IMPROVED OPERATOR MONOTONICITY OF AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for all T a positive operator on a complex Hilbert space H.

Let $A > 0$ and assume that there exist positive numbers $d > c > 0$ such that $d \geq B - A \geq c > 0$, then, we show that,

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \geq \frac{c}{d}[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d + \|A\|)] \geq 0.$$

As a consequence we derive that

$$f(A)A^{-1} - f(B)B^{-1} \geq \frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(d+\|A\|)}{d+\|A\|}\right) \geq 0,$$

if f is operator monotone on $[0,\infty)$ with $f(0) = 0$ and

$$\begin{aligned} & f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) \\ & \geq \frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(d+\|A\|)}{(d+\|A\|)^2}\right] - \frac{cf'_+(0)}{\|A\|(d+\|A\|)} \geq 0 \end{aligned}$$

provided that f is operator convex on $[0,\infty)$ with $f(0) = 0$. Some examples of interest are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.1}$$

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where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.1).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0, \infty)$ such that (1.2) holds. If f is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f'_+(0)$, the right derivative, in (1.1).

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m > 0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0, \infty)$

$$\begin{aligned} f(B) - f(A) &\geq f(\|A\| + m) - f(\|A\|) \\ &\geq f(\|B\|) - f(\|B\| - m) > 0. \end{aligned} \tag{1.4}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) &\geq f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\ &\geq f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{1.5}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [8].

By taking $f(t) = t^r$, $r \in (0, 1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality [5]

$$\begin{aligned} B^r - A^r &\geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r - \|A\|^r \\ &\geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r > 0 \end{aligned} \tag{1.6}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\ln B-\ln A\geq\ln\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-\ln(\|A\|)$$
$$\geq\ln(\|B\|)-\ln\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0. \tag{1.7}$$

Notice that the inequalities between the first and third terms in (1.6) and (1.7) were obtained earlier by M. S. Moslehian and H. Najafi in [7].

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Motivated by the above results, in this paper we show that

$$\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\geq\frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(d+\|A\|)\right]\geq 0,$$

where $A>0$ and provided that there exist positive numbers $d>c>0$ such that $d\geq B-A\geq c>0$. As a consequence, we derive the following alternative lower bound to the one provided by Furuta's result in (1.4),

$$f(A)A^{-1}-f(B)B^{-1}\geq\frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|}-\frac{f(d+\|A\|)}{d+\|A\|}\right)\geq 0,$$

if f is operator monotone on $[0,\infty)$ with $f(0)=0$ and

$$f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$\geq\frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2}-\frac{f(d+\|A\|)}{(d+\|A\|)^2}\right]-\frac{cf'_+(0)}{\|A\|(d+\|A\|)}\geq 0$$

provided that f is operator convex on $[0,\infty)$ with $f(0)=0$. Some examples of interest are also given.

2. Preliminary Facts

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{2.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_{0}^{\infty}\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{2.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{2.4}$$

Now, assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{2.5}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{2.6}$$

for $T>0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{2.7}$$

We define the *upper incomplete Gamma function* as [9]

$$\Gamma(a,z):=\int_{z}^{\infty}t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_{0}^{\infty}t^{a-1}e^{-t}dt \text{ for } \operatorname{Re}a>0.$$

We have the integral representation [10]

$$\Gamma(a,z)=\frac{z^{a}e^{-z}}{\Gamma(1-a)}\int_{0}^{\infty}\frac{t^{-a}e^{-t}}{z+t}dt \tag{2.8}$$

for $\operatorname{Re}a<1$ and $|\operatorname{ph}z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (2.8) we have

$$\mathcal{D}(w_{\cdot^{-a}e^{-\cdot}})(t)=\int_{0}^{\infty}\frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^{t}\Gamma(a,t) \tag{2.9}$$

for $a<1$ and $t>0$.

