OPERATOR MONOTONICITY OF AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $B \geq A > 0$, then $\mathcal{D}(w, \mu)(B) \leq \mathcal{D}(w, \mu)(A)$, namely $\mathcal{D}(w, \mu)$ is operator monotone decreasing on $(0, \infty)$. From this we derive that, if $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$, then the function $[f(0) - f(t)] t^{-1}$ is operator monotone on $(0, \infty)$. Also, if $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$, then the function $\left[f(0) + f'_+(0) t - f(t)\right] t^{-2}$ is operator monotone on $(0, \infty)$. Some examples for integral transforms $\mathcal{D}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.1).

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda \right) A+\lambda B\right) \leq \left(\geq \right) \left(1-\lambda \right) f\left(A\right) +\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty) \rightarrow \mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right) =a+bt+ct^{2}+\int_{0}^{\infty }\frac{t^{2}\lambda }{t+\lambda }d\mu \left(\lambda \right) , \tag{1.3}$$

where a, $b\in \mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f_{+}^{\prime}(0)$, the right derivative, in (1.1).

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\geq m>0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$\begin{aligned} f\left(B\right) -f\left(A\right) &\geq f\left(\left\Vert A\right\Vert +m\right) -f\left(\left\Vert A\right\Vert \right) \\ &\geq f\left(\left\Vert B\right\Vert \right) -f\left(\left\Vert B\right\Vert -m\right) >0. \end{aligned} \tag{1.4}$$

If $B>A>0$, then

$$\begin{aligned} f\left(B\right) -f\left(A\right) &\geq f\left(\left\Vert A\right\Vert +\frac{1}{\left\Vert \left(B-A\right) ^{-1}\right\Vert }\right) -f\left(\left\Vert A\right\Vert \right) \\ &\geq f\left(\left\Vert B\right\Vert \right) -f\left(\left\Vert B\right\Vert -\frac{1}{\left\Vert \left(B-A\right) ^{-1}\right\Vert }\right) >0. \end{aligned} \tag{1.5}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [8].

By taking $f(t)=t^{r}$, $r\in (0,1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality [5]

$$\begin{aligned} B^{r}-A^{r} &\geq \left(\left\Vert A\right\Vert +\frac{1}{\left\Vert \left(B-A\right) ^{-1}\right\Vert }\right) ^{r}-\left\Vert A\right\Vert ^{r} \\ &\geq \left\Vert B\right\Vert ^{r}-\left(\left\Vert B\right\Vert -\frac{1}{\left\Vert \left(B-A\right) ^{-1}\right\Vert }\right) ^{r}>0 \end{aligned} \tag{1.6}$$

provided $B>A>0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\ln B-\ln A\geq\ln\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-\ln(\|A\|) \tag{1.7}$$
$$\geq\ln(\|B\|)-\ln\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0.$$

Notice that the inequalities between the first and third terms in (1.6) and (1.7) were obtained earlier by M. S. Moslehian and H. Najafi in [7].

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda),$$

where the integral is assumed to exist for all T a positive operator on a complex Hilbert space H.

We show among others that, if $B\geq A>0$, then $\mathcal{D}(w,\mu)(B)\leq\mathcal{D}(w,\mu)(A)$, namely $\mathcal{D}(w,\mu)$ is operator monotone decreasing on $(0,\infty)$. From this we derive that, if $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $[f(0)-f(t)]t^{-1}$ is operator monotone on $(0,\infty)$. Also, if $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function $\left[f(0)+f'_+(0)t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$. Some examples for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Basic Identities

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{2.1}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{2.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{2.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^r=\frac{\sin(r\pi)}{\pi}t\mathcal{D}(w_r)(t),\ t>0. \tag{2.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{2.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{2.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{2.8}$$

for $T>0$.

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (2.3) exists and is finite for all $t>0$.

Theorem 3. *For all A, $B>0$ we have the representation*

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\&=-\int_0^\infty\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}dt\right)\\&\times w(\lambda)d\mu(\lambda).\end{aligned} \tag{2.9}$$

Proof. Observe that, for all A, $B>0$

$$\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)=\int_0^\infty w(\lambda)\left[(\lambda+B)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda). \tag{2.10}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.11}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}\left(f_{C,D}(t)\right)dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.12}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 \left((1-t)C+tD\right)^{-1}(D-C)\left((1-t)C+tD\right)^{-1}dt. \tag{2.13}$$

