NEW INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we prove that, if $f:(0,\infty)\to\mathbb{R}$ is operator monotone on $(0,\infty)$, then for all A, B such that

$$0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$$

for some positive constants α, β, γ, δ,

$$0\leq(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta}\leq f(B)-f(A)\leq(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}.$$

In particular, we have the refinement and reverse of the celebrated Löwner-Heinz inequality

$$0<(\gamma-\beta)\frac{\delta^r-\beta^r}{\delta-\beta}\leq B^r-A^r\leq(\delta-\alpha)\frac{\gamma^r-\alpha^r}{\gamma-\alpha}$$

for all $r\in(0,1]$.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq0$) if $\langle Tx,x\rangle\geq0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{ts}{t+s}dm(s), \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty\frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [5]. The function ln is also operator monotone on $(0,\infty)$.

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\geq m1_H>0$. In 2015, [4], T. Furuta obtained the following result for any

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non-constant operator monotone function f on $[0,\infty)$

$$f(B)-f(A)\geq f(\|A\|+m)-f(\|A\|) \geq f(\|B\|)-f(\|B\|-m)>0. \tag{1.2}$$

If $B>A>0$, then

$$\begin{aligned} f(B)-f(A) &\geq f\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-f(\|A\|) \\ &\geq f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0. \end{aligned} \tag{1.3}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t)=t^r$, $r\in(0,1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$\begin{aligned} B^r-A^r &\geq \left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r-\|A\|^r \\ &\geq \|B\|^r-\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r>0 \end{aligned} \tag{1.4}$$

provided $B>A>0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\begin{aligned} \ln B-\ln A &\geq \ln\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-\ln(\|A\|) \\ &\geq \ln(\|B\|)-\ln\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0. \end{aligned} \tag{1.5}$$

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, in this paper we prove that, if $f:(0,\infty)\to\mathbb{R}$ is operator monotone on $(0,\infty)$, then for all A, B such that

$$0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$$

for some positive constants α, β, γ, δ,

$$0\leq(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta}\leq f(B)-f(A)\leq(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}.$$

In particular, we have the refinement and reverse of the celebrated Löwner-Heinz inequality

$$0<(\gamma-\beta)\frac{\delta^r-\beta^r}{\delta-\beta}\leq B^r-A^r\leq(\delta-\alpha)\frac{\gamma^r-\alpha^r}{\gamma-\alpha}$$

for all $r\in(0,1]$.

2. Main Results

We start to the following identity of interest:

Lemma 1. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.1). Then for all* A, $B>0$ *we have*

$$\begin{aligned} f(B)-f(A) &= b(B-A) \\ &\quad+\int_0^\infty s^2\left[\int_0^1 ((1-t)A+tB+s)^{-1}\right. \\ &\quad\left.\times (B-A)((1-t)A+tB+s)^{-1}dt\right]dm(s). \end{aligned} \tag{2.1}$$

Proof. Since the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.1), then for A, $B>0$ we have the representation

$$\begin{aligned} & f(B)-f(A) \\ &= b(B-A)+\int_0^\infty s\left[B(B+s)^{-1}-A(A+s)^{-1}\right]dm(s). \end{aligned} \tag{2.2}$$

Observe that for $s>0$

$$\begin{aligned} & B(B+s)^{-1}-A(A+s)^{-1} \\ &= (B+s-s)(B+s)^{-1}-(A+s-s)(A+s)^{-1} \\ &= (B+s)(B+s)^{-1}-s(B+s)^{-1}-(A+s)(A+s)^{-1}+s(A+s)^{-1} \\ &= s\left[(A+s)^{-1}-(B+s)^{-1}\right]. \end{aligned}$$

Therefore, (2.2) becomes, see also [4]

$$f(B)-f(A)=b(B-A)+\int_0^\infty s^2\left[(A+s)^{-1}-(B+s)^{-1}\right]dm(s). \tag{2.3}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt. \tag{2.6}$$

Now, if we replace in (2.6) $C=A+s$ and $D=B+s$ for $s>0$, then

$$(2.7)\quad \begin{aligned}&(A+s)^{-1}-(B+s)^{-1}\\&=\int_0^1((1-t)A+tB+s)^{-1}(B-A)((1-t)A+tB+s)^{-1}dt.\end{aligned}$$

By the representation (2.3), we derive (2.1). □

Theorem 2. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.1). If*

$$0<\alpha\le A\le\beta<\gamma\le B\le\delta \tag{2.8}$$

for some positive constants α, β, γ, δ, *then*

$$(2.9)\quad \begin{aligned}0&\le(\gamma-\beta)\left(\frac{f(\delta)-f(\beta)}{\delta-\beta}-b\right)\\&\le f(B)-f(A)-b(B-A)\\&\le(\delta-\alpha)\left(\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}-b\right).\end{aligned}$$

Proof. From (2.8) we have

$$0<\gamma-\beta\le B-A\le\delta-\alpha,$$

which implies that

$$\begin{aligned}0&\le(\gamma-\beta)((1-t)A+tB+s)^{-2}\\&\le((1-t)A+tB+s)^{-1}(B-A)((1-t)A+tB+s)^{-1}\\&\le(\delta-\alpha)((1-t)A+tB+s)^{-2}\end{aligned}$$

for all $t\in[0,1]$ and $s\ge0$.

