SEVERAL INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS ON FINITE INTERVALS

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Abstract. In this paper we show that, if the function $f:(-1,1)\to\mathbb{R}$ is operator monotone in $(-1,1)$, then there exists a positive measure μ on $[-1,1]$ such that

$$[f(B)-f(A)](B-A)$$
$$=f'(0)\int_{-1}^{1}\left(\int_0^1\left[(1-\lambda((1-t)A+tB))^{-1}(B-A)\right]^2dt\right)d\mu(\lambda)$$

for all A, B with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset(-1,1)$. Some necessary and sufficient conditions for the operators A, B with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset(-1,1)$ such that the inequality

$$f(B)B+f(A)A\geq f(A)B+f(B)A$$

holds for any operator monotone function f on $(-1,1)$ are also given.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{ts}{t+s}dm(s)\tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty\frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [8].

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In [5], T. Furuta observed that for $\alpha_j \in [0,1]$, $j = 1, ..., n$ the functions

$$g(t) := \left(\sum_{j=1}^{n} t^{-\alpha_j} \right)^{-1} \text{ and } h(t) = \sum_{j=1}^{n} \left(1 + t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0, \infty)$.

Let $f(t)$ be a continuous function $(0,\infty) \to (0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [5] or [10].

Consider the family of functions defined on $(0,\infty)$ and $p \in [-1,2] \setminus \{0,1\}$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

and

$$f_0(t) := \frac{t}{1-t} \ln t, \; f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean), } f_{1/2}(t) = \sqrt{t} \text{ (geometric mean).}$$

In [4] the authors showed that f_p is operator monotone for $1 \le p \le 2$.

In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p - 1}$$

is an operator monotone function for $p \in (0,1]$, [5].

It is well known that the logarithmic function ln is operator monotone and in [5] the author obtained that the functions

$$f(t) = t(1+t)\ln\left(1 + \frac{1}{t}\right), \; g(t) = \frac{1}{(1+t)\ln\left(1+\frac{1}{t}\right)}$$

are also operator monotone functions on $(0,\infty)$.

The case of operator monotone functions on finite intervals is as follows [2, p. 134]:

Theorem 2. *Let f be a nonconstant operator monotone function on $(-1,1)$. Then there exists a unique probability measure μ on $[-1,1]$ such that*

$$f(t) = f(0) + f'(0) \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda) \tag{1.2}$$

for $t \in (-1,1)$, where $f'(0) > 0$.

It is well known that the function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(t) = \tan t$ is operator monotone on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the function g is operator monotone on $(0,\infty)$ if and only if the function $f : (-1,1) \to \mathbb{R}$, $f(t) := g\left(\frac{1+t}{1-t}\right)$ is operator monotone on $(-1,1)$, [12]. Therefore, the functions $f(t) = \left(\frac{1+t}{1-t}\right)^r$ and $f(t) = \ln\left(\frac{1+t}{1-t}\right)$ are operator monotone on $(-1,1)$.

For recent operator inequalities related to operator monotone functions, see [1], [11] and [13].

In this paper we show that, if the function $f:(-1,1)\to\mathbb{R}$ is operator monotone in $(-1,1)$, then there exists a positive measure μ on $[-1,1]$ such that

$$\begin{aligned}&[f(B)-f(A)](B-A)\\&=f'(0)\int_{-1}^{1}\left(\int_0^1\left[(1-\lambda((1-t)A+tB))^{-1}(B-A)\right]^2dt\right)d\mu(\lambda)\end{aligned}$$

for all A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$. Some necessary and sufficient conditions for the operators A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$ such that the inequality

$$f(B)B+f(A)A\geq f(A)B+f(B)A$$

holds for any operator monotone function f on $(-1,1)$ are also given.