For $a=0$ in (2.9) we get

$$\mathcal{D}(w_{e^{-\cdot}})(t)=\int_{0}^{\infty}\frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^{t}\Gamma(0,t)=e^{t}E_1(t) \tag{2.10}$$

for $t>0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du. \tag{2.11}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (2.9) we have

$$\begin{aligned} \mathcal{D}\left(w_{\cdot^{n-1} e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1} e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(n) t^{n-1} e^t \Gamma(1-n,t) \\ &= (n-1)! t^{n-1} e^t \Gamma(1-n,t). \end{aligned} \tag{2.12}$$

If we define the generalized exponential integral [11] by

$$E_p(z) := z^{p-1} \Gamma(1-p,z) = z^{p-1} \int_z^\infty \frac{e^{-t}}{t^p} dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [11, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} E_1(z) + \frac{e^{-z}}{(n-1)!} \sum_{k=0}^{n-2} (n-k-2)! (-z)^k,$$

we get

$$\begin{aligned} &\mathcal{D}\left(w_{\cdot^{n-1} e^{-\cdot}}\right)(t) \\ &= (n-1)! e^t E_n(t) \\ &= (n-1)! e^t \left[\frac{(-t)^{n-1}}{(n-1)!} E_1(t) + \frac{e^{-t}}{(n-1)!} \sum_{k=0}^{n-2} (n-k-2)! (-t)^k \right] \\ &= \sum_{k=0}^{n-2} (-1)^k (n-k-2)! t^k + (-1)^{n-1} t^{n-1} e^t E_1(t) \end{aligned} \tag{2.13}$$

for $n \geq 2$ and $t > 0$.

If $T > 0$, then we have

$$\mathcal{D}\left(w_{\cdot^{-a} e^{-\cdot}}\right)(T) = \int_0^\infty \lambda^{-a} e^{-\lambda} (t+\lambda)^{-1} d\lambda = \Gamma(1-a) T^{-a} \exp(T) \Gamma(a,T) \tag{2.14}$$

for $a < 1$.

In particular,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T) = \int_0^\infty e^{-\lambda} (T+\lambda)^{-1} d\lambda = \exp(T) E_1(T) \tag{2.15}$$

and, for $n \geq 2$

$$\begin{aligned} &\mathcal{D}\left(w_{\cdot^{n-1} e^{-\cdot}}\right)(t) \\ &= \int_0^\infty \lambda^{n-1} e^{-\lambda} (T+\lambda)^{-1} d\lambda \\ &= \sum_{k=0}^{n-2} (-1)^k (n-k-2)! T^k + (-1)^{n-1} T^{n-1} \exp(T) E_1(T), \end{aligned} \tag{2.16}$$

where $T > 0$.

For $n=2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T)=\int_0^\infty \lambda e^{-\lambda}(T+\lambda)^{-1}d\lambda=1-T\exp(T)E_1(T) \tag{2.17}$$

for $T>0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda):=\frac{1}{\lambda+a}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t):=\int_0^\infty \frac{1}{(\lambda+t)(\lambda+a)}d\lambda=\frac{\ln t-\ln a}{t-a} \tag{2.18}$$

for all $a>0$ and $t>0$ with $t\neq a$.

From this, we get

$$\ln t=\ln a+(t-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a>0$.

If $T>0$, then

$$\begin{aligned}\ln T&=\ln a+(T-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)\\&=\ln a+(T-a)\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda.\end{aligned} \tag{2.19}$$

Let $a>0$. Assume that either $0<T<a$ or $T>a$, then by (2.20) we get

$$(\ln T-\ln a)(T-a)^{-1}=\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda. \tag{2.20}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda):=\frac{1}{\lambda^2+a^2}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t)&:=\int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)}d\lambda\\&=\frac{\pi t}{2a\left(t^2+a^2\right)}-\frac{\ln t-\ln a}{t^2+a^2}\end{aligned}$$

for $t>0$ and $a>0$.

