SUBADDITIVITY OF AN INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if B, $A > 0$, then

$$\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B) \geq \mathcal{D}(w,\mu)(A+B),$$

namely $\mathcal{D}(w,\mu)$ is operator subadditive on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $[f(t) - f(0)]t^{-1}$ is operator subadditive on $(0,\infty)$. Also, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function

$$\left[f(t) - f(0) - f'_+(0)t\right] t^{-2}$$

is operator subadditive on $(0,\infty)$. Some examples for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.1}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0,\infty)$, then $a = f(0)$ in (1.1).

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq\left(\geq\right)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right)=a+bt+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu\left(\lambda\right), \tag{1.3}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f_{+}^{\prime}(0)$, the right derivative, in (1.1).

Assume that A, $B\geq0$. In the recent paper [7], Moslehian and Najafi showed that $AB+BA$ is positive if and only if the following *operator subadditivity property* holds

$$f\left(A+B\right)\leq f\left(A\right)+f\left(B\right) \tag{1.4}$$

for all nonnegative operator monotone functions on $[0,\infty)$. For some interesting consequences of this result see [7].

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin\left(r\pi\right)}{\pi}\int_{0}^{\infty}\frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}\left(w,\mu\right)\left(t\right):=\int_{0}^{\infty}\frac{w\left(\lambda\right)}{\lambda+t}d\mu\left(\lambda\right),\ t>0, \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}\left(w\right)\left(t\right):=\int_{0}^{\infty}\frac{w\left(\lambda\right)}{\lambda+t}d\lambda,\ t>0. \tag{1.6}$$

Now, assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}\left(w,\mu\right)\left(T\right):=\int_{0}^{\infty}w\left(\lambda\right)\left(\lambda+T\right)^{-1}d\mu\left(\lambda\right), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}\left(w\right)\left(T\right):=\int_{0}^{\infty}w\left(\lambda\right)\left(\lambda+T\right)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t > 0. \tag{1.9}$$

We define the *upper incomplete Gamma function* as [10]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [11]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{1.10}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (1.10) we have

$$\mathcal{D}(w_{\cdot^{-a}e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{1.11}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (1.11) we get

$$\mathcal{D}(w_{e^{-\cdot}})(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{1.12}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{1.13}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (1.11) we have

$$\begin{aligned}\mathcal{D}(w_{\cdot^{n-1}e^{-\cdot}})(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{1.14}$$

If we define the generalized exponential integral [12] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [12, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we get

$$\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!t^{k}+(-1)^{n-1}t^{n-1}e^{t}E_{1}(t) \tag{1.15}$$

for $n\geq 2$ and $t>0$.

If $T>0$, then we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda^{-a}e^{-\lambda}(t+\lambda)^{-1}d\lambda=\Gamma(1-a)T^{-a}\exp(T)\Gamma(a,T) \tag{1.16}$$

for $a<1$.

In particular,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T)=\int_{0}^{\infty}e^{-\lambda}(T+\lambda)^{-1}d\lambda=\exp(T)E_{1}(T) \tag{1.17}$$

and, for $n\geq 2$

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)&=\int_{0}^{\infty}\lambda^{n-1}e^{-\lambda}(T+\lambda)^{-1}d\lambda\\&=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!T^{k}+(-1)^{n-1}T^{n-1}\exp(T)E_{1}(T),\end{aligned} \tag{1.18}$$

where $T>0$.

For $n=2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda e^{-\lambda}(T+\lambda)^{-1}d\lambda=1-T\exp(T)E_{1}(T) \tag{1.19}$$

for $T>0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda):=\frac{1}{\lambda+a}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t):=\int_{0}^{\infty}\frac{1}{(\lambda+t)(\lambda+a)}d\lambda=\frac{\ln t-\ln a}{t-a} \tag{1.20}$$

for all $a>0$ and $t>0$ with $t\neq a$.

From this, we get

$$\ln t=\ln a+(t-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a>0$.

