INEQUALITIES FOR THE DIFFERENCE $A^{-1}g(A) - B^{-1}g(B)$ WHEN g IS OPERATOR MONOTONE ON $[0, \infty)$

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Abstract. In this paper we show that, if $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ with $g(0) = 0$, $A \geq 0$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$A^{-1}g(A) - B^{-1}g(B) \geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} \right] 1_H$$
$$\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} \right] 1_H > 0$$

and

$$A^{-1}g(A) - B^{-1}g(B)$$
$$\geq c \left(\frac{g(\|A\|)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{d(d + \|A\|)} \right) 1_H \geq 0.$$

Some applications for particular functions of interest are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $[0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{ts}{t + s} dm(s) \tag{1.1}$$

where $b \geq 0$ and a positive measure m on $[0, \infty)$ such that

$$\int_0^\infty \frac{s}{1 + s} dm(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [5]. Also the function ln is operator monotone on the open interval $(0, \infty)$. Let $f(t)$ be a continuous function $(0, \infty) \to (0, \infty)$. It is known that $f(t)$ is

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operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [3] or [7].

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m1_H > 0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0, \infty)$

$$\begin{aligned} f(B) - f(A) &\geq [f(\|A\| + m) - f(\|A\|)] 1_H \\ &\geq [f(\|B\|) - f(\|B\| - m)] 1_H > 0. \end{aligned} \tag{1.2}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) &\geq \left[f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right) - f(\|A\|) \right] 1_H \\ &\geq \left[f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right) \right] 1_H > 0. \end{aligned} \tag{1.3}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t) = t^r$, $r \in (0, 1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$\begin{aligned} B^r - A^r &\geq \left[\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r - \|A\|^r \right] 1_H \\ &\geq \left[\|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r \right] 1_H > 0 \end{aligned} \tag{1.4}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\begin{aligned} \ln B - \ln A &\geq \left[\ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right) - \ln(\|A\|) \right] 1_H \\ &\geq \left[\ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right) \right] 1_H > 0. \end{aligned} \tag{1.5}$$

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, in this paper we show that, if $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ with $g(0) = 0$, $A \geq 0$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$\begin{aligned} A^{-1} g(A) - B^{-1} g(B) &\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} \right] 1_H \\ &\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} \right] 1_H > 0 \end{aligned}$$

and

$$\begin{aligned} & A^{-1}g(A)-B^{-1}g(B) \\ & \geq c\left(\frac{g(\|A\|)}{(d+\|A\|)\|A\|}-\frac{g(d+\|A\|)-g(\|A\|)}{d(d+\|A\|)}\right)1_H \geq 0. \end{aligned}$$

Some applications for particular functions of interest are also given.

2. Main Results

We start with the following lemma that is of interest in itself.

Lemma 1. *Assume that* $g:[0,\infty)\to\mathbb{R}$ *is operator monotone on* $[0,\infty)$. *Then the function* $f:(0,\infty)\to\mathbb{R}$,

$$f(t):=\frac{g(0)-g(t)}{t} \tag{2.1}$$

is operator monotone on $(0,\infty)$. *If* $g(0)=0$, *then* $f(t)=-g(t)t^{-1}$ *is operator monotone on* $(0,\infty)$.

Proof. Since g is operator monotone on $[0,\infty)$, then there exists $b\geq 0$ and w is a positive measure satisfying

$$\int_0^\infty \frac{\lambda}{1+\lambda}dw(\lambda)<\infty$$

such that [1, p. 144-145]

$$g(t)=g(0)+bt+\int_0^\infty \frac{\lambda t}{t+\lambda}dw(\lambda). \tag{2.2}$$

We have for $t>0$ that

$$h(t):=\frac{g(t)-g(0)}{t}-b=\int_0^\infty \frac{\lambda}{t+\lambda}dw(\lambda)$$

Therefore for all A, $B>0$

$$h(B)-h(A)=\int_0^\infty \lambda\left[(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\right]dw(\lambda). \tag{2.3}$$

Let T, $S>0$. The function $g(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S):=\lim_{t\to 0}\left[\frac{g(T+tS)-g(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(t):=g((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$g(D)-g(C)=\int_0^1 \frac{d}{dt}(g_{C,D}(t))dt=\int_0^1 \nabla g_{(1-t)C+tD}(D-C)dt. \tag{2.5}$$

If we write this equality for the function $g(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt. \tag{2.6}$$

