REVERSE AND IMPROVED INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we show that that, if $f:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$, $A\geq 0$ and there exist positive numbers $d>c>0$ such that the condition $d1_H\geq B-A\geq c1_H>0$ is satisfied, then

$$d\frac{f(c)-f(0)}{c}1_H\geq f(B)-f(A)\geq c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\geq 0.$$

We also obtain refinements and a reverse of Löwner-Heinz celebrated inequality,

$$\begin{aligned}\|B-A\|\left\|(B-A)^{-1}\right\|^{1-r}1_H &\geq B^r-A^r\\ &\geq \frac{(\|B-A\|+\|A\|)^r-\|A\|^r}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H\\ &\geq \frac{\|B\|^r-\|A\|^r}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H>0\end{aligned}$$

provided that $B>A\geq 0$ and $r\in(0,1]$.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $[0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{ts}{t+s}dm(s)\tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $[0,\infty)$ such that

$$\int_0^\infty\frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:[0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [5].

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Let $f(t)$ be a continuous function $(0,\infty) \to (0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [3] or [7].

Consider the family of functions defined on $(0,\infty)$ and $p \in [-1,2] \setminus \{0,1\}$ by

$$f_p(t) := \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$$

and

$$f_0(t) := \frac{t}{1-t}\ln t,$$

$$f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean), } f_{1/2}(t) = \sqrt{t} \text{ (geometric mean).}$$

In [2] the authors showed that f_p is operator monotone for $1 \le p \le 2$.

In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p-1}$$

is an operator monotone function for $p \in (0,1]$, [3].

It is well known that the logarithmic function ln is operator monotone and in [3] the author obtained that the functions

$$f(t) = t(1+t)\ln\left(1+\frac{1}{t}\right),\ g(t) = \frac{1}{(1+t)\ln\left(1+\frac{1}{t}\right)}$$

are also operator monotone functions on $(0,\infty)$.

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \ge m1_H > 0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$f(B) - f(A) \ge f(\|A\| + m) - f(\|A\|) \ge f(\|B\|) - f(\|B\| - m) > 0. \tag{1.2}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) &\ge f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\ &\ge f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{1.3}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t) = t^r$, $r \in (0,1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$B^r - A^r \geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r - \|A\|^r \geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r > 0 \tag{1.4}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\ln B - \ln A \geq \ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right) - \ln(\|A\|) \geq \ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right) > 0. \tag{1.5}$$

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, we show in this paper that, if $f : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$d\frac{f(c) - f(0)}{c} 1_H \geq f(B) - f(A) \geq c\frac{f(d + \|A\|) - f(\|A\|)}{d} 1_H \geq 0. \tag{1.6}$$

Some examples of interest, including a refinement and a reverse of Löwner-Heinz inequality, are also provided.

2. Main Results

We have:

Theorem 2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ given by representation (1.1). Let $A \geq 0$ and assume that there exist positive numbers $d > c > 0$ such that*

$$d1_H \geq B - A \geq c1_H > 0. \tag{2.1}$$

Then

$$d\left(\frac{f(c) - f(0)}{c} - b \right) 1_H \geq f(B) - f(A) - b(B - A) \geq c\left(\frac{f(d + \|A\|) - f(\|A\|)}{d} - b \right) 1_H \geq 0. \tag{2.2}$$

Proof. Since the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$, then f can be written as in the equation (1.1) and for A, $B \geq 0$ we have the representation

$$f(B) - f(A) = b(B - A) + \int_0^\infty s\left[B(B + s1_H)^{-1} - A(A + s1_H)^{-1} \right] dm(s). \tag{2.3}$$

Observe that for $s>0$

$$\begin{aligned}
&B\left(B+s1_H\right)^{-1}-A\left(A+s1_H\right)^{-1}\\
&=\left(B+s1_H-s1_H\right)\left(B+s1_H\right)^{-1}-\left(A+s1_H-s1_H\right)\left(A+s1_H\right)^{-1}\\
&=\left(B+s1_H\right)\left(B+s1_H\right)^{-1}-s1_H\left(B+s1_H\right)^{-1}\\
&-\left(A+s1_H\right)\left(A+s1_H\right)^{-1}+s1_H\left(A+s1_H\right)^{-1}\\
&=1_H-s1_H\left(B+s1_H\right)^{-1}-1_H+s1_H\left(A+s1_H\right)^{-1}\\
&=s\left[\left(A+s1_H\right)^{-1}-\left(B+s1_H\right)^{-1}\right].
\end{aligned}$$

