + tB) + \lambda^2 ((1 - t) A + tB)^2
\]

\[
= 1 + 2\lambda((1 - t) A + tB)
\]

\[
+ \lambda^2 \left[ (1 - t)A^2 + t((1 - t)AB + t(1 - t)BA + t^2B^2) \right] \geq 0
\]

for \( t \in [0, 1] \), \( \lambda \in [-1, 1] \).

If we take the integral over \( t \) in this inequality, then we get

\[
\int_0^1 (1 - \lambda((1 - t) A + tB))^{-2} \, dt
\]

\[
\geq 1 + \lambda(A + B) + \frac{1}{3} \lambda^2 \left( A^2 + B^2 + \frac{AB + BA}{2} \right) \geq 0
\]

for \( \lambda \in [-1, 1] \).

Taking the integral over \( \mu(\lambda) \) we get

\[
\int_{-1}^1 \left[ \int_0^1 (1 - \lambda((1 - t) A + tB))^{-2} \, dt \right] d\mu(\lambda)
\]

\[
\geq 1 + \left( \int_{-1}^1 \lambda \mu(\lambda) \right) (A + B)
\]

\[
+ \frac{1}{3} \left( \int_{-1}^1 \lambda^2 \mu(\lambda) \right) \left( A^2 + B^2 + \frac{AB + BA}{2} \right) \geq 0,
\]

which, by the second inequality in (2.9) gives the desired result (2.10). \(\square\)
In the following, in order to simplify terminology, when we write \( T \geq 0 \) we automatically assume that the operator \( T \) is selfadjoint.

We give now some necessary and sufficient conditions for the operators \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1,1) \) such that the inequality \( (B - A) (f(B) - f(A)) \geq 0 \) holds.

**Theorem 6.** Let \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1,1) \). The following statements are equivalent:

(i) For all for \( \lambda \in [-1,1] \),

\[
(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) \geq 2.
\]

(ii) For all for \( \lambda \in [-1,1] \),

\[
\int_0^1 \left[ (1 - \lambda [(1-t)B + \lambda tA])^{-1} (B - A) \right]^2 dt \geq 0.
\]

(iii) For all for \( \lambda \in [-1,1] \),

\[
(\kappa_\lambda (B) - \kappa_\lambda (A)) (B - A) \geq 0,
\]

where \( \kappa_\lambda (t) = t (1 - \lambda t)^{-1}, t \in [-1,1]. \)

(iv) For all operator monotone function \( f \) on \((-1,1)\),

\[
(f(B) - f(A)) (B - A) \geq 0.
\]

(v) For all operator monotone function \( f \) on \((-1,1)\),

\[
(B - A) (f(B) - f(A)) \geq 0.
\]

**Proof.** Consider the continuous function \( f \) defined on an interval \( I \) for which the corresponding operator function is Gâteaux differentiable and for \( C, D \) selfadjoint operators with spectra in \( I \) we consider the auxiliary function defined on \([0,1]\) by

\[
f_{C,D}(t) = f ((1-t)C + tD), \quad t \in [0,1].
\]

If \( f_{C,D} \) is Gâteaux differentiable on the segment \([C,D] := \{(1-t)C + tD, \ t \in [0,1]\}\), then we have, by the properties of the Bochner integral, that

\[
f(D) - f(C) = \int_0^1 \frac{d}{dt} (f_{C,D}(t)) \ dt = \int_0^1 \nabla f_{(1-t)C+tD} (D-C) \ dt.
\]

If we write this equality for the function \( f(t) = -t^{-1} \) and \( C, D > 0 \), then we get the representation

\[
C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D-C) ((1-t)C + tD)^{-1} dt.
\]

From (2.14) we have

\[
(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1}
\]

\[
= \int_0^1 ((1-t) (1 - \lambda B) + t (1 - \lambda A))^{-1} ((1 - \lambda A) - (1 - \lambda B))
\]

\[
\times ((1-t) (1 - \lambda B) + t (1 - \lambda A))^{-1} dt
\]

\[
= \lambda \int_0^1 (1 - \lambda [(1-t)B + \lambda tA])^{-1} (B - A)
\]

\[
\times (1 - \lambda [(1-t)B + \lambda tA])^{-1} dt
\]
for \( \lambda \in [-1, 1] \).

If we multiply this equality at right by \( B - A \), we get
\[
\left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A)
\]
\[
= \lambda \int_0^1 (1 - \lambda [(1 - t) B + \lambda t A])^{-1} (B - A) \times (1 - \lambda [(1 - t) B + \lambda t A])^{-1} (B - A) \, dt
\]
\[
= \lambda \int_0^1 \left[(1 - \lambda [(1 - t) B + \lambda t A])^{-1} (B - A) \right]^2 dt
\]
for \( \lambda \in [-1, 1], \lambda \neq 0 \), namely
\[
V_\lambda := \frac{1}{\lambda} \left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A)
\]
\[
= \int_0^1 \left[(1 - \lambda [(1 - t) B + \lambda t A])^{-1} (B - A) \right]^2 dt.
\]

Also,
\[
\left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A)
\]
\[
= \frac{1}{\lambda} \left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (\lambda B - \lambda A)
\]
\[
= \frac{1}{\lambda} \left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (1 - \lambda A - (1 - \lambda B))
\]
\[
= \frac{1}{\lambda} \left[ (1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right],
\]
which gives that
\[
V_\lambda = \frac{1}{\lambda} \left[ (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A)
\]
\[
= \frac{1}{\lambda^2} \left[ (1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right]
\]
for \( \lambda \in [-1, 1], \lambda \neq 0 \).