Now, if we replace in (2.6) $C = B + \lambda 1_H$ and $D = A + \lambda 1_H$ for $\lambda > 0$, then we get

$$\begin{aligned}(B + \lambda 1_H)^{-1} - (A + \lambda 1_H)^{-1} \\ = \int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (A - B) ((1-t)B + tA + \lambda 1_H)^{-1} dt.\end{aligned} \tag{2.7}$$

Therefore, by (2.3),

$$\begin{aligned} h(B) - h(A) &= \int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (A - B) \right. \\ &\quad \left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \\ &= -\int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda). \end{aligned} \tag{2.8}$$

If $B \geq A > 0$, then

$$((1-t)B + tA + \lambda 1_H)^{-1} (B - A) ((1-t)B + tA + \lambda 1_H)^{-1} \geq 0$$

for all $t,\ \lambda > 0$, which implies that

$$\begin{aligned} &\int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\times \left. ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \geq 0, \end{aligned}$$

namely

$$\begin{aligned} f(B) - f(A) = h(A) - h(B) \\ &= \int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \times \left. ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \\ &\geq 0. \end{aligned}$$

Therefore, the function f is operator monotone on $(0, \infty)$. $\square$

Theorem 2. *Assume that $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$. If $A > 0$ and there exists $c > 0$ such that $B - A \geq c 1_H > 0$, then*

$$\begin{aligned} A^{-1} g(A) - B^{-1} g(B) - g(0) \left(A^{-1} - B^{-1}\right) \\ \geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} - g(0) \frac{c}{(\|A\| + c)\|A\|} \right] 1_H \\ \geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} - g(0) \frac{c}{(\|B\| - c)\|B\|} \right] 1_H > 0. \end{aligned} \tag{2.9}$$

If $g(0)=0$, *then*

$$A^{-1}g(A)-B^{-1}g(B)\geq\left[\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+c)}{\|A\|+c}\right]1_H \geq\left[\frac{g(\|B\|-c)}{\|B\|-c}-\frac{g(\|B\|)}{\|B\|}\right]1_H>0. \tag{2.10}$$

Proof. If we write the inequality (1.2) for $f(t)=\frac{g(0)-g(t)}{t}$, $t>0$, which, by Lemma 1, is operator monotone, then we have

$$\begin{aligned} &B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)]\\ &\geq\left[\frac{g(0)-g(\|A\|+c)}{\|A\|+c}-\frac{g(0)-g(\|A\|)}{\|A\|}\right]1_H\\ &\geq\left[\frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-c)}{\|B\|-c}\right]1_H>0. \end{aligned} \tag{2.11}$$

Observe that

$$\begin{aligned} &B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)]\\ &=A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right), \end{aligned}$$

$$\begin{aligned} &\frac{g(0)-g(\|A\|+c)}{\|A\|+c}-\frac{g(0)-g(\|A\|)}{\|A\|}\\ &=\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+c)}{\|A\|+c}-g(0)\frac{c}{(\|A\|+c)\|A\|} \end{aligned}$$

and

$$\begin{aligned} &\frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-c)}{\|B\|-c}\\ &=\frac{g(\|B\|-c)}{\|B\|-c}-\frac{g(\|B\|)}{\|B\|}-g(0)\frac{c}{(\|B\|-c)\|B\|} \end{aligned}$$

and by (2.11) we get (2.9). □

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned} 0&\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\ &\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle \end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}1_H\leq T. \tag{2.12}$$

Corollary 1. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and $B-A>0$, then*

$$\begin{aligned}
(2.13)\qquad & A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right)\\
&\geq\left[\frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}}\right]1_H\\
&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)\|A\|}1_H\\
&\geq\left[\frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|}\right]1_H\\
&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)\|B\|}1_H\\
&>0.
\end{aligned}$$

If $g(0)=0$, then

$$\begin{aligned}
(2.14)\qquad & A^{-1}g(A)-B^{-1}g(B)\\
&\geq\left[\frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}}\right]1_H\\
&\geq\left[\frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|}\right]1_H>0.
\end{aligned}$$

We have the following lower bound as well:

Theorem 3. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. Let $A\geq 0$ and assume that there exist positive numbers $d>c>0$ such that*

$$d1_H\geq B-A\geq c1_H>0. \tag{2.15}$$

Then

$$f(B)-f(A)\geq c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\geq 0. \tag{2.16}$$