Therefore, (2.3) becomes, see also [4]

$$f\left(B\right)-f\left(A\right)=b\left(B-A\right)+\int_0^\infty s^2\left[\left(A+s1_H\right)^{-1}-\left(B+s1_H\right)^{-1}\right]dm\left(s\right). \tag{2.4}$$

Let T, $S>0$. The function $f\left(t\right)=-t^{-1}$ is operator monotone on $\left(0,\infty\right)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T\left(S\right):=\lim_{t\rightarrow 0}\left[\frac{f\left(T+tS\right)-f\left(T\right)}{t}\right]=T^{-1}ST^{-1} \tag{2.5}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $\left[0,1\right]$ by

$$f_{C,D}\left(t\right)=f\left(\left(1-t\right)C+tD\right),\ t\in\left[0,1\right].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $\left[C,D\right]:=\left\{\left(1-t\right)C+tD,\ t\in\left[0,1\right]\right\}$, then we have, by the properties of the Bochner integral, that

$$f\left(D\right)-f\left(C\right)=\int_0^1\frac{d}{dt}\left(f_{C,D}\left(t\right)\right)dt=\int_0^1\nabla f_{\left(1-t\right)C+tD}\left(D-C\right)dt. \tag{2.6}$$

If we write this equality for the function $f\left(t\right)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1\left(\left(1-t\right)C+tD\right)^{-1}\left(D-C\right)\left(\left(1-t\right)C+tD\right)^{-1}dt. \tag{2.7}$$

Now, if we replace in (2.7) $C=A+s1_H$ and $D=B+s1_H$ for $s>0$, then we get

$$\begin{aligned}
&\left(A+s1_H\right)^{-1}-\left(B+s1_H\right)^{-1}\\
&=\int_0^1\left(\left(1-t\right)A+tB+s1_H\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+s1_H\right)^{-1}dt.
\end{aligned} \tag{2.8}$$

By the representation (2.4), we derive the following identity of interest

$$\begin{aligned}
f\left(B\right)-f\left(A\right)&=b\left(B-A\right)\\
&+\int_0^\infty s^2\left[\int_0^1\left(\left(1-t\right)A+tB+s1_H\right)^{-1}\right.\\
&\left.\times\left(B-A\right)\left(\left(1-t\right)A+tB+s1_H\right)^{-1}dt\right]dm\left(s\right)
\end{aligned} \tag{2.9}$$

for A, $B>0$.

From the representation (2.9) we get for $B = x1_H$, $A = 0$ that

$$f(x) - f(0) - bx = \int_0^\infty s^2 \left(\int_0^1 (tx + s1_H)^{-1} x (tx + s1_H)^{-1} dt \right) dm(s),$$

which gives for $x > 0$ that

$$\frac{f(x) - f(0)}{x} - b = \int_0^\infty s^2 \left(\int_0^1 (tx + s)^{-2} dt \right) dm(s). \tag{2.10}$$

Since $0 < c1_H \le B - A \le d1_H$, hence

$$\begin{aligned} &c((1-t)A + tB + s1_H)^{-2} \\ &\le ((1-t)A + tB + s1_H)^{-1} (B - A) ((1-t)A + tB + s1_H)^{-1} \\ &\le d((1-t)A + tB + s1_H)^{-2} \end{aligned}$$

for $t \in [0,1]$ and $s > 0$ and by (2.9) we get

$$\begin{aligned} &c \int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s) \\ &\le f(B) - f(A) - b(B - A) \\ &\le d \int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s). \end{aligned} \tag{2.11}$$

Observe that for $t \in [0,1]$ and $s > 0$, we have

$$\begin{aligned} (1-t)A + tB + s1_H &= A + t(B - A) + s1_H \\ &\ge 0 + tc1_H + s1_H = (tc + s)1_H. \end{aligned}$$

This implies that

$$((1-t)A + tB + s1_H)^{-1} \le (tc + s)^{-1} 1_H.$$

Therefore

$$\begin{aligned} &\int_0^\infty s^2 \left(\int_0^1 ((1-t)A + tB + s1_H)^{-2} dt \right) dm(s) \\ &\le \int_0^\infty s^2 \left(\int_0^1 (tc + s)^{-2} dt \right) dm(s) 1_H \\ &= \left(\frac{f(c) - f(0)}{c} - b \right) 1_H \text{ (by (2.10))} \end{aligned}$$

and by (2.11) we get

$$f(B) - f(A) - b(B - A) \le d \left(\frac{f(c) - f(0)}{c} - b \right) 1_H. \tag{2.12}$$

We also have

$$\begin{aligned} (1-t)A + tB + s1_H &= A + t(B - A) + s1_H \le A + td1_H + s1_H \\ &= (1-t)A + t(d1_H + A) + s1_H. \end{aligned}$$

Since $A \le \|A\| 1_H$ then

$$(1-t)A + t(d1_H + A) + s1_H \le ((1-t)\|A\| + t(d + \|A\|) + s) 1_H,$$

which implies that

$$(1-t)A + tB + s1_H \le ((1-t)\|A\| + t(d + \|A\|) + s) 1_H$$

for $t\in[0,1]$ and $s>0$.