We conclude that
\[
(k_\lambda (B) - k_\lambda (A)) (B - A)
\]
\[
= \frac{1}{\lambda^2} \left[ (1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right]
\]
\[
= \int_0^1 \left[(1 - \lambda [(1 - t) B + \lambda t A])^{-1} (B - A) \right]^2 dt
\]
for \( \lambda \in [-1, 1], \lambda \neq 0 \).

We observe that the equivalence between the statements (i), (ii) and (iii) follows
directly from the identity (2.15).

Since the function \( k_\lambda (t) = t (1 - \lambda t)^{-1} \) is operator monotone on \((-1, 1)\),
the statement (iv) implies (iii).
If (ii) holds, then by representation (2.6) we have
\[
[f(B) - f(A)](B - A) = f'(0) \int_{-1}^{1} \left( \int_{0}^{1} \left[ (1 - \lambda ((1 - t)A + tB))^{-1}(B - A) \right]^2 dt \right) d\mu(\lambda) \geq 0,
\]
which shows that (iv) holds.

Define the operator \( K := (f(B) - f(A))(B - A) \). Since
\[
K^* = [(f(B) - f(A))(B - A)]^* = (B - A)^*(f(B) - f(A))^*
\]
then the fact that \( K \) is selfadjoint is equivalent to
\[
(f(B) - f(A))(B - A) = (B - A)(f(B) - f(A)),
\]
which is also equivalent to the fact that
\[
f(A)B + f(B)A = Bf(A) + Af(B).
\]
These prove the equivalence between (iv) and (v).

Therefore, we can state:

**Corollary 3.** Let \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \). The statement (i) is equivalent to the inequality
\[
(2.16) \quad f(B)B + f(A)A \geq f(A)B + f(B)A,
\]
for all \( f \) an operator monotone function on \((-1, 1)\).

In the note [3] Fujii and Nakamoto showed that the inequality (2.13) does not hold in general for \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \). They also proved that if \( C, D > 0 \) and \( CD^{-1} + DC^{-1} \) is selfadjoint, then
\[
(2.17) \quad CD^{-1} + DC^{-1} \geq 2.
\]

Indeed, as shown in [3], if we put \( T = CD^{-1} \), then \( V = T + T^{-1} \) is selfadjoint by the assumption. Note that the spectrum \( \text{Sp}(T) \) of \( T \) is included in \((0, \infty)\), because \( C, D > 0 \) and \( \text{Sp}(T) = \text{Sp}(C^{1/2}D^{-1}C^{1/2}) \). Since \( \text{Sp}(V) = \{ t + \frac{1}{t}, \ t \in \text{Sp}(T) \} \) by the spectral mapping theorem for rational functions, hence we have \( T + T^{-1} \geq 2 \).

Observe that, in fact we have:

**Proposition 1.** Let \( A, B \) with \( \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \), then the statements (i) and

(i') The operator \( A(1 - \lambda B)^{-1} + B(1 - \lambda A)^{-1} \) is selfadjoint for all \( \lambda \in [-1, 1] \),

are equivalent.

**Proof.** Notice that for all \( \lambda \in [-1, 1] \),
\[
(2.18) \quad (1 - \lambda B)^{-1}(1 - \lambda A) + (1 - \lambda A)^{-1}(1 - \lambda B)
\]
\[
= (1 - \lambda B)^{-1} + (1 - \lambda A)^{-1} - \lambda \left[ (1 - \lambda B)^{-1}A + (1 - \lambda A)^{-1}B \right].
\]
Also, the operator \( (1 - \lambda B)^{-1} + (1 - \lambda A)^{-1} \) is selfadjoint for \( \lambda \in [-1, 1] \).
If the statement \((i)\) holds, then \((1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B)\) is selfadjoint and by (2.18) we must have that \((1 - \lambda B)^{-1} A + (1 - \lambda A)^{-1} B\) is selfadjoint, which shows that

\[
\left((1 - \lambda B)^{-1} A + (1 - \lambda A)^{-1} B\right)^* = A (1 - \lambda B)^{-1} + B (1 - \lambda A)^{-1}
\]
is selfadjoint, namely \((i')\) is true.

If the statement \((i')\) holds, then by (2.18) we get

\[
(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B)
\]
is selfadjoint and by (2.17) for \(C = (1 - \lambda B)^{-1}, D = (1 - \lambda A)^{-1}\) we obtain the inequality (2.11), namely \((i)\) is true. \(\square\)

We define the class of operators

\[
\mathcal{C}_{(-1,1)}(H) := \{(A, B) | \text{ Sp}(A), \text{ Sp}(B) \subset (-1, 1) \text{ and satisfy condition } (i')\}.
\]

We observe that if \((A, B) \in \mathcal{C}_{(-1,1)}(H)\) then \((B, A) \in \mathcal{C}_{(-1,1)}(H)\).