Proof. Since the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then f can be written as in the equation (1.1) and for A, $B\geq 0$ we have the representation

$$\begin{aligned}
(2.17)\qquad & f(B)-f(A)\\
&=b(B-A)+\int_0^\infty s\left[B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\right]dm(s).
\end{aligned}$$

Observe that for $s > 0$

$$\begin{aligned}
&B(B + s1_H)^{-1} - A(A + s1_H)^{-1} \\
&= (B + s1_H - s1_H)(B + s1_H)^{-1} - (A + s1_H - s1_H)(A + s1_H)^{-1} \\
&= (B + s1_H)(B + s1_H)^{-1} - s1_H(B + s1_H)^{-1} \\
&- (A + s1_H)(A + s1_H)^{-1} + s1_H(A + s1_H)^{-1} \\
&= 1_H - s1_H(B + s1_H)^{-1} - 1_H + s1_H(A + s1_H)^{-1} \\
&= s\left[(A + s1_H)^{-1} - (B + s1_H)^{-1}\right].
\end{aligned}$$

Therefore, (2.17) becomes, see also [4]

$$f(B) - f(A) = b(B - A) + \int_0^\infty s^2\left[(A + s1_H)^{-1} - (B + s1_H)^{-1}\right] dm(s). \tag{2.18}$$

Now, if we replace in (2.6) $C = A + s1_H$ and $D = B + s1_H$ for $s > 0$, then we get

$$\begin{aligned}
&(A + s1_H)^{-1} - (B + s1_H)^{-1} \\
&= \int_0^1 ((1-t)A + tB + s1_H)^{-1}(B - A)((1-t)A + tB + s1_H)^{-1} dt.
\end{aligned} \tag{2.19}$$

By the representation (2.18), we derive the following identity of interest

$$\begin{aligned}
f(B) - f(A) &= b(B - A) \\
&+ \int_0^\infty s^2\left[\int_0^1 ((1-t)A + tB + s1_H)^{-1}\right. \\
&\left.\times (B - A)((1-t)A + tB + s1_H)^{-1} dt\right] dm(s)
\end{aligned} \tag{2.20}$$

for A, $B > 0$.

From the representation (2.20) we get for $B = x1_H$, $A = 0$ that

$$f(x) - f(0) - bx = \int_0^\infty s^2\left(\int_0^1 (tx + s1_H)^{-1} x (tx + s1_H)^{-1} dt\right) dm(s),$$

which gives for $x > 0$ that

$$\frac{f(x) - f(0)}{x} - b = \int_0^\infty s^2\left(\int_0^1 (tx + s)^{-2} dt\right) dm(s). \tag{2.21}$$

Since $0 < c1_H \le B - A$, hence

$$\begin{aligned}
&c((1-t)A + tB + s1_H)^{-2} \\
&\le ((1-t)A + tB + s1_H)^{-1}(B - A)((1-t)A + tB + s1_H)^{-1}
\end{aligned}$$

for $t \in [0, 1]$ and $s > 0$ and by (2.20) we get

$$\begin{aligned}
&c\int_0^\infty s^2\left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt\right) dm(s) \\
&\le f(B) - f(A) - b(B - A).
\end{aligned} \tag{2.22}$$

Observe that for $t \in [0, 1]$ and $s > 0$, we have

$$\begin{aligned}
(1-t)A + tB + s1_H &= A + t(B - A) + s1_H \le A + td1_H + s1_H \\
&= (1-t)A + t(d1_H + A) + s1_H.
\end{aligned}$$

Since $A\leq \|A\|\,1_{H}$ then

$$\left(1-t\right)A+t\left(d1_{H}+A\right)+s1_{H}\leq \left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)1_{H},$$

which implies that

$$\left(1-t\right)A+tB+s1_{H}\leq \left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)1_{H}$$

for $t\in[0,1]$ and $s>0$.

This implies that

$$\left(\left(1-t\right)A+tB+s1_{H}\right)^{-1}\geq \left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)^{-1}1_{H}$$

and

$$\left(\left(1-t\right)A+tB+s1_{H}\right)^{-2}\geq \left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)^{-2}1_{H}$$

for $t\in[0,1]$ and $s>0$.