For $a=1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t):=\int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)}d\lambda=\frac{\pi t}{2\left(t^2+1\right)}-\frac{\ln t}{t^2+1}$$

for $t>0$.

If $T>0$ and $a>0$, then

$$\begin{aligned}&\frac{\pi}{2a}T\left(T^2+a^2\right)^{-1}-(\ln T-\ln a)\left(T^2+a^2\right)^{-1}\\&=\int_0^\infty \frac{1}{\left(\lambda^2+a^2\right)}(\lambda+T)^{-1}d\lambda\end{aligned} \tag{2.21}$$

and, in particular,

$$\frac{\pi}{2}T\left(T^2+1\right)^{-1}-\left(T^2+1\right)^{-1}\ln T=\int_0^\infty \frac{1}{\left(\lambda^2+1\right)}(\lambda+T)^{-1}d\lambda. \tag{2.22}$$

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (2.3) exists and is finite for all $t>0$.

Lemma 1. *For all A, $B>0$ we have the representation*

$$\begin{aligned}\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\ =\int_0^\infty\left(\int_0^1(\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)\\ \times w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{2.23}$$

Proof. Observe that, for all A, $B>0$

$$\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)=\int_0^\infty w(\lambda)\left[(\lambda+B)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda).\tag{2.24}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}\tag{2.25}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.\tag{2.26}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt.\tag{2.27}$$

Now, if we take in (2.27) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}&(\lambda+B)^{-1}-(\lambda+A)^{-1}\\ &=\int_0^1((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B)\\ &\times((1-t)(\lambda+B)+t(\lambda+A))^{-1}dt\\ &=\int_0^1(\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}dt\end{aligned}\tag{2.28}$$

and by (2.24) we derive

$$\begin{aligned}&\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}dt\right)\\ &\times w(\lambda)\,d\mu(\lambda),\end{aligned}$$

which, by the change of variable $t=1-s$, gives (2.23). □

Remark 1. *By making use of the examples provided above, we can infer the following identities for $A,\ B>0$,*

$$A^{r-1}-B^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1} \times \left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\lambda, \tag{2.29}$$

and

$$\Gamma(1-a)\left[A^{-a}\exp(A)\Gamma(a,A)-B^{-a}\exp(B)\Gamma(a,B)\right] = \int_0^\infty \lambda^{-a}e^{-\lambda} \times \left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\lambda, \tag{2.30}$$

for $a<1$.

In particular,

$$\exp(A)E_1(A)-\exp(B)E_1(B) = \int_0^\infty e^{-\lambda} \times \left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\lambda \tag{2.31}$$

and

$$B\exp(B)E_1(B)-B\exp(B)E_1(B) = \int_0^\infty \lambda e^{-\lambda} \times \left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\lambda. \tag{2.32}$$

Let $a>0$. Assume that either $0<A,\ B<a$ or $A,\ B>a$, then

$$(\ln A-\ln a)(A-a)^{-1}-(\ln B-\ln a)(B-a)^{-1} = \int_0^\infty \frac{1}{(\lambda+a)} \times \left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\lambda. \tag{2.33}$$

3. Main Results

Our first main result is as follows:

Theorem 3. *Let $A>0$ and assume that there exist positive numbers $d>c>0$ such that*

$$d\geq B-A\geq c>0, \tag{3.1}$$

then

$$\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(d+\|A\|)\right]\geq 0. \tag{3.2}$$

Proof. Since $B-A\geq c$, then by multiplying both sides with $\left(\lambda+sB+\left(1-s\right)A\right)^{-1}$, we get

$$\begin{aligned}&\left(\lambda+sB+\left(1-s\right)A\right)^{-1}\left(B-A\right)\left(\lambda+sB+\left(1-s\right)A\right)^{-1}\\&\geq c\left(\lambda+sB+\left(1-s\right)A\right)^{-2}\end{aligned}$$

for all $s\in[0,1]$ and $\lambda>0$.

