UPPER AND LOWER BOUNDS FOR $f(A)A^{-1}-f(B)B^{-1}$ WHEN f IS OPERATOR MONOTONE ON $[0,\infty)$

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we show that, if $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ with $f(0)=0$ and

$$0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$$

for some positive constants α, β, γ, δ, then

$$\begin{aligned}0&\leq\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right]\leq f(A)A^{-1}-f(B)B^{-1}\\&\leq\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right].\end{aligned}$$

In particular, we obtain the following inequalities for powers of positive operators

$$0\leq\frac{\gamma-\beta}{\delta-\beta}\left(\beta^{r-1}-\delta^{r-1}\right)\leq A^{r-1}-B^{r-1}\leq\frac{\delta-\alpha}{\gamma-\alpha}\left(\alpha^{r-1}-\gamma^{r-1}\right).$$

for $r\in(0,1]$.

The logarithmic inequalities

$$\begin{aligned}0&\leq\frac{\gamma-\beta}{\delta-\beta}\ln\left[\frac{(\beta+1)^{\beta^{-1}}}{(\delta+1)^{\delta^{-1}}}\right]\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\\&\leq\frac{\delta-\alpha}{\gamma-\alpha}\ln\left[\frac{(\alpha+1)^{\alpha^{-1}}}{(\gamma+1)^{\gamma^{-1}}}\right],\end{aligned}$$

are also valid.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq0$) if $\langle Tx,x\rangle\geq0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty\frac{ts}{t+s}dw(s)\tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator inequalities, Löwner-Heinz inequality.

RGMIA Res. Rep. Coll. 23 (2020), Art. 61, 10 pp. Received 27/05/20

where $b \geq 0$ *and a positive measure* w *on* $[0, \infty)$ *such that*

$$\int_0^\infty \frac{s}{1+s} dw(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [5]. The function ln is also operator monotone on $(0, \infty)$.

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m1_H > 0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0, \infty)$

$$\begin{aligned} f(B) - f(A) &\geq f(\|A\| + m) - f(\|A\|) \\ &\geq f(\|B\|) - f(\|B\| - m) > 0. \end{aligned} \tag{1.2}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) &\geq f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\ &\geq f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{1.3}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [8].

By taking $f(t) = t^r$, $r \in (0, 1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$\begin{aligned} B^r - A^r &\geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r - \|A\|^r \\ &\geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r > 0 \end{aligned} \tag{1.4}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\begin{aligned} \ln B - \ln A &\geq \ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - \ln(\|A\|) \\ &\geq \ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{1.5}$$

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [7].

Motivated by the above results, in this paper we show that, if $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ with $f(0) = 0$ and

$$0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$$

for some positive constants α, β, γ, δ, then

$$0 \leq \frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right] \leq f(A)A^{-1}-f(B)B^{-1}$$
$$\leq \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right].$$

In particular, we obtain the following inequalities for powers of positive operators

$$0 \leq \frac{\gamma-\beta}{\delta-\beta}\left(\beta^{r-1}-\delta^{r-1}\right) \leq A^{r-1}-B^{r-1} \leq \frac{\delta-\alpha}{\gamma-\alpha}\left(\alpha^{r-1}-\gamma^{r-1}\right).$$

for $r \in (0,1]$.

The logarithmic inequalities

$$0 \leq \frac{\gamma-\beta}{\delta-\beta}\ln\left[\frac{(\beta+1)^{\beta^{-1}}}{(\delta+1)^{\delta^{-1}}}\right] \leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)$$
$$\leq \frac{\delta-\alpha}{\gamma-\alpha}\ln\left[\frac{(\alpha+1)^{\alpha^{-1}}}{(\gamma+1)^{\gamma^{-1}}}\right],$$

are also valid.