Therefore

$$\begin{aligned}&\int_{0}^{\infty}s^{2}\left(\int_{0}^{1}\left(\left(1-t\right)A+tB+s1_{H}\right)^{-2}dt\right)dm\left(s\right)\\&\geq\int_{0}^{\infty}s^{2}\left(\int_{0}^{1}\left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)^{-2}dt\right)dm\left(s\right)1_{H}\left(\geq 0\right)\\&=\frac{1}{d}\int_{0}^{\infty}s^{2}\left(\int_{0}^{1}\left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)^{-1}\left(d+\|A\|-\|A\|\right)\right.\\&\left.\times\left(\left(1-t\right)\|A\|+t\left(d+\|A\|\right)+s\right)^{-1}dt\right)dm\left(s\right)1_{H}\\&=\frac{1}{d}\left[\left(f\left(d+\|A\|\right)-f\left(\|A\|\right)-bd\right)\right]1_{H}\text{ (by identity (2.21))}\\&=\left(\frac{f\left(d+\|A\|\right)-f\left(\|A\|\right)}{d}-b\right)1_{H}\geq 0.\end{aligned}$$

By (2.22) we get

$$\begin{aligned}f\left(B\right)-f\left(A\right)-b\left(B-A\right)&\\\geq c\int_{0}^{\infty}s^{2}\left(\int_{0}^{1}\left(\left(1-t\right)A+tB+s1_{H}\right)^{-2}dt\right)dm\left(s\right)&\\\geq c\left(\frac{f\left(d+\|A\|\right)-f\left(\|A\|\right)}{d}-b\right)1_{H}\geq 0.&\end{aligned}\tag{2.23}$$

From (2.23) we derive

$$\begin{aligned}f\left(B\right)-f\left(A\right)&\geq b\left(B-A\right)+c\left(\frac{f\left(d+\|A\|\right)-f\left(\|A\|\right)}{d}-b\right)1_{H}\\&=b\left[\left(B-A\right)-c\right]+c\frac{f\left(d+\|A\|\right)-f\left(\|A\|\right)}{d}1_{H}\\&\geq c\frac{f\left(d+\|A\|\right)-f\left(\|A\|\right)}{d}1_{H}\geq 0\end{aligned}$$

since $b\left[\left(B-A\right)-c\right]\geq 0$ and the inequality (2.16) is obtained. □

Corollary 2. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is operator monotone on* $[0,\infty)$. *If* $A \geq 0$ *and* $B - A > 0$, *then*

$$f(B) - f(A) \geq \frac{f(\|B-A\| + \|A\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \geq \frac{f(\|B\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \geq 0. \tag{2.24}$$

The first inequality follows by (2.16) for $d = \|B-A\|$ and $c = \left\|(B-A)^{-1}\right\|^{-1}$. The second and third inequalities are obvious.

Theorem 4. *Assume that* $g : [0,\infty) \to \mathbb{R}$ *is operator monotone on* $[0,\infty)$. *If* $A > 0$ *and there exist positive numbers* $d > c > 0$ *such that the condition (2.15) is satisfied, then*

$$\begin{aligned} & g(0)\left(B^{-1} - A^{-1}\right) + A^{-1}g(A) - B^{-1}g(B) \\ & \geq c\left(\frac{g(\|A\|) - g(0)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{d(d + \|A\|)}\right) 1_H \geq 0. \end{aligned} \tag{2.25}$$

If $g(0) = 0$, *then*

$$\begin{aligned} & A^{-1}g(A) - B^{-1}g(B) \\ & \geq c\left(\frac{g(\|A\|)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{d(d + \|A\|)}\right) 1_H \geq 0. \end{aligned} \tag{2.26}$$

Proof. Since g is operator monotone, then by Lemma 1 the function $f(t) := \frac{g(0)-g(t)}{t}$ is operator monotone on $(0,\infty)$ and by (2.16) we obtain

$$\frac{g(0) - g(B)}{B} - \frac{g(0) - g(A)}{A} \geq c\frac{\frac{g(0)-g(d+\|A\|)}{d+\|A\|} - \frac{g(0)-g(\|A\|)}{\|A\|}}{d} 1_H \geq 0. \tag{2.27}$$