2. Main Results

We have the following representation results that is of interest in itself as well:

Theorem 3. *Let f be a nonconstant operator monotone function on $(-1,1)$ that satisfies the representation (1.2). Then,*

$$\begin{aligned}f(B)-f(A)=f'(0)\int_{-1}^{1}\left(\int_0^1(1-\lambda((1-t)A+tB))^{-1}\right.\\\left.\times(B-A)(1-\lambda((1-t)A+tB))^{-1}dt\right)d\mu(\lambda)\end{aligned}\tag{2.1}$$

for all A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$.

Proof. For C a selfadjoint operator with $\operatorname{Sp}(C)\subset(-1,1)$, then by (1.2) we have

$$f(C)=f(0)+f'(0)\int_{-1}^{1}C(1-\lambda C)^{-1}d\mu(\lambda).$$

Therefore, for A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$,

$$f(B)-f(A)=f'(0)\int_{-1}^{1}\left[B(1-\lambda B)^{-1}-A(1-\lambda A)^{-1}\right]d\mu(\lambda).$$

For $\lambda\in[-1,1]$, $\lambda\neq0$ we put

$$U_\lambda:=B(1-\lambda B)^{-1}-A(1-\lambda A)^{-1}.$$

Then

$$\begin{aligned}\lambda U_\lambda&=\lambda B(1-\lambda B)^{-1}-\lambda A(1-\lambda A)^{-1}\\&=(1-1+\lambda B)(1-\lambda B)^{-1}-(1-1+\lambda A)(1-\lambda A)^{-1}\\&=(1-\lambda B)^{-1}-1-\left((1-\lambda A)^{-1}-1\right)\\&=(1-\lambda B)^{-1}-(1-\lambda A)^{-1},\end{aligned}$$

namely

$$U_\lambda=\frac{(1-\lambda B)^{-1}-(1-\lambda A)^{-1}}{\lambda}$$

for $\lambda\in[-1,1]$, $\lambda\neq0$.

Using the identity

$$X^{-1}-Y^{-1}=X^{-1}(Y-X)Y^{-1},$$

we get

$$(1-\lambda B)^{-1}-(1-\lambda A)^{-1} \tag{2.2}$$
$$=(1-\lambda B)^{-1}(1-\lambda A-1+\lambda B)(1-\lambda A)^{-1}$$
$$=\lambda(1-\lambda B)^{-1}(B-A)(1-\lambda A)^{-1}$$

and then

$$\lambda U_\lambda=\lambda(1-\lambda B)^{-1}(B-A)(1-\lambda A)^{-1}$$

for $\lambda\in[-1,1]$, $\lambda\neq 0$.

Therefore

$$U_\lambda=(1-\lambda B)^{-1}(B-A)(1-\lambda A)^{-1},\ \lambda\in[-1,1]$$

and we get the identity of interest

$$f(B)-f(A)=f'(0)\int_{-1}^{1}(1-\lambda B)^{-1}(B-A)(1-\lambda A)^{-1}d\mu(\lambda).$$

For small t around 0 we have

$$f(A+tC)-f(A)$$
$$=f'(0)\,t\int_{-1}^{1}\left[(1-\lambda(A+tC))^{-1}C(1-\lambda A)^{-1}\right]d\mu(\lambda),$$

for $C\in B(H)$, C selfadjoint and A with $\operatorname{Sp}(A)\subset(-1,1)$, which implies that the Gâteaux derivative in A of f exists and

$$\nabla_A f(C)=f'(0)\int_{-1}^{1}(1-\lambda A)^{-1}C(1-\lambda A)^{-1}d\mu(\lambda). \tag{2.3}$$

Let A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$. Consider the auxiliary function $\varphi_{A,B}:[0,1]\to B(H)$,

$$\varphi_{A,B}(t):=f((1-t)A+tB).$$

Then

$$\frac{d\varphi_{A,B}(t)}{dt}=\nabla_{(1-t)A+tB}f(B-A),\ t\in(-1,1)$$

and by (2.3) we get

$$\frac{d\varphi_{A,B}(t)}{dt}=f'(0)\int_{-1}^{1}(1-\lambda((1-t)A+tB))^{-1} \tag{2.4}$$
$$\times(B-A)(1-\lambda((1-t)A+tB))^{-1}d\mu(\lambda)$$

for $t\in(-1,1)$.

