LIPSCHITZ TYPE INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we prove that, if the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$, then for all $A\geq m_1>0$, $B\geq m_2>0$,

$$\|f(B)-f(A)\|\leq\|B-A\|\begin{cases}\frac{f(m_2)-f(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ f'(m) & \text{if } m_1=m_2=m.\end{cases}$$

If f is operator monotone on $[0,\infty)$ and $f(0)=0$, then also

$$\left\|f(A)A^{-1}-f(B)B^{-1}\right\|\leq\|B-A\|\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \frac{f(m)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}$$

Some aplications related to the midpoint and trapezoid norm inequalities are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

In 1934, K. Löwner [14] had given a definitive characterization of operator monotone functions as follows, see for instance [5, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}dw(\lambda) \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}dw(\lambda)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [12]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [9] and [11].

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Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [6], [7] and Kato in [13], the following inequality holds

$$\||A| - |B|\| \le \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$ with $A \ne B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \le \sqrt{2} \|A - B\|_{HS} \tag{1.3}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \le a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right) \tag{1.4}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \le f'(a) \|A - B\| \tag{1.5}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \ge aI_H > 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [8] and the references therein.

Motivated by the above results, in this paper we prove that, if the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$, then for all $A \ge m_1 > 0$, $B \ge m_2 > 0$,

$$\|f(B) - f(A)\| \le \|B - A\| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} \text{ if } m_1 \ne m_2, \\ \\ f'(m) \text{ if } m_1 = m_2 = m. \end{cases}$$

If f is operator monotone on $[0,\infty)$ and $f(0)=0$, then also

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}\right\|\\ &\leq \|B-A\|\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ \frac{f(m)-f'(m)m}{m^2} \text{ if } m_1=m_2=m.\end{cases}\end{aligned}$$

Some applications related to the midpoint and trapezoid norm inequalities are also provided.

2. Main Results

We start to the following identity of interest:

Lemma 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.1). Then for all A, $B>0$ we have*

$$\begin{aligned}f(B)-f(A)&=b(B-A)\\ &+\int_0^\infty \lambda^2\left[\int_0^1((1-t)A+tB+\lambda)^{-1}\right.\\ &\left.\times(B-A)((1-t)A+tB+\lambda)^{-1}dt\right]dw(\lambda).\end{aligned}\tag{2.1}$$

Proof. Since the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.1), then for A, $B>0$ we have the representation

$$\begin{aligned}&f(B)-f(A)\\ &=b(B-A)+\int_0^\infty \lambda\left[B(B+\lambda)^{-1}-A(A+\lambda)^{-1}\right]dw(\lambda).\end{aligned}\tag{2.2}$$

Observe that for $\lambda>0$

$$\begin{aligned}&B(B+\lambda)^{-1}-A(A+\lambda)^{-1}\\ &=(B+\lambda-\lambda)(B+\lambda)^{-1}-(A+\lambda-\lambda)(A+\lambda)^{-1}\\ &=(B+\lambda)(B+\lambda)^{-1}-\lambda(B+\lambda)^{-1}-(A+\lambda)(A+\lambda)^{-1}+\lambda(A+\lambda)^{-1}\\ &=\lambda\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right].\end{aligned}$$

Therefore, (2.2) becomes, see also [6]

$$f(B)-f(A)=b(B-A)+\int_0^\infty \lambda^2\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]dw(\lambda).\tag{2.3}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}\tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-t)C + tD,\ t \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C + tD)^{-1}\,dt. \tag{2.6}$$

Now, if we replace in (2.6) $C = A + \lambda$ and $D = B + \lambda$ for $\lambda > 0$, then

$$\begin{aligned}(A+\lambda)^{-1} &- (B+\lambda)^{-1} \\ &= \int_0^1 ((1-t)A + tB + \lambda)^{-1}(B-A)((1-t)A + tB + \lambda)^{-1}\,dt.\end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

We have the norm inequality:

Lemma 2. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.1). Then for all $A \geq m_1 > 0$, $B \geq m_2 > 0$ with $m_1 \neq m_2$, we have*

$$\|f(B) - f(A) - b(B-A)\| \leq \|B-A\|\left[\frac{f(m_2) - f(m_1)}{m_2 - m_1} - b\right]. \tag{2.8}$$

Proof. From the identity (2.1) we obtain

$$\begin{aligned}&\|f(B) - f(A) - b(B-A)\| \\ &\leq \int_0^\infty \lambda^2 \left[\left\|\int_0^1 ((1-t)A + tB + \lambda)^{-1}\right.\right. \\ &\left.\left.\times (B-A)((1-t)A + tB + \lambda)^{-1}\,dt\right]\right\| dw(\lambda) \\ &\leq \|B-A\| \int_0^\infty \lambda^2 \left(\int_0^1 \left\|((1-t)A + tB + \lambda)^{-1}\right\|^2 dt\right) dw(\lambda)\end{aligned} \tag{2.9}$$

for all $A,\ B > 0$.