If $T>0$, then

$$\begin{aligned}\ln T&=\ln a+(T-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)\\&=\ln a+(T-a)\int_{0}^{\infty}\frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda.\end{aligned} \tag{1.21}$$

Let $a>0$. Assume that either $0<T<a$ or $T>a$, then by (1.22) we get

$$(\ln T-\ln a)(T-a)^{-1}=\int_{0}^{\infty}\frac{1}{(\lambda+a)}(\lambda+T)^{-1}d\lambda. \tag{1.22}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)} d\lambda \\ &= \frac{\pi t}{2a\left(t^2+a^2\right)} - \frac{\ln t - \ln a}{t^2+a^2}\end{aligned}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)} d\lambda = \frac{\pi t}{2\left(t^2+1\right)} - \frac{\ln t}{t^2+1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\left(T^2+a^2\right)^{-1}\left(\frac{\pi}{2a}T - \ln T + \ln a\right) = \int_0^\infty \frac{1}{\left(\lambda^2+a^2\right)}(\lambda+T)^{-1} d\lambda \tag{1.23}$$

and, in particular,

$$\left(T^2+1\right)^{-1}\left(\frac{\pi}{2}T - \ln T\right) = \int_0^\infty \frac{1}{\left(\lambda^2+1\right)}(\lambda+T)^{-1} d\lambda. \tag{1.24}$$

Motivated by the above results, we show among others that, if B, $A > 0$, then

$$\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B) \geq \mathcal{D}(w,\mu)(A+B),$$

namely $\mathcal{D}(w,\mu)$ is operator subadditive on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $[f(t) - f(0)]\,t^{-1}$ is operator subadditive on $(0,\infty)$. Also, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function $\left[f(t) - f(0) - f'_+(0)\,t\right] t^{-2}$ is operator subadditive on $(0,\infty)$. Some examples for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Subadditivity Property

The following operator subadditivity property holds:

Theorem 3. *For all A, $B > 0$ we have*

$$D(w,\mu)(A) + D(w,\mu)(B) \geq D(w,\mu)(A+B), \tag{2.1}$$

namely $D(w,\mu)$ is operator subadditive.

Proof. For all A, $B > 0$, by using the representation of $D(w,\mu)$, we have

$$\begin{aligned}&D(w,\mu)(A) + D(w,\mu)(B) - D(w,\mu)(A+B) \\ &= \int_0^\infty w(\lambda)\left[(A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}\right] d\mu(\lambda).\end{aligned} \tag{2.2}$$

For $\lambda \geq 0$, define the operator

$$K_\lambda := (A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}.$$

If we multiply both sides of K_λ with $A+B+\lambda$, then we obtain successively

$$
\begin{aligned}
(2.3)\qquad & (A+B+\lambda)\,K_\lambda\,(A+B+\lambda) \\
&= (A+B+\lambda)\,(A+\lambda)^{-1}\,(A+B+\lambda) \\
&+ (A+B+\lambda)\,(B+\lambda)^{-1}\,(A+B+\lambda) - A - B - \lambda \\
&= \left(1 + B\,(A+\lambda)^{-1}\right)(A+\lambda+B) \\
&+ \left(A\,(B+\lambda)^{-1} + 1\right)(A+B+\lambda) - A - B - \lambda \\
&= A + \lambda + B + B + B\,(A+\lambda)^{-1}\,B \\
&+ A\,(B+\lambda)^{-1}\,A + A + A + B + \lambda - A - B - \lambda \\
&= B\,(A+\lambda)^{-1}\,B + A\,(B+\lambda)^{-1}\,A + 2\,(A+B) + \lambda =: L_\lambda.
\end{aligned}
$$

By multiplying both sides of (2.3) with $(A+B+\lambda)^{-1}$ we get

$$
K_\lambda = (A+B+\lambda)^{-1}\,L_\lambda\,(A+B+\lambda)^{-1}. \tag{2.4}
$$

We then have the representation

$$
\begin{aligned}
(2.5)\qquad & D(w,\mu)\,(A) + D(w,\mu)\,(B) - D(w,\mu)\,(A+B) \\
&= \int_0^\infty w(\lambda)\,K_\lambda d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\,(A+B+\lambda)^{-1}\,L_\lambda\,(A+B+\lambda)^{-1}\,d\mu(\lambda)
\end{aligned}
$$

for all $A,\ B > 0$.