2. Main Results

We have the following identity of interest in itself:

Lemma 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$ with the representation (1.1). Then for all A, $B>0$ we have the identity*

$$\begin{aligned} &f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\ &=\int_0^\infty \lambda\left(\int_0^1 \left((1-t)B+tA+\lambda 1_H\right)^{-1}(B-A)\right. \\ &\left.\times\left((1-t)B+tA+\lambda 1_H\right)^{-1}dt\right)dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. Since f is operator monotone on $[0,\infty)$, then there exists $b\geq 0$ and w is a positive measure satisfying

$$\int_0^\infty \frac{\lambda}{1+\lambda}dw(\lambda)<\infty$$

such that [1, p. 144-145]

$$f(t)=f(0)+bt+\int_0^\infty \frac{\lambda t}{t+\lambda}dw(\lambda). \tag{2.2}$$

We have for $t>0$ that

$$g(t):=\frac{f(t)-f(0)}{t}-b=\int_0^\infty \frac{\lambda}{t+\lambda}dw(\lambda).$$

Therefore for all A, $B>0$

$$g(B)-g(A)=\int_0^\infty \lambda\left[(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\right]dw(\lambda). \tag{2.3}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we replace in (2.6) $C=B+\lambda 1_H$ and $D=A+\lambda 1_H$ for $\lambda>0$, then we get

$$\begin{aligned}&(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\\&=\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(A-B)((1-t)B+tA+\lambda 1_H)^{-1}\,dt.\end{aligned} \tag{2.7}$$

Therefore, by (2.3),

$$\begin{aligned}g(B)-g(A)&=\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(A-B)\right.\\&\quad\left.\times((1-t)B+tA+\lambda 1_H)^{-1}\,dt\right)dw(\lambda)\\&=-\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(B-A)\right.\\&\quad\left.\times((1-t)B+tA+\lambda 1_H)^{-1}\,dt\right)dw(\lambda),\end{aligned} \tag{2.8}$$

namely

$$\begin{aligned}&[f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1}\\&=\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(B-A)\right.\\&\left.\times((1-t)B+tA+\lambda 1_H)^{-1}\,dt\right)dw(\lambda),\end{aligned}$$

which is equivalent to (2.1). □

Our main result is as follows:

Theorem 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If*

$$0<\alpha\le A\le\beta<\gamma\le B\le\delta \tag{2.9}$$

for some positive constants α, β, γ, δ, *then*

$$\begin{aligned} 0 &\le \frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right] \\ &\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right]. \end{aligned} \tag{2.10}$$

If $f(0)=0$, *then we have the simpler inequality*

$$\begin{aligned} 0 &\le \frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right] \le f(A)A^{-1}-f(B)B^{-1} \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right]. \end{aligned} \tag{2.11}$$

Proof. From (2.9) we have

$$0<\gamma-\beta\le B-A\le\delta-\alpha,$$

which implies that

$$\begin{aligned} 0 &\le (\gamma-\beta)\left((1-t)B+tA+\lambda\right)^{-2} \\ &\le \left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1} \\ &\le (\delta-\alpha)\left((1-t)B+tA+\lambda\right)^{-2} \end{aligned}$$

for all $t\in[0,1]$ and $\lambda\ge 0$.

By integration over $t\in[0,1]$ we deduce

$$\begin{aligned} 0 &\le (\gamma-\beta)\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt \\ &\le \int_0^1\left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1}dt \\ &\le (\delta-\alpha)\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt \end{aligned}$$

for all $\lambda\ge 0$.

If we multiply this inequality by λ and integrate over the measure $w(\lambda)$, we get

$$\begin{aligned} 0 &\le (\gamma-\beta)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda) \\ &\le \int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1}dt\right)dw(\lambda) \\ &\le (\delta-\alpha)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda), \end{aligned}$$

and, by (2.1) we derive the inequality of interest

$$\begin{aligned} 0 &\le (\gamma-\beta)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda) \\ &\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\ &\le (\delta-\alpha)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda). \end{aligned} \tag{2.12}$$