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\|((1-t)A + tB + \lambda)^{-1}\right\|^2 \leq ((1-t)m_1 + tm_2 + \lambda)^{-2} \tag{2.10}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (2.10) we derive

$$\int_0^\infty \lambda^2 \left(\int_0^1 \left\| ((1-t) A + tB + \lambda)^{-1} \right\|^2 dt \right) dw(\lambda)$$
$$\leq \int_0^\infty \lambda^2 \left(\int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-2} dt \right) dw(\lambda)$$
$$= \frac{1}{m_2 - m_1} \int_0^\infty \lambda^2 \left(\int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-1} \right.$$
$$\left. \times (m_2 - m_1) ((1-t) m_1 + tm_2 + \lambda)^{-1} dt \right) dw(\lambda)$$
$$= \frac{1}{m_2 - m_1} [f(m_2) - f(m_1) - b(m_2 - m_1)] \text{ (by (2.1)).}$$

and by (2.9) we deduce (2.8).

The case $m_2 < m_1$ goes in a similar way. □

Lemma 3. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation (1.1). Then for all A, $B \geq m > 0$,*

$$\|f(B) - f(A) - b(B - A)\| \leq \|B - A\| (f'(m) - b). \tag{2.11}$$

Proof. Let $\epsilon > 0$. Then $B + \epsilon \geq m + \epsilon > 0$. From (2.8) we get

$$\|f(B + \epsilon) - f(A) - b(B - A)\| \leq \|B + \epsilon - A\| \left[\frac{f(m + \epsilon) - f(m)}{m + \epsilon - m} - b \right]$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of f, we get (2.11). □

Theorem 2. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$. Then for all $A \geq m_1 > 0$, $B \geq m_2 > 0$, we have*

$$\|f(B) - f(A)\| \leq \|B - A\| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ f'(m) & \text{if } m_1 = m_2 = m, \text{ see (1.5).} \end{cases} \tag{2.12}$$

Proof. From (2.8) we get

$$\|f(B) - f(A)\| - b\|B - A\| \leq \|f(B) - f(A) - b(B - A)\|$$
$$\leq \|B - A\| \left[\frac{f(m_2) - f(m_1)}{m_2 - m_1} - b \right]$$
$$= \|B - A\| \frac{f(m_2) - f(m_1)}{m_2 - m_1} - b\|B - A\|,$$

which implies the first part of (2.12).

The second part follows in a similar way. □

Remark 1. *Let $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $r \in (0, 1]$. Then by (2.12) we have the power inequalities*

$$\|B^r - A^r\| \leq \|B - A\| \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ rm^{r-1} & \text{if } m_1 = m_2 = m. \end{cases} \tag{2.13}$$

We also can state the logarithmic inequalities

$$\|\ln B-\ln A\|\le\|B-A\|\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \frac{1}{m} & \text{if } m_1=m_2=m.\end{cases}\tag{2.14}$$

If $1>A\ge m_1>0$, $1>B\ge m_2>0$, *then*

$$\left\|(1-B)^{-1}-(1-A)^{-1}\right\|\le\|B-A\|\begin{cases}\frac{1}{(1-m_1)(1-m_2)} & \text{if } m_1\neq m_2,\\ \\ \frac{1}{(1-m)^2} & \text{if } m_1=m_2=m.\end{cases}\tag{2.15}$$

3. Related Results

We also have the following identity of interest:

Lemma 4. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone on* $[0,\infty)$ *with the representation (1.1). Then for all* A, $B>0$ *we have the identity*

$$\begin{aligned}&f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\&=\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(B-A)\right.\\&\left.\times((1-t)B+tA+\lambda 1_H)^{-1}dt\right)dw(\lambda).\end{aligned}\tag{3.1}$$