By integration over $t\in[0,1]$ we deduce

$$\begin{aligned}0&\le(\gamma-\beta)\int_0^1((1-t)A+tB+s)^{-2}dt\\&\le\int_0^1((1-t)A+tB+s)^{-1}(B-A)((1-t)A+tB+s)^{-1}dt\\&\le(\delta-\alpha)\int_0^1((1-t)A+tB+s)^{-2}dt\end{aligned}$$

for all $s\ge0$.

If we multiply this inequality by s^2 and integrate over the measure $m(s)$, we get

$$\begin{aligned}0&\le(\gamma-\beta)\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s)^{-2}dt\right)dm(s)\\&\le\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s)^{-1}(B-A)((1-t)A+tB+s)^{-1}dt\right)dm(s)\\&\le(\delta-\alpha)\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s)^{-2}dt\right)dm(s),\end{aligned}$$

and, by (2.1) we derive the inequality of interest

$$\begin{aligned} 0 &\leq (\gamma-\beta)\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s)^{-2}\,dt\right)dm(s) \\ &\leq f(B)-f(A)-b(B-A) \\ &\leq (\delta-\alpha)\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s)^{-2}\,dt\right)dm(s). \end{aligned} \tag{2.10}$$

From (2.8) we derive that

$$(1-t)A+tB+s \leq (1-t)\beta+t\delta+s,$$

which implies that

$$((1-t)A+tB+s)^{-1} \geq ((1-t)\beta+t\delta+s)^{-1}$$

and

$$((1-t)A+tB+s)^{-2} \geq ((1-t)\beta+t\delta+s)^{-2}$$

for all $t\in[0,1]$ and $s\geq 0$.

Also

$$(1-t)A+tB+s \geq (1-t)\alpha+t\gamma+s,$$

which implies that

$$((1-t)A+tB+s)^{-1} \leq ((1-t)\alpha+t\gamma+s)^{-1}$$

and

$$((1-t)A+tB+s)^{-2} \leq ((1-t)\alpha+t\gamma+s)^{-2}$$

for all $t\in[0,1]$ and $s\geq 0$.

Therefore

$$\begin{aligned} &(\gamma-\beta)\int_0^\infty s^2\left(\int_0^1 ((1-t)\beta+t\delta+s)^{-2}\,dt\right)dm(s) \\ &\leq (\gamma-\beta)\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s)^{-2}\,dt\right)dm(s) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &(\delta-\alpha)\int_0^\infty s^2\left(\int_0^1 ((1-t)A+tB+s)^{-2}\,dt\right)dm(s) \\ &\leq (\delta-\alpha)\int_0^\infty s^2\left(\int_0^1 ((1-t)\alpha+t\gamma+s)^{-2}\,dt\right)dm(s). \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} &(\gamma-\beta)\int_0^\infty s^2\left(\int_0^1 ((1-t)\beta+t\delta+s)^{-2}\,dt\right)dm(s) \\ &= \frac{\gamma-\beta}{\delta-\beta} \\ &\times \int_0^\infty s^2\left(\int_0^1 ((1-t)\beta+t\delta+s)^{-1}(\delta-\beta)((1-t)\beta+t\delta+s)^{-1}\,dt\right)dm(s) \\ &= \frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)-b(\delta-\beta)\right] \text{ (by (2.1))} \end{aligned}$$

and

$$(\delta-\alpha)\int_0^\infty s^2\left(\int_0^1((1-t)\alpha+t\gamma+s)^{-2}dt\right)dm(s)$$
$$=\frac{\delta-\alpha}{\gamma-\alpha}$$
$$\times\int_0^\infty s^2\left(\int_0^1((1-t)\alpha+t\gamma+s)^{-1}(\gamma-\alpha)((1-t)\alpha+t\gamma+s)^{-1}dt\right)dm(s)$$
$$=\frac{\delta-\alpha}{\gamma-\alpha}[f(\gamma)-f(\alpha)-b(\gamma-\alpha)] \text{ (by (2.1))},$$

then (2.11) and (2.12) become

$$\frac{\gamma-\beta}{\delta-\beta}[f(\delta)-f(\beta)-b(\delta-\beta)] \le(\gamma-\beta)\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s)^{-2}dt\right)dm(s) \tag{2.13}$$

and

$$(\delta-\alpha)\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s)^{-2}dt\right)dm(s) \le\frac{\delta-\alpha}{\gamma-\alpha}[f(\gamma)-f(\alpha)-b(\gamma-\alpha)]. \tag{2.14}$$