Now, if we take in (2.13) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}(\lambda+B)^{-1}-(\lambda+A)^{-1} \\ &=\int_0^1 \left((1-t)(\lambda+B)+t(\lambda+A)\right)^{-1}(A-B) \\ &\times\left((1-t)(\lambda+B)+t(\lambda+A)\right)^{-1}dt \\ &=\int_0^1 \left(\lambda+(1-t)B+tA\right)^{-1}(A-B)\left(\lambda+(1-t)B+tA\right)^{-1}dt\end{aligned} \tag{2.14}$$

and by (2.10) we derive (2.9). □

Remark 1. *We observe that if $A,\ B>0$ and $r\in(0,1]$, then by (2.5) we get the identity*

$$\begin{aligned}B^{r-1}-A^{r-1}=-\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\left(\int_0^1 \left(\lambda+(1-t)B+tA\right)^{-1}\right. \\ \left.\times(B-A)\left(\lambda+(1-t)B+tA\right)^{-1}dt\right)d\lambda.\end{aligned} \tag{2.15}$$

If $A,\ B>0$ with $A-1$ and $B-1$ invertible, then

$$\begin{aligned}(B-1)^{-1}\ln B-(A-1)^{-1}\ln A \\ =-\int_0^\infty (\lambda+1)^{-1} \\ \times\left(\int_0^1 \left(\lambda+(1-t)B+tA\right)^{-1}(B-A)\left(\lambda+(1-t)B+tA\right)^{-1}dt\right)d\lambda.\end{aligned} \tag{2.16}$$

Corollary 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function that has the representation (1.1). Then for all $A,\ B>0$ we have the equality*

$$\begin{aligned}B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right) \\ =-\int_0^\infty \lambda\left(\int_0^1 \left(\lambda+(1-t)B+tA\right)^{-1}(B-A)\right. \\ \left.\times\left(\lambda+(1-t)B+tA\right)^{-1}dt\right)d\mu(\lambda).\end{aligned} \tag{2.17}$$

If $f(0)=0$, then we have the simpler equality

$$\begin{aligned}B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right) \\ =-\int_0^\infty \lambda\left(\int_0^1 \left(\lambda+(1-t)B+tA\right)^{-1}(B-A)\right. \\ \left.\times\left(\lambda+(1-t)B+tA\right)^{-1}dt\right)d\mu(\lambda).\end{aligned} \tag{2.18}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.19}$$

where $\ell(\lambda)=\lambda$, $\lambda>0$. Then for $A,\ B>0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)&=[f(B)-f(0)]B^{-1}-[f(A)-f(0)]A^{-1}\\&=B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)\end{aligned}$$

and by (2.9) we derive (2.17). □

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function that has the representation (1.2). Then for all $A,\ B>0$ we have the equality*

$$\begin{aligned}&f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right)-f(0)\left(B^{-2}-A^{-2}\right)\\&=-\int_0^\infty\lambda\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned} \tag{2.20}$$

If $f(0)=0$, then we have the simpler equality

$$\begin{aligned}&f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right)\\&=-\int_0^\infty\lambda\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned} \tag{2.21}$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. Then for $A,\ B>0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)&=f(B)B^{-2}-f'_{+}(0)B^{-1}-f(0)B^{-2}\\&\quad-f(A)A^{-2}+f'_{+}(0)A^{-1}+f(0)A^{-2}\\&=f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right)\\&\quad-f(0)\left(B^{-2}-A^{-2}\right)\end{aligned}$$

and by (2.9) we derive (2.20). □

Remark 2. *Let $a>0$ and $f(t)=(t+a)^p$ with $p\in[-1,0)\cup[1,2]$. This function is operator convex and $f(0)=a^p$, $f'(0)=pa^{p-1}$. Then for all $A,\ B>0$ we have the equality*

$$\begin{aligned}&(B+a)^pB^{-2}-(A+a)^pA^{-2}-pa^{p-1}\left(B^{-1}-A^{-1}\right)-a^p\left(B^{-2}-A^{-2}\right)\\&=-\int_0^\infty\lambda\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda),\end{aligned} \tag{2.22}$$

for some positive measure μ on $(0,\infty)$.

3. Monotonicity Properties

In what follows, we assume that the integral transform defined by (2.3) is well defined for a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$.

Theorem 4. *If $B \geq A > 0$, then*

$$\mathcal{D}(w, \mu)(B) \leq \mathcal{D}(w, \mu)(A), \tag{3.1}$$

namely, the function $\mathcal{D}(w, \mu)(\cdot)$ is operator monotone decreasing on $(0, \infty)$.