Since $A,\ B > 0$ and $\lambda \geq 0$, hence by the definition of L_λ we obtain that $L_\lambda \geq 0$, which, by (2.4) implies that $K_\lambda \geq 0$ and multiplying with $w(\lambda) \geq 0$ and integrating over the measure μ we deduce the desired result (2.1). □

Corollary 1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$. If $A,\ B > 0$, then*

$$
\begin{aligned}
(2.6)\qquad & f(A)\,A^{-1} + f(B)\,B^{-1} - f(A+B)\,(A+B)^{-1} \\
&\geq f(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right],
\end{aligned}
$$

namely, the function $[f(t) - f(0)]\,t^{-1}$ is operator subadditive on $(0,\infty)$.

In particular, if $f(0) = 0$, then

$$
f(A)\,A^{-1} + f(B)\,B^{-1} \geq f(A+B)\,(A+B)^{-1}. \tag{2.7}
$$

Proof. If $f : [0,\infty) \to \mathbb{R}$ is an operator monotone, then by (1.1)

$$
\frac{f(t) - f(0)}{t} - b = \mathcal{D}(\ell,\mu)\,(t),\ \ t > 0
$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$.

By applying Theorem 3 for the $\mathcal{D}(\ell,\mu)$ and performing the required calculations, we deduce

$$\begin{aligned}
0 &\le D(w,\mu)(A) + D(w,\mu)(B) - D(w,\mu)(A+B)\\
&= [f(A) - f(0)]A^{-1} - b + [f(B) - f(0)]B^{-1} - b\\
&- [f(A+B) - f(0)](A+B)^{-1} + b\\
&= f(A)A^{-1} + f(B)B^{-1} - f(A+B)(A+B)^{-1}\\
&- f(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right] - b,
\end{aligned}$$

for all A, $B > 0$, which gives that

$$\begin{aligned}
&f(A)A^{-1} + f(B)B^{-1} - f(A+B)(A+B)^{-1}\\
&\ge f(0)\left(A^{-1} + B^{-1} - (A+B)^{-1}\right) + b\\
&\ge f(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right]
\end{aligned}$$

for all A, $B > 0$ and the inequality (2.6) is obtained. □

Remark 1. *If we take $f(t) = t^r$, $r \in (0,1]$ in (2.7), then we get the power inequality*

$$A^{r-1} + B^{r-1} \ge (A+B)^{r-1} \tag{2.8}$$

for all A, $B > 0$.

If we take $f(t) = \ln(t+1)$ in (2.7), then we get the logarithmic inequality

$$A^{-1}\ln(A+1) + B^{-1}\ln(B+1) \ge (A+B)^{-1}\ln(A+B+1). \tag{2.9}$$

The interested author may state other similar inequalities by using the examples of operator monotone functions from [2], [3] and the references therein.

Corollary 2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$. If A, $B > 0$, then*

$$\begin{aligned}
&f(A)A^{-2} + f(B)B^{-2} - f(A+B)(A+B)^{-2}\\
&\ge f(0)\left[A^{-2} + B^{-2} - (A+B)^{-2}\right]\\
&+ f'_+(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right],
\end{aligned} \tag{2.10}$$

namely, the function $\left[f(t) - f(0) - f'_+(0)t\right]t^{-2}$ is operator subadditive on $(0,\infty)$.

If $f(0) = 0$, then

$$\begin{aligned}
&f(A)A^{-2} + f(B)B^{-2} - f(A+B)(A+B)^{-2}\\
&\ge f'_+(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right].
\end{aligned} \tag{2.11}$$

Proof. If $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t) - f(0) - f'_+(0)t}{t^2} - c = \mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$.

By applying Theorem 3 for the $\mathcal{D}(\ell,\mu)$ and performing the required calculations, we deduce

$$\begin{aligned}
0 &\le D(w,\mu)(A)+D(w,\mu)(B)-D(w,\mu)(A+B)\\
&= f(A)A^{-2}-f(0)A^{-2}-f'_{+}(0)A^{-1}-c\\
&+ f(B)B^{-2}-f(0)B^{-2}-f'_{+}(0)B^{-1}-c\\
&- f(A+B)(A+B)^{-2}+f(0)(A+B)^{-2}+f'_{+}(0)(A+B)^{-1}+c\\
&= f(A)A^{-2}+f(B)B^{-2}-f(A+B)(A+B)^{-2}\\
&- f(0)\left[A^{-2}+B^{-2}-(A+B)^{-2}\right]-f'_{+}(0)\left[A^{-1}+B^{-1}-(A+B)^{-1}\right]-c
\end{aligned}$$

for all $A,\ B>0$.