Since

$$f(B)-f(A)=\varphi_{A,B}(1)-\varphi_{A,B}(0)=\int_0^1\frac{d\varphi_{A,B}(t)}{dt}dt,$$

hence by (2.4) we get

$$f(B)-f(A)=f'(0)\int_0^1\left(\int_{-1}^{1}(1-\lambda((1-t)A+tB))^{-1}\right.$$
$$\left.\times(B-A)(1-\lambda((1-t)A+tB))^{-1}d\mu(\lambda)\right)dt$$

and by Fubini's theorem we get (2.1). □

Corollary 1. *With the assumptions of Theorem 3, we have the norm inequality*

$$\|f(B)-f(A)\| \le f'(0)\|B-A\| \times \int_{-1}^{1}\left(\int_0^1 \left\|(1-\lambda((1-t)A+tB))^{-1}\right\|^2 dt\right) d\mu(\lambda). \tag{2.5}$$

Theorem 4. *Let f be a nonconstant operator monotone function on $(-1,1)$ that satisfies the representation (1.2), then*

$$[f(B)-f(A)](B-A) = f'(0)\int_{-1}^{1}\left(\int_0^1 \left[(1-\lambda((1-t)A+tB))^{-1}(B-A)\right]^2 dt\right) d\mu(\lambda) \tag{2.6}$$

and

$$(B-A)[f(B)-f(A)] = f'(0)\int_{-1}^{1}\left(\int_0^1 \left[(B-A)(1-\lambda((1-t)A+tB))^{-1}\right]^2 dt\right) d\mu(\lambda) \tag{2.7}$$

for all A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$.

Proof. If we multiply the equality (2.1) at the right with $B-A$ we get

$$\begin{aligned}(f(B)-f(A))(B-A) &= f'(0)\int_{-1}^{1}\left(\int_0^1 (1-\lambda((1-t)A+tB))^{-1}\right.\\ &\quad\left.\times(B-A)(1-\lambda((1-t)A+tB))^{-1}(B-A)\,dt\right)d\mu(\lambda)\\ &= f'(0)\int_{-1}^{1}\left(\int_0^1 \left[(1-\lambda((1-t)A+tB))^{-1}(B-A)\right]^2 dt\right) d\mu(\lambda),\end{aligned} \tag{2.8}$$

which proves (2.6).

The proof of (2.7) follows in a similar way. □

We also have:

Theorem 5. *Let f be a nonconstant operator monotone function on $(-1,1)$ that satisfies the representation (1.2), then for A, B with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$ *and*

$$M \ge B-A \ge m,$$

for some real numbers $M>m$, we have

$$\begin{aligned}&Mf'(0)\int_{-1}^{1}\left[\int_0^1 (1-\lambda((1-t)A+tB))^{-2}dt\right]d\mu(\lambda)\\ &\ge f(B)-f(A)\\ &\ge mf'(0)\int_{-1}^{1}\left[\int_0^1 (1-\lambda((1-t)A+tB))^{-2}dt\right]d\mu(\lambda).\end{aligned} \tag{2.9}$$

Proof. If $B-A\ge m$, then

$$\begin{aligned}&(1-\lambda((1-t)A+tB))^{-1}(B-A)(1-\lambda((1-t)A+tB))^{-1}dt\\ &\ge m(1-\lambda((1-t)A+tB))^{-2}\end{aligned}$$

for $t\in[0,1]$ and $\lambda\in[-1,1]$, which implies that

$$f(B)-f(A)\geq mf'(0)\int_{-1}^{1}\left[\int_{0}^{1}(1-\lambda((1-t)A+tB))^{-2}dt\right]d\mu(\lambda).$$