Finally, on making use of (2.10), (2.13) and (2.14), we derive (2.9). □

Since the parameter b may be difficult to find for a given operator monotone function, we can give the following double inequality where this parameter is not involved:

Corollary 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If the condition (2.8) is satisfied, then*

$$0\le(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta}\le f(B)-f(A)\le(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}. \tag{2.15}$$

Proof. From the first inequality in (2.9) we derive

$$\begin{aligned}f(B)-f(A)&\ge b(B-A)+(\gamma-\beta)\left(\frac{f(\delta)-f(\beta)}{\delta-\beta}-b\right)\\&=b[(B-A)-(\gamma-\beta)]+(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta}\\&\ge(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta}\ge0\end{aligned}$$

since $b\ge0$ and $(B-A)-(\gamma-\beta)\ge0$.

From the second inequality in (2.9) we get

$$\begin{aligned}f(B)-f(A)&\le b(B-A)+(\delta-\alpha)\left(\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}-b\right)\\&=b[(B-A)-(\delta-\alpha)]+(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}\\&\le(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}\end{aligned}$$

since $b \geq 0$ and $(B-A)-(\delta-\alpha) \leq 0$. □

Its is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned} 0 &\leq \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ &\leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T. \tag{2.16}$$

Corollary 2. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$. If A, $B > 0$ and $\|A\| \left\|B^{-1}\right\| < 1$, then*

$$\begin{aligned} 0 &< \frac{1 - \|A\| \left\|B^{-1}\right\|}{\|B^{-1}\|} \frac{f(\|B\|) - f(\|A\|)}{\|B\| - \|A\|} \\ &\leq f(B) - f(A) \\ &\leq \left(\|B\| \left\|A^{-1}\right\| - 1\right) \left\|B^{-1}\right\| \frac{f\left(\left\|B^{-1}\right\|^{-1}\right) - f\left(\left\|A^{-1}\right\|^{-1}\right)}{\|A^{-1}\| - \|B^{-1}\|}. \end{aligned} \tag{2.17}$$

Proof. Since $\|A\| \left\|B^{-1}\right\| < 1$, then

$$0 < \left\|A^{-1}\right\|^{-1} \leq A \leq \|A\| < \left\|B^{-1}\right\|^{-1} \leq B \leq \|B\|.$$

By employing the inequality (2.15) for $\alpha = \left\|A^{-1}\right\|^{-1}$, $\beta = \|A\|$, $\gamma = \left\|B^{-1}\right\|^{-1}$ and $\delta = \|B\|$ we get

$$\begin{aligned} 0 &\leq \left(\left\|B^{-1}\right\|^{-1} - \|A\|\right) \frac{f(\|B\|) - f(\|A\|)}{\|B\| - \|A\|} \leq f(B) - f(A) \\ &\leq \left(\|B\| - \left\|A^{-1}\right\|^{-1}\right) \frac{f\left(\left\|B^{-1}\right\|^{-1}\right) - f\left(\left\|A^{-1}\right\|^{-1}\right)}{\|B^{-1}\|^{-1} - \|A^{-1}\|^{-1}}, \end{aligned}$$

which is equivalent to (2.17). □

3. Some Examples

Assume that $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$, then by (2.9) we have the inequalities

$$0 < (\gamma - \beta) \frac{\delta^r - \beta^r}{\delta - \beta} \leq B^r - A^r \leq (\delta - \alpha) \frac{\gamma^r - \alpha^r}{\gamma - \alpha} \tag{3.1}$$

for all $r \in (0, 1]$.

We also have the logarithmic inequalities

$$0 \leq (\gamma - \beta) \frac{\ln\delta - \ln\beta}{\delta - \beta} \leq \ln B - \ln A \leq (\delta - \alpha) \frac{\ln\gamma - \ln\alpha}{\gamma - \alpha}. \tag{3.2}$$

If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then by (2.17)

$$0<\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\frac{\|B\|^{r}-\|A\|^{r}}{\|B\|-\|A\|} \tag{3.3}$$
$$\leq B^{r}-A^{r}$$
$$\leq\frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|^{1-r}}{\|A^{-1}\|^{r}}\frac{\left\|A^{-1}\right\|^{r}-\left\|B^{-1}\right\|^{r}}{\|A^{-1}\|-\|B^{-1}\|}$$

for all $r\in(0,1]$.

We also have the logarithmic inequalities

$$0<\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\frac{\ln\|B\|-\ln\|A\|}{\|B\|-\|A\|} \tag{3.4}$$
$$\leq\ln B-\ln A$$
$$\leq\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|\frac{\ln\left\|A^{-1}\right\|-\ln\left\|B^{-1}\right\|}{\|A^{-1}\|-\|B^{-1}\|}.$$

For other examples of operator monotone functions, see [2], [3], [7] and the references therein.

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