From (2.9) we derive that

$$(1-t)B+tA+\lambda \leq (1-t)\delta+t\beta+\lambda,$$

which implies that

$$((1-t)B+tA+\lambda)^{-1} \geq ((1-t)\delta+t\beta+\lambda)^{-1}$$

and

$$((1-t)B+tA+\lambda)^{-2} \geq ((1-t)\delta+t\beta+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Also

$$(1-t)B+tA+\lambda \geq (1-t)\gamma+t\alpha+\lambda,$$

which implies that

$$((1-t)B+tA+\lambda)^{-1} \leq ((1-t)\gamma+t\alpha+\lambda)^{-1}$$

and

$$((1-t)B+tA+\lambda)^{-2} \leq ((1-t)\gamma+t\alpha+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore

$$\begin{aligned}(2.13)\qquad &(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-2}dt\right)dw(\lambda)\\ &\leq(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda)\end{aligned}$$

and

$$\begin{aligned}(2.14)\qquad &(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda)\\ &\leq(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\gamma+t\alpha+\lambda)^{-2}dt\right)dw(\lambda).\end{aligned}$$

Since

$$\begin{aligned}&(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-2}dt\right)dw(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}\\ &\times\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-1}(\delta-\beta)((1-t)\delta+t\beta+\lambda)^{-1}dt\right)dw(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right]\ \text{(by (2.1))}\end{aligned}$$

and

$$
\begin{aligned}
&(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\alpha+t\gamma+\lambda)^{-2}\,dt\right)dw(\lambda)\\
&=\frac{\delta-\alpha}{\gamma-\alpha}\\
&\times\int_0^\infty \lambda\left(\int_0^1 ((1-t)\gamma+t\alpha+\lambda)^{-1}(\gamma-\alpha)((1-t)\gamma+t\alpha+\lambda)^{-1}\,dt\right)dw(\lambda)\\
&=\frac{\delta-\alpha}{\gamma-\alpha}\left[f(A)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right]\ \text{(by (2.1))},
\end{aligned}
$$

then (2.13) and (2.14) become

$$
\begin{aligned}
&\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right]\\
&\le(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}\,dt\right)dw(\lambda)
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
&(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}\,dt\right)dw(\lambda)\\
&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right].
\end{aligned}
\tag{2.16}
$$

Finally, on making use of (2.12), (2.15) and (2.17), we derive (2.10). □

Its is well known that, if $P \geq 0$, then

$$
|\langle Px,y\rangle|^2 \le \langle Px,x\rangle\langle Py,y\rangle
$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$
\begin{aligned}
0 &\le \langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\
&\le\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle
\end{aligned}
$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$
1\le\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\le\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,
$$

which implies the following operator inequality

$$
\left\|T^{-1}\right\|^{-1}\le T. \tag{2.17}
$$

Corollary 1. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$*. If* A*,* $B>0$ *and* $\|A\|\,\|B^{-1}\|<1$*, then*

$$
\begin{aligned}
(2.18)\qquad 0 &\leq \frac{1-\|A\|\,\|B^{-1}\|}{(\|B\|-\|A\|)\,\|B^{-1}\|} \\
&\times\left[f(\|A\|)\,\|A\|^{-1}-f(\|B\|)\,\|B\|^{-1}-f(0)\left(\|A\|^{-1}-\|B\|^{-1}\right)\right] \\
&\leq f(A)\,A^{-1}-f(B)\,B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\
&\leq \frac{(\|B\|\,\|A^{-1}\|-1)\,\|B^{-1}\|}{\|A^{-1}\|-\|B^{-1}\|} \\
&\times\left[f\left(\|A^{-1}\|^{-1}\right)\|A^{-1}\|-f\left(\|B^{-1}\|^{-1}\right)\|B^{-1}\|\right. \\
&\left.-f(0)\left(\|A^{-1}\|-\|B^{-1}\|\right)\right].
\end{aligned}
$$

If $f(0)=0$*, then we have the simpler inequality*

$$
\begin{aligned}
(2.19)\qquad 0 &\leq \frac{1-\|A\|\,\|B^{-1}\|}{(\|B\|-\|A\|)\,\|B^{-1}\|} \\
&\times\left[f(\|A\|)\,\|A\|^{-1}-f(\|B\|)\,\|B\|^{-1}\right] \\
&\leq f(A)\,A^{-1}-f(B)\,B^{-1} \\
&\leq \frac{(\|B\|\,\|A^{-1}\|-1)\,\|B^{-1}\|}{\|A^{-1}\|-\|B^{-1}\|} \\
&\times\left[f\left(\|A^{-1}\|^{-1}\right)\|A^{-1}\|-f\left(\|B^{-1}\|^{-1}\right)\|B^{-1}\|\right].
\end{aligned}
$$