By integration over $s\in[0,1]$ we get

$$\begin{aligned}&\int_0^1\left(\lambda+sB+\left(1-s\right)A\right)^{-1}\left(B-A\right)\left(\lambda+sB+\left(1-s\right)A\right)^{-1}ds\\&\geq c\int_0^1\left(\lambda+sB+\left(1-s\right)A\right)^{-2}ds\end{aligned}$$

for all $\lambda>0$.

If we multiply this inequality with $w\left(\lambda\right)$ and integrate, then we get

$$\begin{aligned}&\int_0^\infty w\left(\lambda\right)\\&\times\left(\int_0^1\left(\lambda+sB+\left(1-s\right)A\right)^{-1}\left(B-A\right)\left(\lambda+sB+\left(1-s\right)A\right)^{-1}ds\right)d\mu\left(\lambda\right)\\&\geq c\int_0^\infty w\left(\lambda\right)\left(\int_0^1\left(\lambda+sB+\left(1-s\right)A\right)^{-2}ds\right)d\mu\left(\lambda\right).\end{aligned}\tag{3.3}$$

Since $A\leq\|A\|$, then

$$\begin{aligned}\lambda+sB+\left(1-s\right)A&=\lambda+A+s\left(B-A\right)\leq\lambda+\|A\|+sd\\&=\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\end{aligned}$$

for all $s\in[0,1]$ and $\lambda>0$, which implies that

$$\left(\lambda+sB+\left(1-s\right)A\right)^{-1}\geq\left(\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\right)^{-1}$$

and

$$\left(\lambda+sB+\left(1-s\right)A\right)^{-2}\geq\left(\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\right)^{-2}\tag{3.4}$$

for all $s\in[0,1]$ and $\lambda>0$.

From (3.4) we get by integration twice the inequality

$$\begin{aligned}&\int_0^\infty w\left(\lambda\right)\left(\int_0^1\left(\lambda+sB+\left(1-s\right)A\right)^{-2}ds\right)d\mu\left(\lambda\right)\\&\geq\int_0^\infty w\left(\lambda\right)\left(\int_0^1\left(\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\right)^{-2}ds\right)d\mu\left(\lambda\right)\ \ (\geq0)\\&=\frac{1}{d}\int_0^\infty w\left(\lambda\right)\left[\int_0^1\left(\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\right)^{-1}\left(d+\|A\|-\|A\|\right)\right.\\&\left.\times\left(\lambda+\left(1-s\right)\|A\|+s\left(d+\|A\|\right)\right)^{-1}ds\right]d\mu\left(\lambda\right)\\&=\frac{1}{d}\left[\mathcal{D}\left(w,\mu\right)\left(\|A\|\right)-\mathcal{D}\left(w,\mu\right)\left(d+\|A\|\right)\right]\geq0\text{ (by (2.23)).}\end{aligned}\tag{3.5}$$

By utilising (3.3) and (3.5) we obtain

$$\begin{aligned}&\int_0^\infty w(\lambda)\\&\times\left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)(\lambda+sB+(1-s)A)^{-1}ds\right)d\mu(\lambda)\\&\geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(d+\|A\|)\right],\end{aligned}$$

which by the representation (2.23) gives (3.2). □

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}0&\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\&\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequalities

$$\left\|T^{-1}\right\|^{-1}\leq T\leq\|T\|. \tag{3.6}$$

Corollary 1. *Assume that $A>0$ and $B-A>0$. Then*

$$\begin{aligned}\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)&\geq\frac{1}{\|B-A\|\left\|(B-A)^{-1}\right\|}\\&\times\left[\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B-A\|+\|A\|)\right]\\&\geq 0.\end{aligned}\tag{3.7}$$

The proof follows by (3.2) since, by 3.6,

$$0<\left\|(B-A)^{-1}\right\|^{-1}\leq B-A\leq\|B-A\|.$$

We can state the following result for operator monotone functions on $[0,\infty)$:

Proposition 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$. If A, $B>0$ satisfy condition (3.1), then*

$$\begin{aligned}&f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|}-\frac{f(d+\|A\|)}{d+\|A\|}\right)-\frac{cf(0)}{\|A\|(d+\|A\|)}\geq 0.\end{aligned}\tag{3.8}$$

If $f(0)=0$, then we have the simpler inequality

$$f(A)A^{-1}-f(B)B^{-1}\geq\frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|}-\frac{f(d+\|A\|)}{d+\|A\|}\right)\geq 0. \tag{3.9}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is an operator monotone, then by (1.1)

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),\ t>0$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$.