Proof. From $B - A \geq 0$, by multiplying both sides with $(\lambda + (1-t)B + tA)^{-1}$ for $t \in [0, 1]$ and $\lambda > 0$, we get

$$(\lambda + (1-t)B + tA)^{-1}(B - A)(\lambda + (1-t)B + tA)^{-1} \geq 0,$$

which gives, by integration over $t \in [0, 1]$, that

$$\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B - A)(\lambda + (1-t)B + tA)^{-1} dt \geq 0,$$

for all $\lambda > 0$.

Now, if we multiply this inequality by $w(\lambda) > 0$ and integrate over the positive measure $d\mu(\lambda)$, we get

$$\int_0^\infty w(\lambda)\left(\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B - A)(\lambda + (1-t)B + tA)^{-1} dt\right) d\mu(\lambda)$$
$$\geq 0,$$

and by representation (2.9), we deduce (3.1). □

Corollary 3. *Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$. Then for all $B \geq A > 0$ we have*

$$A^{-1}f(A) - B^{-1}f(B) \geq f(0)\left(A^{-1} - B^{-1}\right), \tag{3.2}$$

namely the function $[f(0) - f(t)]t^{-1}$ is operator monotone on $(0, \infty)$.

In particular, if $f(0) = 0$, then

$$A^{-1}f(A) \geq B^{-1}f(B) \tag{3.3}$$

for all $B \geq A > 0$, namely $-f(t)t^{-1}$ is operator monotone on $(0, \infty)$.

Proof. It follows by Theorem 4 by observing that, if $f : [0, \infty) \to \mathbb{R}$ is operator monotone, then by (1.1)

$$\frac{f(t) - f(0)}{t} - b = \mathcal{D}(\ell, \mu)(t), \; t > 0$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$. □

Corollary 4. *Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$. Then for all $B \geq A > 0$ we have*

$$f(A)A^{-2} - f(B)B^{-2} \geq f'_+(0)\left(A^{-1} - B^{-1}\right) + f(0)\left(A^{-2} - B^{-2}\right) \tag{3.4}$$

namely the function $\left[f(0) + f'_+(0)t - f(t)\right]t^{-2}$ is operator monotone on $(0, \infty)$.

In particular, if $f(0) = 0$, then

$$f(A)A^{-2} - f(B)B^{-2} \geq f'_+(0)\left(A^{-1} - B^{-1}\right) \tag{3.5}$$

for all $B \geq A > 0$, namely $\left[f'_+(0)t - f(t)\right]t^{-2}$ is operator monotone on $(0, \infty)$

Proof. It follows by Theorem 4 by observing that, if $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t)-f(0)-f'_{+}(0)\,t}{t^{2}}-c=\mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$. □

Remark 3. *Let $a>0$ and $p\in[-1,0)\cup[1,2]$. Then for all $B\geq A>0$ we have the inequality*

$$(A+a)^{p}A^{-2}-(B+a)^{p}B^{-2}\geq pa^{p-1}\left(A^{-1}-B^{-1}\right)+a^{p}\left(A^{-2}-B^{-2}\right). \tag{3.6}$$

4. Related Inequalities

We start with the following inequalities that can be derived from Furuta's inequalities (1.4).

Proposition 1. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and there exists $m>0$ such that $B-A\geq m>0$, then*

$$\begin{aligned}&A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m}-g(0)\frac{m}{(\|A\|+m)\|A\|}\\&\geq\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}-g(0)\frac{m}{(\|B\|-m)\|B\|}\geq 0.\end{aligned}\tag{4.1}$$

If $g(0)=0$, then

$$\begin{aligned}A^{-1}g(A)-B^{-1}g(B)&\geq\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m}\\&\geq\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}\geq 0.\end{aligned}\tag{4.2}$$

Proof. If we write the inequality (1.4) for $f(t)=\frac{g(0)-g(t)}{t}$, $t>0$, which, by Corollary 3, is operator monotone, then we have

$$\begin{aligned}&B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)]\\&\geq\frac{g(0)-g(\|A\|+m)}{\|A\|+m}-\frac{g(0)-g(\|A\|)}{\|A\|}\\&\geq\frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-m)}{\|B\|-m}>0.\end{aligned}\tag{4.3}$$