Proof. Since $\|A\|\,\|B^{-1}\|<1$, then

$$0<\|A^{-1}\|^{-1}\leq A\leq\|A\|<\|B^{-1}\|^{-1}\leq B\leq\|B\|.$$

By employing the inequality (2.10) for $\alpha=\|A^{-1}\|^{-1}$, $\beta=\|A\|$, $\gamma=\|B^{-1}\|^{-1}$ and $\delta=\|B\|$ we get

$$
\begin{aligned}
0 &\leq \frac{\|B^{-1}\|^{-1}-\|A\|}{\|B\|-\|A\|} \\
&\times\left[f(\|A\|)\,\|A\|^{-1}-f(\|B\|)\,\|B\|^{-1}-f(0)\left(\|A\|^{-1}-\|B\|^{-1}\right)\right] \\
&\leq f(A)\,A^{-1}-f(B)\,B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\
&\leq \frac{\|B\|-\|A^{-1}\|^{-1}}{\|B^{-1}\|^{-1}-\|A^{-1}\|^{-1}} \\
&\times\left[f\left(\|A^{-1}\|^{-1}\right)\|A^{-1}\|-f\left(\|B^{-1}\|^{-1}\right)\|B^{-1}\|-f(0)\left(\|A^{-1}\|-\|B^{-1}\|\right)\right],
\end{aligned}
$$

which is equivalent to (2.19). □

3. Some Examples

If $0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$ for some positive constants α, β, γ, δ, then (by 2.11)

$$
(3.1)\qquad 0\leq\frac{\gamma-\beta}{\delta-\beta}\left(\beta^{r-1}-\delta^{r-1}\right)\leq A^{r-1}-B^{r-1}\leq\frac{\delta-\alpha}{\gamma-\alpha}\left(\alpha^{r-1}-\gamma^{r-1}\right).
$$

for $r\in(0,1]$.

If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then by (2.19) we get

$$\begin{aligned}0&\le\frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[\left(\|A\|^{r-1}-\|B\|^{r-1}\right)\right]\\&\le A^{r-1}-B^{r-1}\\&\le\frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left(\left\|A^{-1}\right\|^{1-r}-\left\|B^{-1}\right\|^{1-r}\right).\end{aligned}\tag{3.2}$$

for $r\in(0,1]$.

The function $f(t)=\ln(t+1)$ is operator monotone on $[0,\infty)$ and $f(0)=0$. If we write the inequality (2.11) for this function, we derive

$$\begin{aligned}0&\le\frac{\gamma-\beta}{\delta-\beta}\ln\left[\frac{(\beta+1)^{\beta^{-1}}}{(\delta+1)^{\delta^{-1}}}\right]\le A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\ln\left[\frac{(\alpha+1)^{\alpha^{-1}}}{(\gamma+1)^{\gamma^{-1}}}\right],\end{aligned}\tag{3.3}$$

provided that $0<\alpha\le A\le\beta<\gamma\le B\le\delta$ for some positive constants α, β, γ, δ.

If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then by (2.19) we get

$$\begin{aligned}0&\le\frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\\&\times\left[\|A\|^{-1}\ln(\|A\|+1)-\|B\|^{-1}\ln(\|B\|+1)\right]\\&\le A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\\&\le\frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\\&\times\left[\left\|A^{-1}\right\|\ln\left(\left\|A^{-1}\right\|^{-1}+1\right)-\left\|B^{-1}\right\|\ln\left(\left\|B^{-1}\right\|^{-1}+1\right)\right].\end{aligned}\tag{3.4}$$

For other examples of operator monotone functions see also [2], [4] and the references therein.

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[3] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[4] T. Furuta, Precise lower bound of $f(A)-f(B)$ for $A>B>0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[5] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[6] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[7] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[8] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.