From this we get

$$\begin{aligned}
& f(A)A^{-2}+f(B)B^{-2}-f(A+B)(A+B)^{-2}\\
&\ge f(0)\left[A^{-2}+B^{-2}-(A+B)^{-2}\right]+f'_{+}(0)\left[A^{-1}+B^{-1}-(A+B)^{-1}\right]+c\\
&\ge f(0)\left[A^{-2}+B^{-2}-(A+B)^{-2}\right]+f'_{+}(0)\left[A^{-1}+B^{-1}-(A+B)^{-1}\right],
\end{aligned}$$

which proves (2.11). □

Remark 2. *Let $a>0$ and $p\in[-1,0)\cup[1,2]$. Then for all $A,\ B>0$ we have the power inequality*

$$\begin{aligned}
(2.12)\qquad & (A+a)^{p}A^{-2}+(B+a)^{p}B^{-2}-(A+B+a)^{p}(A+B)^{-2}\\
&\ge a^{p}\left[A^{-2}+B^{-2}-(A+B)^{-2}\right]+pa^{p-1}\left[A^{-1}+B^{-1}-(A+B)^{-1}\right].
\end{aligned}$$

We also have the logarithmic inequality

$$\begin{aligned}
(2.13)\qquad & (A+B)^{-2}\ln(A+B+1)-A^{-2}\ln(A+1)-B^{-2}\ln(B+1)\\
&\ge (A+B)^{-1}-B^{-1}-A^{-1}
\end{aligned}$$

for all $A,\ B>0$.

Using the transform (1.16) we have for $A,\ B>0$ that

$$\begin{aligned}
(2.14)\qquad & A^{-a}\exp(A)\Gamma(a,A)+B^{-a}\exp(B)\Gamma(a,B)\\
&\ge (A+B)^{-a}\exp(A+B)\Gamma(a,A+B)
\end{aligned}$$

for $a<1$.

In particular, we have

$$(2.15)\qquad \exp(A)E_1(A)+\exp(B)E_1(B)\ge\exp(A+B)E_1(A+B)$$

and

$$(2.16)\quad (A+B)\exp(A+B)E_1(A+B)\ge 1+A\exp(A)E_1(A)+B\exp(B)E_1(B)$$

for all $A,\ B>0$.

Using the (1.22) we also have

$$\begin{aligned}
(2.17)\qquad & (\ln A-\ln a)(A-a)^{-1}+(\ln B-\ln a)(B-a)^{-1}\\
&\ge (\ln(A+B)-\ln a)(A+B-a)^{-1}
\end{aligned}$$

for all $A,\ B>a>0$.

3. Reverse Inequalities

We define the difference $\mathcal{D}(w,\mu)(\cdot,\cdot)$ for positive numbers t, s by

$$\mathcal{D}(w,\mu)(t,s) := D(w,\mu)(t) + D(w,\mu)(s) - D(w,\mu)(t+s) \geq 0$$

and the difference for positive operators A, B,

$$\mathcal{D}(w,\mu)(A,B) := D(w,\mu)(A) + D(w,\mu)(B) - D(w,\mu)(A+B) \geq 0$$

for a continuous and positive function $w(\lambda)$, $\lambda > 0$, and μ a positive measure on $(0,\infty)$ such that the integral (1.5) exists for all $t \geq 0$.

We also have the following reverse inequality:

Theorem 4. *Assume that there exists positive constants α, β, γ and δ such that*

$$0 < \alpha \leq A \leq \beta \text{ and } 0 < \gamma \leq B \leq \delta. \tag{3.1}$$

Then

$$0 \leq \mathcal{D}(w,\mu)(A,B) \leq \mathcal{D}(w,\mu)(\alpha,\gamma) + \left(\frac{\beta+\delta}{\alpha+\gamma} - 1\right) D(w,\mu)(\beta+\delta). \tag{3.2}$$

Proof. Observe that

$$(A+\lambda)^{-1} \leq (\alpha+\lambda)^{-1}, \ (B+\lambda)^{-1} \leq (\gamma+\lambda)^{-1}$$

and

$$(\beta+\delta+\lambda)^{-1} \leq (A+B+\lambda)^{-1}$$

i.e.,

$$-(A+B+\lambda)^{-1} \leq -(\beta+\delta+\lambda)^{-1},$$

which give

$$\begin{aligned}&(A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}\\ &\leq (\alpha+\lambda)^{-1} + (\gamma+\lambda)^{-1} - (\beta+\delta+\lambda)^{-1}\end{aligned}$$

namely

$$\begin{aligned}&(A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}\\ &\leq (\alpha+\lambda)^{-1} + (\gamma+\lambda)^{-1} - (\alpha+\gamma+\lambda)^{-1} + (\alpha+\gamma+\lambda)^{-1} - (\beta+\delta+\lambda)^{-1}\end{aligned}$$

for all $\lambda \geq 0$.