Observe that

$$\begin{aligned}&B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)]\\&=A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right),\end{aligned}$$

$$\begin{aligned}&\frac{g(0)-g(\|A\|+m)}{\|A\|+m}-\frac{g(0)-g(\|A\|)}{\|A\|}\\&=\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m}-g(0)\frac{m}{(\|A\|+m)\|A\|}\end{aligned}$$

and

$$\begin{aligned}&\frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-m)}{\|B\|-m}\\&=\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}-g(0)\frac{m}{(\|B\|-m)\|B\|}\end{aligned}$$

and by (4.3) we get (4.1). □

Remark 4. *If we take $g(t)=t^r$, $r\in(0,1]$ in (4.2), then we get*

$$A^{r-1}-B^{r-1}\geq\|A\|^{r-1}-(\|A\|+m)^{r-1}\geq(\|B\|-m)^{r-1}-\|B\|^{r-1}>0, \tag{4.4}$$

provided $A>0$ and $B-A\geq m>0$.

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}0&\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\&\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}1_H\leq T. \tag{4.5}$$

Corollary 5. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and $B-A>0$, then*

$$\begin{aligned}&A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}}\\&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)\|A\|}\\&\geq\frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|}\\&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)\|B\|}\\&\geq 0.\end{aligned} \tag{4.6}$$

If $g(0)=0$, *then*

$$A^{-1}g(A)-B^{-1}g(B) \geq \frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}} \geq \frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|} \geq 0. \tag{4.7}$$

Remark 5. *If we take* $g(t)=t^r$, $r\in(0,1]$ *in (4.7), then we get*

$$A^{r-1}-B^{r-1} \geq \|A\|^{r-1}-\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1} \geq \left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1}-\|B\|^{r-1}>0, \tag{4.8}$$

where $A>0$ *and* $B-A>0$.

Proposition 2. *Assume that* $h:[0,\infty)\to\mathbb{R}$ *is operator convex on* $[0,\infty)$. *If* $A>0$ *and there exists* $m>0$ *such that* $B-A\geq m>0$, *then*

$$\begin{aligned} &h(A)A^{-2}-h(B)B^{-2}-h(0)\left(A^{-2}-B^{-2}\right)+h'_+(0)(B-A) \\ &\geq h(\|A\|)\|A\|^{-2}-h(\|A\|+m)(\|A\|+m)^{-2} \\ &-h(0)\left(\|A\|^{-2}-(\|A\|+m)^{-2}\right)-h'_+(0)\left(\|A\|^{-1}-(\|A\|+m)^{-1}\right) \\ &\geq h(\|B\|-m)(\|B\|-m)^{-2}-h(\|B\|)\|B\|^{-2} \\ &-h(0)\left((\|B\|-m)^{-2}-\|B\|^{-2}\right)-h'_+(0)\left((\|B\|-m)^{-1}-\|B\|^{-1}\right) \\ &\geq 0 \end{aligned} \tag{4.9}$$

If $h(0)=0$, *then*

$$\begin{aligned} &h(A)A^{-2}-h(B)B^{-2}+h'_+(0)(B-A) \\ &\geq h(\|A\|)\|A\|^{-2}-h(\|A\|+m)(\|A\|+m)^{-2} \\ &-h'_+(0)\left(\|A\|^{-1}-(\|A\|+m)^{-1}\right) \\ &\geq h(\|B\|-m)(\|B\|-m)^{-2}-h(\|B\|)\|B\|^{-2} \\ &-h'_+(0)\left((\|B\|-m)^{-1}-\|B\|^{-1}\right) \\ &\geq 0 \end{aligned} \tag{4.10}$$

Proof. If we write the inequality (1.4) for $f(t) = \left[h(0) + h'_+(0)t - h(t)\right]t^{-2}$, $t > 0$, which, by Corollary 4, is operator monotone, then we have

$$\begin{aligned}
&\left[h(0) + h'_+(0)B - h(B)\right]B^{-2} - \left[h(0) + h'_+(0)A - h(A)\right]A^{-2} \\
&\geq \left[h(0) + h'_+(0)(\|A\| + m) - h(\|A\| + m)\right](\|A\| + m)^{-2} \\
&- \left[h(0) + h'_+(0)\|A\| - h(\|A\|)\right](\|A\|)^{-2} \\
&\geq \left[h(0) + h'_+(0)\|B\| - h(\|B\|)\right]\|B\|^{-2} \\
&- \left[h(0) + h'_+(0)(\|B\| - m) - h(\|B\| - m)\right](\|B\| - m)^{-2} \\
&> 0.
\end{aligned} \tag{4.11}$$