Proof. Since f is operator monotone on $[0,\infty)$, then there exists $b\ge 0$ and w is a positive measure such that

$$f(t)=f(0)+bt+\int_0^\infty\frac{\lambda t}{t+\lambda}dw(\lambda).\tag{3.2}$$

We have for $t>0$ that

$$g(t):=\frac{f(t)-f(0)}{t}-b=\int_0^\infty\frac{\lambda}{t+\lambda}dw(\lambda).$$

Therefore for all A, $B>0$

$$g(B)-g(A)=\int_0^\infty\lambda\left[(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\right]dw(\lambda).\tag{3.3}$$

Now, if we replace in (2.6) $C=B+\lambda 1_H$ and $D=A+\lambda 1_H$ for $\lambda>0$, then we get

$$\begin{aligned}&(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\\&=\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(A-B)((1-t)B+tA+\lambda 1_H)^{-1}dt.\end{aligned}\tag{3.4}$$

Therefore, by (3.3) and (3.4) we derive

$$\begin{aligned}g(B)-g(A)&=\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(A-B)\right.\\&\left.\times((1-t)B+tA+\lambda 1_H)^{-1}dt\right)dw(\lambda)\\&=-\int_0^\infty\lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(B-A)\right.\\&\left.\times((1-t)B+tA+\lambda 1_H)^{-1}dt\right)dw(\lambda),\end{aligned}$$

namely

$$\begin{aligned}&[f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1}\\&=\int_0^\infty \lambda\left(\int_0^1((1-t)B+tA+\lambda 1_H)^{-1}(B-A)\right.\\&\left.\times((1-t)B+tA+\lambda 1_H)^{-1}dt\right)dw(\lambda),\end{aligned}$$

which is equivalent to (3.1). □

Theorem 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. Then for all $A\geq m_1>0$, $B\geq m_2>0$, we have*

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\right\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}\right) & \text{if } m_1\neq m_2,\\ \\ \frac{f(m)-f(0)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{3.5}$$

If $f(0)=0$, then we have the simpler inequalities

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}\right\|\\&\leq\|B-A\|\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \frac{f(m)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{3.6}$$

Proof. Assume that $m_2>m_1>0$. By (3.1) we have

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\right\|\\&\leq\|B-A\|\int_0^\infty\lambda\left(\int_0^1\left\|((1-t)B+tA+\lambda 1_H)^{-1}\right\|^2dt\right)dw(\lambda).\end{aligned}\tag{3.7}$$

We have

$$(1-t)B+tA+\lambda 1_H\geq(1-t)m_2+tm_1+\lambda 1_H$$

for $t\in[0,1]$ and $\lambda\geq 0$, which implies that

$$((1-t)B+tA+\lambda 1_H)^{-1}\leq((1-t)m_2+tm_1+\lambda 1_H)^{-1}$$

and

$$\left\|((1-t)B+tA+\lambda 1_H)^{-1}\right\|^2\leq((1-t)m_2+tm_1+\lambda 1_H)^{-2}$$

for $t\in[0,1]$ and $\lambda\geq 0$.

By integrating this inequality we obtain

$$\begin{aligned}
&\int_0^\infty \lambda\left(\int_0^1 \left\|\left((1-t)B+tA+\lambda 1_H\right)^{-1}\right\|^2 dt\right)dw(\lambda)\\
&\le \int_0^\infty \lambda\left(\int_0^1 \left((1-t)m_2+tm_1+\lambda 1_H\right)^{-2} dt\right)dw(\lambda)\\
&= \frac{1}{m_2-m_1}\int_0^\infty \lambda\left(\int_0^1 \left((1-t)m_2+tm_1+\lambda 1_H\right)^{-1}(m_2-m_1)\right.\\
&\times \left.\left((1-t)m_2+tm_1+\lambda 1_H\right)^{-1} dt\right)dw(\lambda)\\
&= \frac{1}{m_2-m_1}\left[f(m_1)m_1^{-1}-f(m_2)m_2^{-1}-f(0)\left(m_1^{-1}-m_2^{-1}\right)\right] \text{ (by (3.1))}\\
&= \frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}
\end{aligned}$$

and by (3.7) we derive (3.5).

The case when $m_1 > m_2 > 0$ follows in a similar way.