Observe that

$$\begin{aligned} & \frac{g(0) - g(B)}{B} - \frac{g(0) - g(A)}{A} \\ & = g(0)\left(B^{-1} - A^{-1}\right) + A^{-1}g(A) - B^{-1}g(B) \end{aligned}$$

and

$$\begin{aligned} & \frac{g(0) - g(d + \|A\|)}{d + \|A\|} - \frac{g(0) - g(\|A\|)}{\|A\|} \\ & = \frac{[g(0) - g(d + \|A\|)]\|A\| - [g(0) - g(\|A\|)](d + \|A\|)}{(d + \|A\|)\|A\|} \\ & = \frac{g(0)\|A\| - g(d + \|A\|)\|A\| - g(0)d + g(\|A\|)d - g(0)\|A\| + g(\|A\|)\|A\|}{(d + \|A\|)\|A\|} \\ & = \frac{g(\|A\|)d - g(0)d + g(\|A\|)\|A\| - g(d + \|A\|)\|A\|}{(d + \|A\|)\|A\|} \\ & = d\frac{g(\|A\|) - g(0)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{(d + \|A\|)}, \end{aligned}$$

which gives

$$\frac{\frac{g(0)-g(d+\|A\|)}{d+\|A\|}-\frac{g(0)-g(\|A\|)}{\|A\|}}{d}=\frac{g\left(\|A\|\right)-g\left(0\right)}{\left(d+\|A\|\right)\|A\|}-\frac{g\left(d+\|A\|\right)-g\left(\|A\|\right)}{d\left(d+\|A\|\right)}.$$

Then by (2.27) we get (2.25). □

Corollary 3. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and $B-A>0$, then*

$$\begin{aligned}(2.28)\qquad & g\left(0\right)\left(B^{-1}-A^{-1}\right)+A^{-1}g\left(A\right)-B^{-1}g\left(B\right)\\ & \geq\left\|\left(B-A\right)^{-1}\right\|^{-1}\\ & \times\left(\frac{g\left(\|A\|\right)-g\left(0\right)}{\left(\|B-A\|+\|A\|\right)\|A\|}-\frac{g\left(\|B-A\|+\|A\|\right)-g\left(\|A\|\right)}{\|B-A\|\left(\|B-A\|+\|A\|\right)}\right)1_H\\ & \geq 0.\end{aligned}$$

If $g\left(0\right)=0$, then

$$\begin{aligned}(2.29)\qquad & A^{-1}g\left(A\right)-B^{-1}g\left(B\right)\\ & \geq\left\|\left(B-A\right)^{-1}\right\|^{-1}\\ & \times\left(\frac{g\left(\|A\|\right)}{\left(\|B-A\|+\|A\|\right)\|A\|}-\frac{g\left(\|B-A\|+\|A\|\right)-g\left(\|A\|\right)}{\|B-A\|\left(\|B-A\|+\|A\|\right)}\right)1_H\\ & \geq 0.\end{aligned}$$

3. Some Examples

Consider the function $g\left(t\right)=t^r$, $r\in(0,1]$. This function is operator monotone and by (2.10) we have

$$\begin{aligned}(3.1)\qquad A^{r-1}-B^{r-1} & \geq\left[\|A\|^{r-1}-\left(\|A\|+c\right)^{r-1}\right]1_H\\ & \geq\left[\left(\|B\|-c\right)^{r-1}-\|B\|^{r-1}\right]1_H>0\end{aligned}$$

provided that $A>0$ and $B-A\geq c1_H>0$.

If $A>0$ and $B-A>0$, then

$$\begin{aligned}(3.2)\qquad A^{r-1}-B^{r-1} & \geq\left[\|A\|^{r-1}-\left(\|A\|+\left\|\left(B-A\right)^{-1}\right\|^{-1}\right)^{r-1}\right]1_H\\ & \geq\left[\left(\|B\|-\left\|\left(B-A\right)^{-1}\right\|^{-1}\right)^{r-1}-\|B\|^{r-1}\right]1_H>0.\end{aligned}$$

From (2.16) we obtain

$$(3.3)\qquad B^r-A^r\geq c\frac{\left(d+\|A\|\right)^r-\|A\|^r}{d}1_H\geq 0$$

provided that there exist positive numbers $d>c>0$ such that condition (2.15) is satisfied.

If $A>0$ and $B-A>0$, then

$$B^{r}-A^{r}\geq\left\|(B-A)^{-1}\right\|^{-1}\frac{(\|B-A\|+\|A\|)^{r}-\|A\|^{r}}{\|B-A\|}1_{H}\geq\left\|(B-A)^{-1}\right\|^{-1}\frac{\|B\|^{r}-\|A\|^{r}}{\|B-A\|}1_{H}\geq0. \tag{3.4}$$

From (2.25) we have

$$A^{r-1}-B^{r-1}\geq c\left(\frac{\|A\|^{r-1}}{d+\|A\|}-\frac{(d+\|A\|)^{r}-\|A\|^{r}}{d(d+\|A\|)}\right)1_{H}\geq0, \tag{3.5}$$

provided that there exist positive numbers $d>c>0$ such that condition (2.15) is satisfied.