If $M\geq B-A$, then in a similar way

$$Mf'(0)\int_{-1}^{1}\left[\int_{0}^{1}(1-\lambda((1-t)A+tB))^{-2}dt\right]d\mu(\lambda)\geq f(B)-f(A)$$

and the theorem is proved. □

Corollary 2. *With the assumptions of Theorem 5 and if $B-A\geq m>0$, then*

$$f(B)-f(A)\geq mf'(0)\left[1+\left(\int_{-1}^{1}\lambda d\mu(\lambda)\right)(A+B)+\frac{1}{3}\left(\int_{-1}^{1}\lambda^{2}d\mu(\lambda)\right)\left(A^{2}+B^{2}+\frac{AB+BA}{2}\right)\right]\geq 0. \tag{2.10}$$

Proof. Since for $x<1$, then

$$\frac{1}{1-x}\geq 1+x,$$

which implies that

$$(1-x)^{-2}\geq(1+x)^{2}=1+2x+x^{2}\geq 0.$$

We then get

$$\begin{aligned}&(1-\lambda((1-t)A+tB))^{-2}\\&\geq 1+2\lambda((1-t)A+tB)+\lambda^{2}((1-t)A+tB)^{2}\\&=1+2\lambda((1-t)A+tB)\\&+\lambda^{2}\left[(1-t)^{2}A^{2}+t(1-t)AB+t(1-t)BA+t^{2}B^{2}\right]\geq 0\end{aligned}$$

for $t\in[0,1]$, $\lambda\in[-1,1]$.

If we take the integral over t in this inequality, then we get

$$\begin{aligned}&\int_{0}^{1}(1-\lambda((1-t)A+tB))^{-2}dt\\&\geq 1+\lambda(A+B)+\frac{1}{3}\lambda^{2}\left(A^{2}+B^{2}+\frac{AB+BA}{2}\right)\geq 0\end{aligned}$$

for $\lambda\in[-1,1]$.

Taking the integral over $\mu(\lambda)$ we get

$$\begin{aligned}&\int_{-1}^{1}\left[\int_{0}^{1}(1-\lambda((1-t)A+tB))^{-2}dt\right]d\mu(\lambda)\\&\geq 1+\left(\int_{-1}^{1}\lambda d\mu(\lambda)\right)(A+B)\\&+\frac{1}{3}\left(\int_{-1}^{1}\lambda^{2}d\mu(\lambda)\right)\left(A^{2}+B^{2}+\frac{AB+BA}{2}\right)\geq 0,\end{aligned}$$

which, by the second inequality in (2.9) gives the desired result (2.10). □

In the following, in order to simplify terminology, when we write $T \geq 0$ we automatically assume that the operator T is selfadjoint.

We give now some necessary and sufficient conditions for the operators A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$ such that the inequality $(B-A)(f(B)-f(A)) \geq 0$ holds.

Theorem 6. *Let A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$. The following statements are equivalent:*

(i) *For all for $\lambda \in [-1,1]$,*

$$(1-\lambda B)^{-1}(1-\lambda A)+(1-\lambda A)^{-1}(1-\lambda B) \geq 2. \tag{2.11}$$

(ii) *For all for $\lambda \in [-1,1]$,*

$$\int_0^1 \left[(1-\lambda[(1-t)B+\lambda tA])^{-1}(B-A)\right]^2 dt \geq 0.$$

(iii) *For all $\lambda \in [-1,1]$*

$$(\kappa_\lambda(B)-\kappa_\lambda(A))(B-A) \geq 0,$$

where $\kappa_\lambda(t)=t(1-\lambda t)^{-1}$, $t \in [-1,1]$.