Observe that

$$\begin{aligned}
&\left[h(0) + h'_+(0)B - h(B)\right]B^{-2} - \left[h(0) + h'_+(0)A - h(A)\right]A^{-2} \\
&= h(A)A^{-2} - h(B)B^{-2} - h(0)\left(A^{-2} - B^{-2}\right) + h'_+(0)(B - A),
\end{aligned}$$

$$\begin{aligned}
&\left[h(0) + h'_+(0)(\|A\| + m) - h(\|A\| + m)\right](\|A\| + m)^{-2} \\
&- \left[h(0) + h'_+(0)\|A\| - h(\|A\|)\right]\|A\|^{-2} \\
&= h(\|A\|)\|A\|^{-2} - h(\|A\| + m)(\|A\| + m)^{-2} \\
&- h(0)\left(\|A\|^{-2} - (\|A\| + m)^{-2}\right) - h'_+(0)\left(\|A\|^{-1} - (\|A\| + m)^{-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left[h(0) + h'_+(0)\|B\| - h(\|B\|)\right]\|B\|^{-2} \\
&- \left[h(0) + h'_+(0)(\|B\| - m) - h(\|B\| - m)\right](\|B\| - m)^{-2} \\
&= h(\|B\| - m)(\|B\| - m)^{-2} - h(\|B\|)\|B\|^{-2} \\
&- h(0)\left((\|B\| - m)^{-2} - \|B\|^{-2}\right) - h'_+(0)\left((\|B\| - m)^{-1} - \|B\|^{-1}\right)
\end{aligned}$$

and by (4.11) we derive the desired inequality (4.9). □

Remark 6. *If $A > 0$ and $B - A > 0$, then we can take $m = \left\|(B - A)^{-1}\right\|^{-1}$ in Proposition 2 to obtain other norm inequalities. The details are omitted.*

The function $h(t) := -\ln(t + 1)$ is operator convex with $h(0) = 0$ and $h'(0) = -1$. Then by (4.10) we get

$$\begin{aligned}
&B^{-2}\ln(B + 1) - A^{-2}\ln(A + 1) - (B - A) \\
&\geq (\|A\| + m)^{-2}\ln(\|A\| + m + 1) - \|A\|^{-2}\ln(\|A\| + 1) \\
&+ \|A\|^{-1} - (\|A\| + m)^{-1} \\
&\geq \|B\|^{-2}\ln(\|B\| + 1) - (\|B\| - m)^{-2}\ln(\|B\| - m + 1) \\
&+ (\|B\| - m)^{-1} - \|B\|^{-1} \\
&> 0
\end{aligned} \tag{4.12}$$

provided that $A > 0$ and $B - A \geq m > 0$.

5. More Examples of Interest

We define the *upper incomplete Gamma function* as [9]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a>0.$$

We have the integral representation [10]

$$\Gamma(a,z)=\frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{5.1}$$

for $\operatorname{Re} a<1$ and $|\operatorname{ph} z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (5.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{5.2}$$

for $a<1$ and $t>0$.

For $a=0$ in (5.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t)=\int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^t\Gamma(0,t)=e^tE_1(t) \tag{5.3}$$

for $t>0$, where

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du. \tag{5.4}$$

Let $a=1-n$, with n a natural number with $n\geq 0$, then by (5.2) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)&=\int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(n)t^{n-1}e^t\Gamma(1-n,t)\\&=(n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{5.5}$$

If we define the generalized exponential integral [11] by

$$E_p(z):=z^{p-1}\Gamma(1-p,z)=z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t)=E_n(t)$$

for $n\geq 1$ and $t>0$.

Using the identity [11, Eq 8.19.7], for $n\geq 2$

$$E_n(z)=\frac{(-z)^{n-1}}{(n-1)!}E_1(z)+\frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-z)^k,$$

we get

$$\begin{aligned}(5.6)\qquad &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)\\ &=(n-1)!e^{t}E_{n}(t)\\ &=(n-1)!e^{t}\left[\frac{(-t)^{n-1}}{(n-1)!}E_{1}(t)+\frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-t)^{k}\right]\\ &=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!t^{k}+(-1)^{n-1}t^{n-1}e^{t}E_{1}(t)\end{aligned}$$

for $n\geq 2$ and $t>0$.

If $T>0$, then we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda^{-a}e^{-\lambda}(t+\lambda)^{-1}d\lambda=\Gamma(1-a)T^{-a}\exp(T)\,\Gamma(a,T)\tag{5.7}$$

for $a<1$.