If $A>0$ and $B-A>0$, then

$$A^{r-1}-B^{r-1}\geq\left\|(B-A)^{-1}\right\|^{-1}\times\left(\frac{\|A\|^{r-1}}{\|B-A\|+\|A\|}-\frac{(\|B-A\|+\|A\|)^{r}-\|A\|^{r}}{\|B-A\|(\|B-A\|+\|A\|)}\right)1_{H}\geq0. \tag{3.6}$$

Consider the function $g(t)=\ln(t+1)$, which is operator monotone on $[0,\infty)$ and $g(0)=0$. By Lemma 1 we get that the function $f(t)=-t^{-1}\ln(t+1)$ is operator monotone on $(0,\infty)$.

From (2.10) we get

$$A^{-1}\ln(A+1_{H})-B^{-1}\ln(B+1_{H})\geq\left(\frac{\ln(\|A\|+1)}{\|A\|}-\frac{\ln(\|A\|+1+c)}{\|A\|+c}\right)1_{H}\geq\left(\frac{\ln(\|B\|+1-c)}{\|B\|-c}-\frac{\ln(\|B\|+1)}{\|B\|}\right)1_{H}>0 \tag{3.7}$$

provided that $A>0$ and $B-A\geq c1_{H}>0$.

If $A>0$ and $B-A>0$, then

$$A^{-1}\ln(A+1_{H})-B^{-1}\ln(B+1_{H})\geq\left(\frac{\ln(\|A\|+1)}{\|A\|}-\frac{\ln\left(\|A\|+1+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}}\right)1_{H}\geq\left(\frac{\ln\left(\|B\|+1-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{\ln(\|B\|+1)}{\|B\|}\right)1_{H}>0. \tag{3.8}$$

From (2.16) we derive

$$\ln(B+1_{H})-\ln(A+1_{H})\geq c\frac{\ln(d+\|A\|+1)-\ln(\|A\|+1)}{d}1_{H}\geq0 \tag{3.9}$$

provided that there exist positive numbers $d > c > 0$ such that the condition (2.15) is satisfied.

If $A > 0$ and $B - A > 0$, then

$$\begin{aligned} & \ln(B + 1_H) - \ln(A + 1_H) \\ & \geq \left\| (B - A)^{-1} \right\|^{-1} \frac{\ln(\|B - A\| + \|A\| + 1) - \ln(\|A\| + 1)}{\|B - A\|} 1_H \\ & \geq \left\| (B - A)^{-1} \right\|^{-1} \frac{\ln(\|B\| + 1) - \ln(\|A\| + 1)}{\|B - A\|} 1_H \geq 0. \end{aligned} \tag{3.10}$$

From (2.25) we have

$$\begin{aligned} & A^{-1} \ln(A + 1) - B^{-1} \ln(B + 1) \\ & \geq c \left(\frac{\ln(\|A\| + 1)}{(d + \|A\|) \|A\|} - \frac{\ln(d + \|A\| + 1) - \ln(\|A\| + 1)}{d(d + \|A\|)} \right) 1_H \geq 0 \end{aligned} \tag{3.11}$$

provided that there exist positive numbers $d > c > 0$ such that the condition (2.15) is satisfied.

Finally, from (2.29) we derive

$$\begin{aligned} & A^{-1} \ln(A + 1_H) - B^{-1} \ln(B + 1_H) \\ & \geq \left\| (B - A)^{-1} \right\|^{-1} \\ & \times \left(\frac{\ln(\|A\| + 1)}{(\|B - A\| + \|A\|) \|A\|} - \frac{\ln(\|B - A\| + \|A\| + 1) - \ln(\|A\| + 1)}{\|B - A\| (\|B - A\| + \|A\|)} \right) 1_H \\ & \geq 0, \end{aligned} \tag{3.12}$$

provided $A > 0$ and $B - A > 0$.

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[3] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra and its Applications* **429** (2008) 972–980.

[4] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[5] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[6] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[7] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.* **246** (1980) 205–224.

[8] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[9] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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