(iv) *For all operator monotone function f on $(-1,1)$,*

$$(f(B)-f(A))(B-A) \geq 0. \tag{2.12}$$

(v) *For all operator monotone function f on $(-1,1)$,*

$$(B-A)(f(B)-f(A)) \geq 0. \tag{2.13}$$

Proof. Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t \in [0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-t)C+tD,\ t \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.14}$$

From (2.14) we have

$$\begin{aligned}
&(1-\lambda B)^{-1}-(1-\lambda A)^{-1}\\
&=\int_0^1 ((1-t)(1-\lambda B)+t(1-\lambda A))^{-1}((1-\lambda A)-(1-\lambda B))\\
&\times((1-t)(1-\lambda B)+t(1-\lambda A))^{-1}\,dt\\
&=\lambda\int_0^1 (1-\lambda[(1-t)B+\lambda tA])^{-1}(B-A)\\
&\times(1-\lambda[(1-t)B+\lambda tA])^{-1}\,dt
\end{aligned}$$

for $\lambda \in [-1, 1]$.

If we multiply this equality at right by $B - A$, we get

$$\begin{aligned}
&\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](B-A) \\
&= \lambda \int_0^1 (1-\lambda[(1-t)B+\lambda t A])^{-1}(B-A) \\
&\times (1-\lambda[(1-t)B+\lambda t A])^{-1}(B-A)\,dt \\
&= \lambda \int_0^1 \left[(1-\lambda[(1-t)B+\lambda t A])^{-1}(B-A)\right]^2 dt
\end{aligned}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$, namely

$$\begin{aligned}
V_\lambda &:= \frac{1}{\lambda}\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](B-A) \\
&= \int_0^1 \left[(1-\lambda[(1-t)B+\lambda t A])^{-1}(B-A)\right]^2 dt.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](B-A) \\
&= \frac{1}{\lambda}\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](\lambda B - \lambda A) \\
&= \frac{1}{\lambda}\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](1-\lambda A - (1-\lambda B)) \\
&= \frac{1}{\lambda}\left[(1-\lambda B)^{-1}(1-\lambda A) + (1-\lambda A)^{-1}(1-\lambda B) - 2\right],
\end{aligned}$$

which gives that

$$\begin{aligned}
V_\lambda &= \frac{1}{\lambda}\left[(1-\lambda B)^{-1} - (1-\lambda A)^{-1}\right](B-A) \\
&= \frac{1}{\lambda^2}\left[(1-\lambda B)^{-1}(1-\lambda A) + (1-\lambda A)^{-1}(1-\lambda B) - 2\right]
\end{aligned}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

We conclude that

$$\begin{aligned}
&(\kappa_\lambda(B) - \kappa_\lambda(A))(B-A) \\
&= \frac{1}{\lambda^2}\left[(1-\lambda B)^{-1}(1-\lambda A) + (1-\lambda A)^{-1}(1-\lambda B) - 2\right] \\
&= \int_0^1 \left[(1-\lambda[(1-t)B+\lambda t A])^{-1}(B-A)\right]^2 dt
\end{aligned} \tag{2.15}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

We observe that the equivalence between the statements (i), (ii) and (iii) follows directly from the identity (2.15).

Since the function $\kappa_\lambda(t) = t(1-\lambda t)^{-1}$ is operator monotone on $(-1, 1)$, the statement (iv) implies (iii).

If (ii) holds, then by representation (2.6) we have

$$\begin{aligned}&[f(B)-f(A)](B-A)\\&=f'(0)\int_{-1}^{1}\left(\int_{0}^{1}\left[(1-\lambda((1-t)A+tB))^{-1}(B-A)\right]^{2}dt\right)d\mu(\lambda)\\&\geq 0,\end{aligned}$$

which shows that (iv) holds.

Define the operator $K:=(f(B)-f(A))(B-A)$. Since

$$\begin{aligned}K^{*}&=[(f(B)-f(A))(B-A)]^{*}=(B-A)^{*}(f(B)-f(A))^{*}\\&=(B-A)(f(B)-f(A))\end{aligned}$$

then the fact that K is selfadjoint is equivalent to

$$(f(B)-f(A))(B-A)=(B-A)(f(B)-f(A)),$$

which is also equivalent to the fact that

$$f(A)B+f(B)A=Bf(A)+Af(B).$$

These prove the equivalence between (iv) and (v). □

Therefore, we can state:

Corollary 3. *Let A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$. The statement (i) is equivalent to the inequality*

$$f(B)B+f(A)A\geq f(A)B+f(B)A, \tag{2.16}$$

for all f an operator monotone function on $(-1,1)$.