If we multiply with $w(\lambda) \geq 0$ and integrate, then by (2.2) we get

$$\begin{aligned}&D(w,\mu)(A) + D(w,\mu)(B) - D(w,\mu)(A+B)\\ &\leq D(w,\mu)(\alpha) + D(w,\mu)(\gamma) - D(w,\mu)(\alpha+\gamma)\\ &+ \int_0^\infty w(\lambda)\left[(\alpha+\gamma+\lambda)^{-1} - (\beta+\delta+\lambda)^{-1}\right] d\mu(\lambda),\end{aligned}$$

namely

$$\begin{aligned}\mathcal{D}(w,\mu)(A,B) &\leq \mathcal{D}(w,\mu)(\alpha,\gamma)\\ &+ \int_0^\infty w(\lambda)\left[(\alpha+\gamma+\lambda)^{-1} - (\beta+\delta+\lambda)^{-1}\right] d\mu(\lambda).\end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left[(\alpha+\gamma+\lambda)^{-1}-(\beta+\delta+\lambda)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\left[\frac{\beta+\delta-\alpha-\gamma}{(\alpha+\gamma+\lambda)(\beta+\delta+\lambda)}\right]d\mu(\lambda)\\
&=(\beta+\delta-\alpha-\gamma)\int_0^\infty \frac{w(\lambda)}{(\alpha+\gamma+\lambda)(\beta+\delta+\lambda)}d\mu(\lambda).
\end{aligned}$$

Observe that

$$\frac{1}{\alpha+\gamma+\lambda}\le\frac{1}{\alpha+\gamma}\text{ for }\lambda\ge 0,$$

which implies that

$$\begin{aligned}
\int_0^\infty \frac{w(\lambda)}{(\alpha+\gamma+\lambda)(\beta+\delta+\lambda)}d\mu(\lambda)&\le\frac{1}{\alpha+\gamma}\int_0^\infty\frac{w(\lambda)}{\beta+\delta+\lambda}d\mu(\lambda)\\
&=\frac{1}{\alpha+\gamma}D(w,\mu)(\beta+\delta).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left[(\alpha+\gamma+\lambda)^{-1}-(\beta+\delta+\lambda)^{-1}\right]d\mu(\lambda)\\
&\le\left(\frac{\beta+\delta}{\alpha+\gamma}-1\right)D(w,\mu)(\beta+\delta).
\end{aligned}\tag{3.4}$$

By making use of (3.3) and (3.4) we derive (3.2). □

The case of operator monotone functions is as follows:

Corollary 3. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$ with $f(0)=0$. If A, $B>0$ satisfy the condition (3.1), then*

$$\begin{aligned}
0&\le f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}\\
&\le f(\alpha)\alpha^{-1}+f(\gamma)\gamma^{-1}-f(\alpha+\gamma)(\alpha+\gamma)^{-1}\\
&+\left(\frac{\beta+\delta}{\alpha+\gamma}-1\right)f(\beta+\delta)(\beta+\delta)^{-1}.
\end{aligned}\tag{3.5}$$

Proof. From (3.2) we get for

$$\frac{f(t)}{t}-b=\mathcal{D}(\ell,\mu)(t),\ t>0$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$, that

$$\begin{aligned}
0&\le f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}\\
&\le f(\alpha)\alpha^{-1}+f(\gamma)\gamma^{-1}-f(\alpha+\gamma)(\alpha+\gamma)^{-1}\\
&+\left(\frac{\beta+\delta}{\alpha+\gamma}-1\right)\left(\frac{f(\beta+\delta)}{\beta+\delta}-b\right)\\
&\le f(\alpha)\alpha^{-1}+f(\gamma)\gamma^{-1}-f(\alpha+\gamma)(\alpha+\gamma)^{-1}\\
&+\left(\frac{\beta+\delta}{\alpha+\gamma}-1\right)\frac{f(\beta+\delta)}{\beta+\delta},
\end{aligned}$$