This implies that

$$((1-t)A+tB+s1_H)^{-1}\geq((1-t)\|A\|+t(d+\|A\|)+s)^{-1}1_H$$

and

$$((1-t)A+tB+s1_H)^{-2}\geq((1-t)\|A\|+t(d+\|A\|)+s)^{-2}1_H$$

for $t\in[0,1]$ and $s>0$.

Therefore

$$\begin{aligned}&\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s1_H)^{-2}dt\right)dm(s)\\&\geq\int_0^\infty s^2\left(\int_0^1((1-t)\|A\|+t(d+\|A\|)+s)^{-2}dt\right)dm(s)1_H\ (\geq0)\\&=\frac{1}{d}\int_0^\infty s^2\left(\int_0^1((1-t)\|A\|+t(d+\|A\|)+s)^{-1}(d+\|A\|-\|A\|)\right.\\&\left.\times((1-t)\|A\|+t(d+\|A\|)+s)^{-1}dt\right)dm(s)1_H\\&=\frac{1}{d}[(f(d+\|A\|)-f(\|A\|)-bd)]1_H\text{ (by identity (2.10))}\\&=\left(\frac{f(d+\|A\|)-f(\|A\|)}{d}-b\right)1_H\geq0.\end{aligned}$$

By (2.11) we get

$$\begin{aligned}&f(B)-f(A)-b(B-A)\\&\geq c\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s1_H)^{-2}dt\right)dm(s)\\&\geq c\left(\frac{f(d+\|A\|)-f(\|A\|)}{d}-b\right)1_H\geq0.\end{aligned}\tag{2.13}$$

The inequalities (2.12) and (2.13 imply (2.2). □

From the first inequality in (2.2) we get

$$d\frac{f(c)-f(0)}{c}1_H-b[d1_H-(B-A)]\geq f(B)-f(A)$$

and since $d1_H-(B-A)\geq0$ and $b\geq0$, hence

$$d\frac{f(c)-f(0)}{c}1_H\geq d\frac{f(c)-f(0)}{c}1_H-b[d1_H-(B-A)].$$

From the second inequality in (2.2) we have

$$\begin{aligned}f(B)-f(A)&\geq b[(B-A)-c]+c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\\&\geq c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\geq0\end{aligned}$$

since $b[(B-A)-c1_H]\geq0$.

Therefore we have the following result which does not contain the value b:

Corollary 1. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone on* $[0,\infty)$, $A\geq 0$ *and that there exist positive numbers* $d>c>0$ *such that the condition (2.1) is satisfied, then*

$$d\frac{f(c)-f(0)}{c}1_H\geq f(B)-f(A)\geq c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\geq 0. \tag{2.14}$$

Remark 1. *If we take in (2.14),* $f(t)=t^r$, $r\in(0,1]$, *then we get*

$$dc^{r-1}1_H\geq B^r-A^r\geq c\frac{(d+\|A\|)^r-\|A\|^r}{d}1_H\geq 0, \tag{2.15}$$

provided that the condition (2.1) is satisfied and $A\geq 0$.

We also have the logarithmic inequality

$$\ln B-\ln A\geq c\frac{\ln(d+\|A\|)-\ln(\|A\|)}{d}1_H\geq 0, \tag{2.16}$$

provided that $A>0$ *and the condition (2.1) is satisfied.*

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}0&\leq\langle x,x\rangle^2=\langle T^{-1}Tx,x\rangle^2=\langle Tx,T^{-1}x\rangle^2\\&\leq\langle Tx,x\rangle\langle TT^{-1}x,T^{-1}x\rangle=\langle Tx,x\rangle\langle x,T^{-1}x\rangle\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\langle x,T^{-1}x\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\langle x,T^{-1}x\rangle=\langle Tx,x\rangle\|T^{-1}\|,$$

which implies the following operator inequality

$$\frac{1}{\|T^{-1}\|}1_H\leq T. \tag{2.17}$$

Corollary 2. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone on* $[0,\infty)$ *and* $B>A\geq 0$, *then*

$$\begin{aligned}&\|B-A\|\left\|(B-A)^{-1}\right\|\left[f\left(\left\|(B-A)^{-1}\right\|^{-1}\right)-f(0)1_H\right]\\&\geq f(B)-f(A)\\&\geq\frac{f(\|B-A\|+\|A\|)-f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H\\&\geq\frac{f(\|B\|)-f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H\geq 0.\end{aligned} \tag{2.18}$$