By the inequality (3.2) we have

$$\begin{aligned}&[f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1}\\&\geq\frac{c}{d}\left[\frac{f(\|A\|)-f(0)}{\|A\|}-\frac{f(d+\|A\|)-f(0)}{d+\|A\|}\right]\geq 0,\end{aligned}$$

which is equivalent to (3.8). □

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$, $A>0$ and $B-A>0$. Then*

$$\begin{aligned}&f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left(\frac{f(\|A\|)}{\|A\|}-\frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}\right)\\&-\frac{f(0)}{\|A\|\left\|(B-A)^{-1}\right\|(\|B-A\|+\|A\|)}\\&\geq 0.\end{aligned}\tag{3.10}$$

If $f(0)=0$, then

$$\begin{aligned}&f(A)A^{-1}-f(B)B^{-1}\\&\geq\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left(\frac{f(\|A\|)}{\|A\|}-\frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}\right)\geq 0.\end{aligned}\tag{3.11}$$

In the case of operator convex functions, we have:

Proposition 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. If A, $B>0$ satisfy condition (3.1), then*

$$\begin{aligned}&f(A)A^{-2}-f(B)B^{-2}-f(0)\left(A^{-2}-B^{-2}\right)-f'_+(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2}-\frac{f(d+\|A\|)}{(d+\|A\|)^2}\right]-\frac{cf(0)(d+2\|A\|)}{\|A\|^2(d+\|A\|)^2}\\&-\frac{cf'_+(0)}{\|A\|(d+\|A\|)}\\&\geq 0.\end{aligned}\tag{3.12}$$

If $f(0)=0$, then

$$\begin{aligned}&f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)\\&\geq\frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2}-\frac{f(d+\|A\|)}{(d+\|A\|)^2}\right]-\frac{cf'_+(0)}{\|A\|(d+\|A\|)}\geq 0.\end{aligned}\tag{3.13}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$.

By the inequality (3.2) we have

$$\begin{aligned}
&\left[f(A) - f(0) - f'_+(0)A\right]A^{-2} - \left[f(B) - f(0) - f'_+(0)B\right]B^{-2} \\
&\geq \frac{c}{d}\left[\frac{f(\|A\|) - f(0) - f'_+(0)\|A\|}{\|A\|^2}\right. \\
&\left.-\frac{f(d+\|A\|) - f(0) - f'_+(0)(d+\|A\|)}{(d+\|A\|)^2}\right] \\
&\geq 0,
\end{aligned}$$

which is equivalent to (3.12). □

Corollary 3. *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is an operator convex function on* $[0, \infty)$, $A > 0$ *and* $B - A > 0$. *Then*

$$\begin{aligned}
&f(A)A^{-2} - f(B)B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right) \\
&\geq \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(\|B-A\| + \|A\|)}{(\|B-A\| + \|A\|)^2}\right] \\
&- \frac{f(0)(\|B-A\| + 2\|A\|)}{\left\|(B-A)^{-1}\right\|\|A\|^2(\|B-A\| + \|A\|)^2} \\
&- \frac{f'_+(0)}{\left\|(B-A)^{-1}\right\|\|A\|(\|B-A\| + \|A\|)} \\
&\geq 0.
\end{aligned} \tag{3.14}$$

If $f(0) = 0$, *then*

$$\begin{aligned}
&f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) \\
&\geq \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(\|B-A\| + \|A\|)}{(\|B-A\| + \|A\|)^2}\right] \\
&- \frac{f'_+(0)}{\left\|(B-A)^{-1}\right\|\|A\|(\|B-A\| + \|A\|)} \\
&\geq 0.
\end{aligned} \tag{3.15}$$

4. Some Examples

In this section we give some example of the above general inequalities that hold for some particular operator monotone or operator convex functions of interest.