which proves the desired result (3.5). □

Remark 3. *If A, $B > 0$ satisfy the condition (3.1) and $r \in (0,1]$, then we have the reverse power inequality*

$$0 \leq A^{r-1} + B^{r-1} - (A+B)^{r-1} \leq \alpha^{r-1} + \gamma^{r-1} - (\alpha+\gamma)^{r-1} + \left(\frac{\beta+\delta}{\alpha+\gamma} - 1\right)(\beta+\delta)^{r-1}. \tag{3.6}$$

The case of operator convex functions is as follows:

Corollary 4. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$ with $f(0) = 0$. If A, $B > 0$ satisfy the condition (3.1), then*

$$\begin{aligned} 0 &\leq f(A)\,A^{-2} + f(B)\,B^{-2} - f(A+B)\,(A+B)^{-2} \\ &- f'_+(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right] \\ &\leq f(\alpha)\,\alpha^{-2} + f(\gamma)\,\gamma^{-2} - f(\alpha+\gamma)\,(\alpha+\gamma)^{-2} \\ &- f'_+(0)\left[\alpha^{-1} + \gamma^{-1} - (\alpha+\gamma)^{-1}\right] \\ &+ \left(\frac{\beta+\delta}{\alpha+\gamma} - 1\right)\frac{f(\beta+\delta) - f'_+(0)(\beta+\delta)}{(\beta+\delta)^2}. \end{aligned} \tag{3.7}$$

Proof. Follows by (3.2) observing, by (1.3), we have that

$$\frac{f(t) - f(0) - f'_+(0)\,t}{t^2} - c = \mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$. □

Remark 4. *If A, $B > 0$ satisfy the condition (3.1), then by taking $f(t) = -\ln(t+1)$ in (3.7), we obtain*

$$\begin{aligned} 0 &\leq (A+B)^{-2}\ln(A+B+1) - A^{-2}\ln(A+1) - B^{-2}\ln(B+1) \\ &- (A+B)^{-1} + A^{-1} + B^{-1} \\ &\leq (\alpha+\gamma)^{-2}\ln(\alpha+\gamma+1) - \alpha^{-2}\ln(\alpha+1) - \gamma^{-2}\ln(\gamma+1) \\ &- (\alpha+\gamma)^{-1} + \alpha^{-1} + \gamma^{-1} \\ &+ \left(\frac{\beta+\delta}{\alpha+\gamma} - 1\right)\frac{(\beta+\delta) - \ln(\beta+\delta+1)}{(\beta+\delta)^2}. \end{aligned} \tag{3.8}$$

Using the transform (1.16) we have for A, $B > 0$ satisfying the condition (3.1) that

$$\begin{aligned} 0 &\leq A^{-a}\exp(A)\,\Gamma(a,A) + B^{-a}\exp(B)\,\Gamma(a,B) \\ &- (A+B)^{-a}\exp(A+B)\,\Gamma(a,A+B) \\ &\leq \alpha^{-a}\exp(\alpha)\,\Gamma(a,\alpha) + \gamma^{-a}\exp(\gamma)\,\Gamma(a,\gamma) \\ &- (\alpha+\gamma)^{-a}\exp(\alpha+\gamma)\,\Gamma(a,\alpha+\gamma) \\ &+ \left(\frac{\beta+\delta}{\alpha+\gamma} - 1\right)(\beta+\delta)^{-a}\exp(\beta+\delta)\,\Gamma(a,\beta+\delta) \end{aligned} \tag{3.9}$$

for $a<1$. In particular, we have

$$
\begin{aligned}
0 & \leq \exp(A+B) E_1(A+B)-\exp(A) E_1(A)-\exp(B) E_1(B) \\
& \leq \exp(\alpha+\gamma) E_1(\alpha+\gamma)-\exp(\alpha) E_1(\alpha)-\exp(\gamma) E_1(\gamma) \\
& +\left(\frac{\beta+\delta}{\alpha+\gamma}-1\right) \exp(\beta+\delta) E_1(\beta+\delta) .
\end{aligned} \tag{3.10}
$$

The interested reader may state other similar inequalities by using the examples of transforms presented in the introduction. We omit the details.

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