In the note [3] Fujii and Nakamoto showed that the inequality (2.13) does not hold in general for A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$. They also proved that if C, $D>0$ and $CD^{-1}+DC^{-1}$ is *selfadjoint*, then

$$CD^{-1}+DC^{-1}\geq 2. \tag{2.17}$$

Indeed, as shown in [3], if we put $T=CD^{-1}$, then $V=T+T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\operatorname{Sp}(T)$ of T is included in $(0,\infty)$, because C, $D>0$ and $\operatorname{Sp}(T)=\operatorname{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\operatorname{Sp}(V)=\left\{t+\frac{1}{t},\ t\in\operatorname{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T+T^{-1}\geq 2$.

Observe that, in fact we have:

Proposition 1. *Let A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$, then the statements (i) and*

(i') The operator $A(1-\lambda B)^{-1}+B(1-\lambda A)^{-1}$ is selfadjoint for all $\lambda\in[-1,1]$, are equivalent.

Proof. Notice that for all $\lambda\in[-1,1]$,

$$\begin{aligned}&(1-\lambda B)^{-1}(1-\lambda A)+(1-\lambda A)^{-1}(1-\lambda B)\\&=(1-\lambda B)^{-1}+(1-\lambda A)^{-1}-\lambda\left[(1-\lambda B)^{-1}A+(1-\lambda A)^{-1}B\right].\end{aligned} \tag{2.18}$$

Also, the operator $(1-\lambda B)^{-1}+(1-\lambda A)^{-1}$ is selfadjoint for $\lambda\in[-1,1]$.

If the statement *(i)* holds, then $(1-\lambda B)^{-1}(1-\lambda A)+(1-\lambda A)^{-1}(1-\lambda B)$ is selfadjoint and by (2.18) we must have that $(1-\lambda B)^{-1}A+(1-\lambda A)^{-1}B$ is selfadjoint, which shows that

$$\left((1-\lambda B)^{-1}A+(1-\lambda A)^{-1}B\right)^{*}=A(1-\lambda B)^{-1}+B(1-\lambda A)^{-1}$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (2.18) we get

$$(1-\lambda B)^{-1}(1-\lambda A)+(1-\lambda A)^{-1}(1-\lambda B)$$

is selfadjoint and by (2.17) for $C=(1-\lambda B)^{-1}$, $D=(1-\lambda A)^{-1}$ we obtain the inequality (2.11), namely *(i)* is true. □

We define the class of operators

$$\mathfrak{Cl}_{(-1,1)}(H):=\{(A,B)\mid \operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)\text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B)\in\mathfrak{Cl}_{(-1,1)}(H)$ then $(B,A)\in\mathfrak{Cl}_{(-1,1)}(H)$.

Also if $AB=BA$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$ then $V_\lambda:=(1-\lambda B)^{-1}(1-\lambda A)$ and $V_\lambda^{-1}=(1-\lambda A)^{-1}(1-\lambda B)$, $\lambda\in[-1,1]$, are selfadjoint and since $V_\lambda+V_\lambda^{-1}\geq 2$, $\lambda\in[-1,1]$ we derive that $(A,B)\in\mathfrak{Cl}_{(-1,1)}(H)$. Therefore, if $\mathfrak{Co}_{(-1,1)}(H)$ is the class of all pairs of *commutative operators* A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$, then we have

$$\emptyset\neq\mathfrak{Co}_{(-1,1)}(H)\subset\mathfrak{Cl}_{(-1,1)}(H). \tag{2.19}$$

3. Discrete Inequalities

We have the following Chebychev type operator inequality:

Proposition 2. *Assume that the function* $f:(-1,1)\to\mathbb{R}$ *is operator monotone in* $(-1,1)$*. If* $g:I\to(-1,1)$ *is continuous, then for all selfadjoint operators* A_k*,* $k=1,\dots,n$ *with spectra in* I *such that* $(g(A_k),g(A_j))\in\mathfrak{Cl}_{(-1,1)}(H)$ *for all* k*,* $j=1,\dots,n$ *and* $p_k\geq 0$*,* $k=1,\dots,n$ *with* $\sum_{k=1}^n p_k=1$*, we have*

$$\sum_{k=1}^{n}p_k(f\circ g)(A_k)g(A_k)\geq\sum_{k=1}^{n}p_kg(A_k)\sum_{k=1}^{n}p_k(f\circ g)(A_k). \tag{3.1}$$

If A_k*,* $k=1,\dots,n$*, with* $\operatorname{Sp}(A_k)\subset(-1,1)$*,* $(A_k,A_j)\in\mathfrak{Cl}_{(-1,1)}(H)$ *for all* k*,* $j=1,\dots,n$ *and* $p_k\geq 0$*,* $k=1,\dots,n$ *with* $\sum_{k=1}^n p_k=1$*, then*

$$\sum_{k=1}^{n}p_kf(A_k)A_k\geq\sum_{k=1}^{n}p_kA_k\sum_{k=1}^{n}p_kf(A_k). \tag{3.2}$$

Proof. From (2.16) we get

$$\begin{aligned}&(f\circ g)(A_k)g(A_k)+(f\circ g)(A_j)g(A_j)\\&\geq(f\circ g)(A_j)g(A_k)+(f\circ g)(A_k)g(A_j)\end{aligned} \tag{3.3}$$

for all k, $j\in\{1,\dots,n\}$.

If we multiply (3.3) by $p_k p_j \geq 0$ and sum over k and j from 1 to n, we get

$$\sum_{k=1}^{n}\sum_{j=1}^{n} p_k p_j (f\circ g)(A_k)\, g(A_k) + \sum_{k=1}^{n}\sum_{j=1}^{n} p_k p_j (f\circ g)(A_j)\, g(A_j)$$
$$\geq \sum_{k=1}^{n}\sum_{j=1}^{n} p_k p_j (f\circ g)(A_j)\, g(A_k) + \sum_{k=1}^{n}\sum_{j=1}^{n} p_k p_j (f\circ g)(A_k)\, g(A_j),$$

which is equivalent to

$$\sum_{k=1}^{n} p_k (f\circ g)(A_k)\, g(A_k) + \sum_{j=1}^{n} p_j (f\circ g)(A_j)\, g(A_j)$$
$$\geq \sum_{j=1}^{n} p_j (f\circ g)(A_j) \sum_{k=1}^{n} p_k g(A_k) + \sum_{k=1}^{n} p_k (f\circ g)(A_k) \sum_{j=1}^{n} p_j g(A_j)$$

that is equivalent to the first part of (3.1). □

Remark 1. *If A_k, $k=1,...,n$, with $\operatorname{Sp}(A_k)\subset(-1,1)$, $(A_k,A_j)\in\mathfrak{Cl}_{(-1,1)}(H)$ for all k, $j=1,...,n$ and $p_k\geq 0$, $k=1,...,n$ with $\sum_{k=1}^{n} p_k = 1$, then by (3.2),*

$$\sum_{k=1}^{n} p_k A_k \tan(A_k) \geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \tan(A_k). \tag{3.4}$$

Also for $r\in(0,1)$,

$$\sum_{k=1}^{n} p_k A_k \left[(1_H + A_k)(1_H - A_k)^{-1}\right]^r \geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \left[(1_H + A_k)(1_H - A_k)^{-1}\right]^r. \tag{3.5}$$

We also have

$$\sum_{k=1}^{n} p_k A_k \ln\left[(1_H + A_k)(1_H - A_k)^{-1}\right] \geq \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k \ln\left[(1_H + A_k)(1_H - A_k)^{-1}\right]. \tag{3.6}$$

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