Also if \(AB = BA\) with \(\text{ Sp}(A), \text{ Sp}(B) \subset (-1, 1)\) then \(V = (1 - \lambda B)^{-1} (1 - \lambda A)\) and \(V^{-1} = (1 - \lambda A)^{-1} (1 - \lambda B), \lambda \in [-1, 1],\) are selfadjoint and since \(V + V^{-1} \geq 2, \lambda \in [-1, 1]\) we derive that \((A, B) \in \mathcal{C}_{(-1,1)}(H)\). Therefore, if \(\mathcal{C}_0(-1,1)(H)\) is the class of all pairs of commutative operators \(A, B\) with \(\text{ Sp}(A), \text{ Sp}(B) \subset (-1, 1)\), then we have

\[
(2.19) \quad \emptyset \neq \mathcal{C}_0(-1,1)(H) \subset \mathcal{C}_{(-1,1)}(H).
\]

3. Discrete Inequalities

We have the following Chebychev type operator inequality:

**Proposition 2.** Assume that the function \(f : (-1,1) \to \mathbb{R}\) is operator monotone in \((-1,1)\). If \(g : I \to (-1,1)\) is continuous, then for all selfadjoint operators \(A_k, k = 1,\ldots,n\) with spectra in \(I\) such that \((g(A_k), g(A_j)) \in \mathcal{C}_{(-1,1)}(H)\) for all \(k, j = 1,\ldots,n\) and \(p_k \geq 0, k = 1,\ldots,n\) with \(\sum_{k=1}^n p_k = 1\), we have

\[
(3.1) \quad \sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) \geq \sum_{k=1}^n p_k g(A_k) \sum_{k=1}^n p_k (f \circ g)(A_k).
\]

If \(A_k, k = 1,\ldots,n,\) with \(\text{ Sp}(A_k) \subset (-1,1), (A_k, A_j) \in \mathcal{C}_{(-1,1)}(H)\) for all \(k, j = 1,\ldots,n\) and \(p_k \geq 0, k = 1,\ldots,n\) with \(\sum_{k=1}^n p_k = 1\), then

\[
(3.2) \quad \sum_{k=1}^n p_k f(A_k) A_k \geq \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k f(A_k).
\]

**Proof.** From (2.16) we get

\[
(3.3) \quad (f \circ g)(A_k) g(A_k) + (f \circ g)(A_j) g(A_j)
\]

\[
\geq (f \circ g)(A_j) g(A_k) + (f \circ g)(A_k) g(A_j)
\]

for all \(k, j \in \{1,\ldots,n\}\).
If we multiply (3.3) by $p_k p_j \geq 0$ and sum over $k$ and $j$ from 1 to $n$, we get
\[
\sum_{k=1}^{n} \sum_{j=1}^{n} p_k p_j (f \circ g) (A_k) g (A_k) + \sum_{k=1}^{n} \sum_{j=1}^{n} p_k p_j (f \circ g) (A_j) g (A_j) \\
\geq \sum_{k=1}^{n} \sum_{j=1}^{n} p_k p_j (f \circ g) (A_k) g (A_k) + \sum_{k=1}^{n} \sum_{j=1}^{n} p_k p_j (f \circ g) (A_k) g (A_j), 
\]
which is equivalent to
\[
\sum_{k=1}^{n} p_k (f \circ g) (A_k) g (A_k) + \sum_{j=1}^{n} p_j (f \circ g) (A_j) g (A_j) \\
\geq \sum_{j=1}^{n} p_j (f \circ g) (A_j) \sum_{k=1}^{n} p_k g (A_k) + \sum_{k=1}^{n} p_k (f \circ g) (A_k) \sum_{j=1}^{n} p_j g (A_j)
\]
that is equivalent to the first part of (3.1).

\[\square\]

**Remark 1.** If $A_k, k = 1, \ldots, n$, with $\text{Sp} (A_k) \subset (-1, 1)$, $(A_k, A_j) \in \mathcal{C}(-1, 1) (H)$ for all $k, j = 1, \ldots, n$ and $p_k \geq 0$, $k = 1, \ldots, n$ with $\sum_{k=1}^{n} p_k = 1$, then by (3.2),
\[
(3.4) \quad \sum_{k=1}^{n} p_k A_k \tan (A_k) \geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \tan (A_k).
\]

Also for $r \in (0, 1),$
\[
(3.5) \quad \sum_{k=1}^{n} p_k A_k \left[ (1_H + A_k) (1_H - A_k)^{-1} \right]^r \\
\geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \left[ (1_H + A_k) (1_H - A_k)^{-1} \right]^r.
\]

We also have
\[
(3.6) \quad \sum_{k=1}^{n} p_k A_k \ln \left[ (1_H + A_k) (1_H - A_k)^{-1} \right] \\
\geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \ln \left[ (1_H + A_k) (1_H - A_k)^{-1} \right].
\]

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