Proof. Since $B-A>0$, hence by (2.17) we get

$$\frac{1}{\left\|(B-A)^{-1}\right\|}1_H\leq B-A\leq\|B-A\|1_H.$$

So, if we write the inequality (2.14) for $c = \frac{1}{\left\|(B-A)^{-1}\right\|}$ and $d = \|B - A\|$, then we get

$$\begin{aligned}(2.19)\quad & \|B-A\| \left\|(B-A)^{-1}\right\| \left[f\left(\left\|(B-A)^{-1}\right\|^{-1}\right) - f(0)\, 1_H\right] \\ & \geq f(B) - f(A) \\ & \geq \frac{f(\|B-A\| + \|A\|) - f(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \geq 0.\end{aligned}$$

Also, we have $\|B - A\| + \|A\| \geq \|B\|$ and since f is nondecreasing, then

$$(2.20)\quad f(\|B-A\| + \|A\|) - f(\|A\|) \geq f(\|B\|) - f(\|A\|) \geq 0.$$

By (2.19) and (2.20) we derive (2.18). □

3. Some Examples

Assume that $B > A \geq 0$ and $r \in (0, 1]$. Then by (2.18) we have, for the operator monotone function $f(t) = t^r$ on $[0, \infty)$, the following refinement and reverse of Löwner–Heinz inequality

$$\begin{aligned}(3.1)\quad \|B-A\| \left\|(B-A)^{-1}\right\|^{1-r} 1_H & \geq B^r - A^r \\ & \geq \frac{(\|B-A\| + \|A\|)^r - \|A\|^r}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \\ & \geq \frac{\|B\|^r - \|A\|^r}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0.\end{aligned}$$

If we use the third inequality in (2.18) for the operator monotone function ln on $(0, \infty)$, then we get

$$\begin{aligned}(3.2)\quad \ln B - \ln A & \geq \frac{\ln(\|B-A\| + \|A\|) - \ln(\|A\|)}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H \\ & \geq \frac{\ln\|B\| - \ln\|A\|}{\left\|(B-A)^{-1}\right\| \|B-A\|} 1_H > 0\end{aligned}$$

for $B > A > 0$.

Consider the function

$$f_0(t) := \begin{cases} \frac{t}{1-t} \ln t \text{ for } t > 0 \\ \\ 0 \text{ for } t = 0, \end{cases}$$

which operator monotone on $[0,\infty)$. By the use of (2.18) we then have

$$\frac{\|B-A\|}{\left\|(B-A)^{-1}\right\|^{-1}-1}\ln\left\|(B-A)^{-1}\right\|1_H \tag{3.3}$$
$$\geq B(1-B)^{-1}\ln B - A(1-A)^{-1}\ln A$$
$$\geq \frac{\frac{\|B\|}{1-\|B\|}\ln\|B\| - \frac{\|A\|}{1-\|A\|}\ln\|A\|}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H > 0$$

for $B > A > 0$ and $\|A\|, \|B\|, \left\|(B-A)^{-1}\right\| \neq 1$.

The function $f(t) = \ln(t+1)$ is also operator monotone on $[0,\infty)$, then by (2.18) we have

$$\|B-A\|\left\|(B-A)^{-1}\right\|\ln\left(\left\|(B-A)^{-1}\right\|^{-1}+1\right)1_H \tag{3.4}$$
$$\geq \ln(B+1) - \ln(A+1)$$
$$\geq \frac{\ln(\|B-A\|+\|A\|+1) - \ln(\|A\|+1)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H$$
$$\geq \frac{\ln(\|B\|+1) - \ln(\|A\|+1)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H > 0$$

for $B > A \geq 0$.

Consider the function $f_{-1}(t) = \frac{2t}{1+t}$, $t \in [0,\infty)$, which is operator monotone, then by (2.18) we derive

$$\frac{\|B-A\|}{1+\left\|(B-A)^{-1}\right\|^{-1}}1_H \tag{3.5}$$
$$\geq B(1+B)^{-1} - A(1+A)^{-1}$$
$$\geq \frac{\|B\|-\|A\|}{\left\|(B-A)^{-1}\right\|\|B-A\|(1+\|B\|)(1+\|A\|)}1_H > 0,$$

for $B > A \geq 0$.

The interested reader may state other similar inequalities by employing the operator monotone functions presented in Introduction. We omit the details.

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