If $m_1 = m_2 = m$, then we use a similar argument to the one in the proof of Lemma 3 and deduce the second branch of (3.5). □

Remark 2. *Let $A \ge m_1 > 0$, $B \ge m_2 > 0$ and $r \in (0,1]$. Then by (3.6) we have the power inequalities*

$$\left\|A^{r-1}-B^{r-1}\right\| \le \|B-A\| \begin{cases} \frac{m_1^{r-1}-m_2^{r-1}}{m_2-m_1} & \text{if } m_1 \ne m_2,\\[2ex] \frac{1-r}{m^{2-r}} & \text{if } m_1=m_2=m.\end{cases} \tag{3.8}$$

If we take $f(t)=\ln(t+1)$, then we get by (3.6) that

$$\begin{aligned}
&\left\|A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\right\| \\
&\le \|B-A\| \begin{cases} \frac{m_1^{-1}\ln(m_1+1)-m_2^{-1}\ln(m_2+1)}{m_2-m_1} & \text{if } m_1 \ne m_2,\\[2ex] \frac{(m+1)\ln(m+1)-m}{m^2(m+1)} & \text{if } m_1=m_2=m.\end{cases}
\end{aligned} \tag{3.9}$$

4. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. Then for all A, $B \ge m > 0$,*

$$\left\|\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right\| \le \frac{1}{4}f'(m)\|B-A\|. \tag{4.1}$$

If $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ with $f(0)=0$ and A, $B \ge m > 0$, then

$$\begin{aligned}
&\left\|\int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right)\right\|\\
&\le \frac{f(m)-f'(m)m}{4m^2}\|B-A\|.
\end{aligned} \tag{4.2}$$

Proof. Since $A,\ B\geq m$, hence $\frac{A+B}{2}\geq m>0$ and $(1-t)\,A+tB\geq m>0$ for all $t\in[0,1]$ and by (2.12)

$$\left\Vert f\left((1-t)\,A+tB\right)-f\left(\frac{A+B}{2}\right)\right\Vert \leq f'(m)\left\Vert(1-t)\,A+tB-\frac{A+B}{2}\right\Vert = f'(m)\left|t-\frac{1}{2}\right|\left\Vert B-A\right\Vert \tag{4.3}$$

for all $t\in[0,1]$.

Taking the integral in (4.3), we get

$$\begin{aligned}&\left\Vert\int_0^1 f\left((1-t)\,A+tB\right)dt-f\left(\frac{A+B}{2}\right)\right\Vert\\ &\leq\int_0^1\left\Vert f\left((1-t)\,A+tB\right)-f\left(\frac{A+B}{2}\right)\right\Vert dt\\ &\leq f'(m)\left\Vert B-A\right\Vert\int_0^1\left|t-\frac{1}{2}\right|dt=\frac{1}{4}f'(m)\left\Vert B-A\right\Vert\end{aligned}$$

and the inequality (4.1).

The inequality (4.2) follows in a similar way from (3.6). □

Assume that $A,\ B\geq m>0$, then by Proposition 1 we obtain the following power inequalities

$$\left\Vert\int_0^1\left((1-t)\,A+tB\right)^r dt-\left(\frac{A+B}{2}\right)^r\right\Vert\leq\frac{1}{4}rm^{r-1}\left\Vert B-A\right\Vert \tag{4.4}$$

and

$$\left\Vert\int_0^1\left((1-t)\,A+tB\right)^{r-1} dt-\left(\frac{A+B}{2}\right)^{r-1}\right\Vert\leq\frac{1-r}{4m^{2-r}}\left\Vert B-A\right\Vert, \tag{4.5}$$

where $r\in(0,1]$.

We have the logarithmic inequalities

$$\left\Vert\int_0^1\ln\left((1-t)\,A+tB\right)dt-\ln\left(\frac{A+B}{2}\right)\right\Vert\leq\frac{1}{4m}\left\Vert B-A\right\Vert \tag{4.6}$$

and

$$\begin{aligned}&\left\Vert\int_0^1\left((1-t)\,A+tB\right)^{-1}\ln\left((1-t)\,A+tB+1\right)dt\right.\\ &\left.-\left(\frac{A+B}{2}\right)^{-1}\ln\left(\frac{A+B}{2}+1\right)\right\Vert\\ &\leq\frac{(m+1)\ln(m+1)-m}{4m^2(m+1)}\left\Vert B-A\right\Vert.\end{aligned} \tag{4.7}$$

We also have the trapezoid type inequalities:

Proposition 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. Then for all $A,\ B\geq m>0$,*

$$\left\Vert\frac{f(A)+f(B)}{2}-\int_0^1 f\left((1-t)\,A+tB\right)dt\right\Vert\leq\frac{1}{4}f'(m)\left\Vert B-A\right\Vert. \tag{4.8}$$

If $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *with* $f(0)=0$ *and* $A,\ B\geq m>0$, *then*

$$\left\|\frac{f(A)A^{-1}+f(B)B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt\right\| \leq\frac{f(m)-f'(m)m}{4m^2}\|B-A\|. \tag{4.9}$$

Proof. Since $A,\ B\geq m$, hence $(1-s)A+s\frac{A+B}{2},\ s\frac{A+B}{2}+(1-s)B\geq m>0$ for all $s\in[0,1]$ and by (2.12) we get

$$\left\|f(A)-f\left((1-s)A+s\frac{A+B}{2}\right)\right\|\leq\frac{1}{2}f'(m)\|B-A\|s \tag{4.10}$$

and

$$\left\|f(B)-f\left(s\frac{A+B}{2}+(1-s)B\right)\right\|\leq\frac{1}{2}f'(m)\|B-A\|s. \tag{4.11}$$

From (4.10) and (4.11) we derive by addition, division by 2 and triangle inequality that

$$\left\|\frac{f(A)+f(B)}{2}-\frac{1}{2}\left[f\left((1-s)A+s\frac{A+B}{2}\right)+f\left(s\frac{A+B}{2}+(1-s)B\right)\right]\right\| \leq\frac{1}{2}f'(m)\|B-A\|s$$

for all $s\in[0,1]$.

By taking the integral and using its properties, we derive

$$\left\|\frac{f(A)+f(B)}{2}-\frac{1}{2}\int_0^1\left[f\left((1-s)A+s\frac{A+B}{2}\right)+f\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds\right\| \leq\frac{1}{2}f'(m)\|B-A\|\int_0^1 s\,ds=\frac{1}{4}f'(m)\|B-A\|. \tag{4.12}$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1 f\left((1-t)A+t\frac{A+B}{2}\right)dt=\int_0^{1/2}f((1-s)A+sB)\,ds$$

and by the change of variable $t=1-v$ we have

$$\frac{1}{2}\int_0^1 f\left(t\frac{A+B}{2}+(1-t)A\right)dt=\frac{1}{2}\int_0^1 f\left((1-v)\frac{A+B}{2}+vB\right)dv.$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1 f\left((1-v)\frac{A+B}{2}+vB\right)dv=\int_{1/2}^1 f((1-s)A+sB)\,ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[f\left((1-s)A+s\frac{A+B}{2}\right)+f\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds\\
&=\int_0^{1/2} f\left((1-s)A+sB\right)dt+\int_{1/2}^1 f\left((1-s)A+sB\right)ds\\
&=\int_0^1 f\left((1-s)A+sB\right)ds
\end{aligned}$$

and by (4.12) we deduce the desired result (4.8).

The inequality (4.9) follows in a similar way and we omit the details. □

Assume that $A, B \geq m > 0$, then by Proposition 2 we obtain the following power inequalities

$$\left\|\frac{A^r+B^r}{2}-\int_0^1 \left((1-t)A+tB\right)^r dt\right\| \leq \frac{1}{4}rm^{r-1}\left\|B-A\right\| \tag{4.13}$$

and

$$\left\|\frac{A^{r-1}+B^{r-1}}{2}-\int_0^1 \left((1-t)A+tB\right)^{r-1} dt\right\| \leq \frac{1-r}{4m^{2-r}}\left\|B-A\right\|, \tag{4.14}$$

where $r \in (0,1]$.

We have the logarithmic inequalities

$$\left\|\frac{\ln A+\ln B}{2}-\int_0^1 \ln\left((1-t)A+tB\right) dt\right\| \leq \frac{1}{4m}\left\|B-A\right\| \tag{4.15}$$

and

$$\begin{aligned}
&\left\|\frac{A^{-1}\ln(A+1)+B^{-1}\ln(B+1)}{2}\right.\\
&\left.-\int_0^1 \left((1-t)A+tB\right)^{-1}\ln\left((1-t)A+tB+1\right)dt\right\|\\
&\leq \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)}\left\|B-A\right\|.
\end{aligned} \tag{4.16}$$

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