If we take $f(t) = t^r$, $r \in (0, 1]$ in (3.9), then we get

$$A^{r-1} - B^{r-1} \geq \frac{c}{d}\left(\|A\|^{r-1} - (d + \|A\|)^{r-1}\right) > 0, \tag{4.1}$$

provided A, $B > 0$ satisfy condition (3.1).

If $A>0$ and $B-A>0$, then

$$A^{r-1}-B^{r-1} \geq \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left[\|A\|^{r-1}-(\|B-A\|+\|A\|)^{r-1}\right] \geq 0. \tag{4.2}$$

If we take $f(t)=-\ln(t+1)$, which is operator convex on $[0,\infty)$, then by (3.13) we get

$$B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1} \geq \frac{c}{d}\left[\frac{\ln(d+\|A\|+1)}{(d+\|A\|)^{2}}-\frac{\ln(\|A\|+1)}{\|A\|^{2}}\right]+\frac{c}{\|A\|(d+\|A\|)} \geq 0, \tag{4.3}$$

provided that A, $B\geq 0$ and satisfy condition (3.1).

If $A\geq 0$ and $B-A>0$, then

$$\begin{aligned} &B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1} \\ &\geq \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left[\frac{\ln(\|B-A\|+\|A\|+1)}{(\|B-A\|+\|A\|)^{2}}-\frac{\ln(\|A\|+1)}{\|A\|^{2}}\right] \\ &+\frac{1}{\left\|(B-A)^{-1}\right\|\|A\|(\|B-A\|+\|A\|)} \\ &\geq 0. \end{aligned} \tag{4.4}$$

Assume that A, $B>0$ and satisfy condition (3.1) for $d>c>0$, then

$$\begin{aligned} &A^{-a}\exp(A)\Gamma(a,A)-B^{-a}\exp(B)\Gamma(a,B) \\ &\geq \frac{c}{d}\Big[\|A\|^{-a}\exp(\|A\|)\Gamma(a,\|A\|) \\ &-(d+\|A\|)^{-a}\exp(d+\|A\|)\Gamma(a,d+\|A\|)\Big] \\ &\geq 0 \end{aligned} \tag{4.5}$$

for $a<1$.

In particular, we have

$$\exp(A)E_1(A)-\exp(B)E_1(B) \geq \frac{c}{d}\left[\exp(\|A\|)E_1(\|A\|)-\exp(d+\|A\|)E_1(d+\|A\|)\right] \geq 0 \tag{4.6}$$

and

$$\begin{aligned} &B\exp(B)E_1(B)-A\exp(A)E_1(A) \\ &\geq \frac{c}{d}\left[(d+\|A\|)\exp(d+\|A\|)E_1(d+\|A\|)-\|A\|\exp(\|A\|)E_1(\|A\|)\right] \\ &\geq 0. \end{aligned} \tag{4.7}$$

Let $a > 0$. Assume that $A, B > a$ and there exists $d > c > 0$ such that (3.1) holds, then by (2.20) we get

$$\begin{aligned}(4.8)\qquad & (\ln A - \ln a)(A - a)^{-1} - (\ln B - \ln a)(B - a)^{-1} \\ & \geq \frac{c}{d}\Big[(\ln \|A\| - \ln a)(\|A\| - a)^{-1} \\ & - (\ln(d + \|A\|) - \ln a)(d + \|A\| - a)^{-1}\Big] \\ & \geq 0.\end{aligned}$$

The interested author may state other similar inequalities by using the examples of operator monotone functions from [2], [3] and the references therein.

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