In particular,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T)=\int_{0}^{\infty}e^{-\lambda}(T+\lambda)^{-1}d\lambda=\exp(T)\,E_{1}(T)\tag{5.8}$$

and, for $n\geq 2$

$$\begin{aligned}(5.9)\qquad &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(T)\\ &=\int_{0}^{\infty}\lambda^{n-1}e^{-\lambda}(T+\lambda)^{-1}d\lambda\\ &=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!T^{k}+(-1)^{n-1}T^{n-1}\exp(T)\,E_{1}(T),\end{aligned}$$

where $T>0$.

For $n=2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda e^{-\lambda}(T+\lambda)^{-1}d\lambda=1-T\exp(T)\,E_{1}(T)\tag{5.10}$$

for $T>0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda):=\frac{1}{\lambda+a}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t):=\int_{0}^{\infty}\frac{1}{(\lambda+t)(\lambda+a)}d\lambda=\frac{\ln t-\ln a}{t-a}\tag{5.11}$$

for all $a>0$ and $t>0$ with $t\neq a$.

From this, we get

$$\ln t=\ln a+(t-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a>0$.

If $T>0$, then

$$\begin{aligned}(5.12)\qquad \ln T&=\ln a+(T-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)\\ &=\ln a+(T-a)\int_{0}^{\infty}\frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda.\end{aligned}$$

Let $a>0$. Assume that either $0<T<a$ or $T>a$, then by (5.13) we get

$$(\ln T-\ln a)(T-a)^{-1}=\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda. \tag{5.13}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda):=\frac{1}{\lambda^2+a^2}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t)&:=\int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)}d\lambda\\&=\frac{\pi t}{2a\left(t^2+a^2\right)}-\frac{\ln t-\ln a}{t^2+a^2}\end{aligned}$$

for $t>0$ and $a>0$.

For $a=1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t):=\int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)}d\lambda=\frac{\pi t}{2\left(t^2+1\right)}-\frac{\ln t}{t^2+1}$$

for $t>0$.

If $T>0$ and $a>0$, then

$$\begin{aligned}&\frac{\pi}{2a}T\left(T^2+a^2\right)^{-1}-(\ln T-\ln a)\left(T^2+a^2\right)^{-1}\\&=\int_0^\infty \frac{1}{\left(\lambda^2+a^2\right)}(\lambda+T)^{-1}d\lambda\end{aligned} \tag{5.14}$$

and, in particular,

$$\frac{\pi}{2a}T\left(T^2+1\right)^{-1}-\left(T^2+1\right)^{-1}\ln T=\int_0^\infty \frac{1}{\left(\lambda^2+1\right)}(\lambda+T)^{-1}d\lambda. \tag{5.15}$$

Proposition 3. *Let $B\geq A>0$ and $a<1$, then*

$$A^{-a}\exp(A)\Gamma(a,A)\geq B^{-a}\exp(B)\Gamma(a,B). \tag{5.16}$$

In particular,

$$\exp(A)E_1(A)\geq \exp(B)E_1(B) \tag{5.17}$$

and

$$B\exp(B)E_1(B)\geq A\exp(A)E_1(A). \tag{5.18}$$

The proof follows by Theorem 4 and the identity (5.7).

Proposition 4. *Let $B\geq A>a>0$ or $a>B\geq A>0$, then*

$$(\ln A-\ln a)(A-a)^{-1}\geq(\ln B-\ln a)(B-a)^{-1}. \tag{5.19}$$

If $B\geq A>1>0$ or $1>B\geq A>0$, then

$$(A-1)^{-1}\ln A\geq(B-1)^{-1}\ln B. \tag{5.20}$$

The proof follows by Theorem 4 and the identity (5.13).

Proposition 5. *Let $B\geq A>0$ and $a>0$, then*

$$\begin{aligned}&(\ln B-\ln a)\left(B^2+a^2\right)^{-1}-(\ln A-\ln a)\left(A^2+a^2\right)^{-1}\\&\geq\frac{\pi}{2a}\left[B\left(B^2+a^2\right)^{-1}-A\left(A^2+a^2\right)^{-1}\right].\end{aligned} \tag{5.21}$$

In particular, for $a = 1$,

$$\left(B^2+1\right)^{-1}\ln B-\left(A^2+1\right)^{-1}\ln A\geq\frac{\pi}{2}\left[B\left(B^2+1\right)^{-1}-A\left(A^2+1\right)^{-1}\right]. \tag{5.22}$$

The proof follows by Theorem 4 and the identity (5.14).

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[1] Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2] DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.