OPERATOR CONVEXITY OF AN INTEGRAL TRANSFORM WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty w(\lambda)(\lambda + t)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for $t > 0$.

We show among others that $\mathcal{D}(w, \mu)$ is operator convex on $(0, \infty)$. From this we derive that, if $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$, then the function $[f(0) - f(t)] t^{-1}$ is operator convex on $(0, \infty)$. Also, if $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$, then the function $\left[f(0) + f'_+(0) t - f(t)\right] t^{-2}$ is operator convex on $(0, \infty)$. Some lower and upper bounds for the Jensen's difference

$$\frac{\mathcal{D}(w, \mu)(A) + \mathcal{D}(w, \mu)(B)}{2} - \mathcal{D}(w, \mu)\left(\frac{A + B}{2}\right)$$

under some natural assumptions for the positive operators A and B are given. Examples for power, exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$. We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.1).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{OC}$$

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in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where $a,\ b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.1).

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.4}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.5}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.6}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.7}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

From (1.6) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(T) \tag{1.9}$$

where $T > 0$ and from (1.7)

$$(T - 1)^{-1} \ln T = \mathcal{D}(w_{\ln})(T) \tag{1.10}$$

provided $T > 0$ and $T - 1$ is invertible.

In this paper, we show among others that $\mathcal{D}(w, \mu)$ is operator convex on $(0, \infty)$. From this we derive that, if $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$, then the function $[f(0) - f(t)] t^{-1}$ is operator convex on $(0, \infty)$. Also, if $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$, then the function $\left[f(0) + f'_+(0) t - f(t)\right] t^{-2}$ is operator convex on $(0, \infty)$. Some lower and upper bounds for the Jensen's difference

$$\frac{\mathcal{D}(w, \mu)(A) + \mathcal{D}(w, \mu)(B)}{2} - \mathcal{D}(w, \mu)\left(\frac{A + B}{2}\right)$$

under some natural assumptions for the positive operators A and B are given. Examples for power, exponential and logarithmic functions are also provided.

2. Preliminary Results

We start with the following elementary identity that give a simple proof for the fact that the function $f(t) = t^{-1}$ is operator convex on $(0, \infty)$, see for instance [6, p. 8]:

Lemma 1. *For any A, $B > 0$ we have*

$$\frac{A^{-1} + B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} = \frac{\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)}{2} \geq 0. \tag{2.1}$$

If more assumptions are made for the operators A and B, then one can obtain the following lower and upper bounds:

Corollary 1. *Assume that $0 < \alpha \leq A \leq \beta$ and $0 < \gamma \leq B \leq \delta$ for some constants α, β, γ, δ. Then*

$$\frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 \leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} \leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2. \tag{2.2}$$

Proof. We have $\beta^{-1} \leq A^{-1} \leq \alpha^{-1}$ and $\delta^{-1} \leq B^{-1} \leq \gamma^{-1}$, which gives

$$\beta^{-1} + \delta^{-1} \leq A^{-1} + B^{-1} \leq \alpha^{-1} + \gamma^{-1}$$

namely

$$\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} \leq \left(A^{-1} + B^{-1}\right)^{-1} \leq \left(\beta^{-1} + \delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(A^{-1} - B^{-1}\right)$ and dividing by 2, we get

$$\begin{aligned}\frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^{2} &\leq \frac{\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)}{2} \\ &\leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^{2}.\end{aligned}$$

□

We know that for $T > 0$, we have the operator inequalities

$$0 < \left\|T^{-1}\right\|^{-1} \leq T \leq \|T\|. \tag{2.3}$$

Indeed, it is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^{2} \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned}0 \leq \langle x, x\rangle^{2} &= \left\langle T^{-1}Tx, x\right\rangle^{2} = \left\langle Tx, T^{-1}x\right\rangle^{2} \\ &\leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle\end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T.$$

The second inequality in (2.3) is obvious.

Remark 1. *If A, $B > 0$ and $B - A > 0$, then by taking $\alpha = \left\|A^{-1}\right\|^{-1}$, $\beta = \|A\|$, $\gamma = \left\|B^{-1}\right\|^{-1}$ and $\delta = \|B\|$ in (2.2), we get*

$$\begin{aligned}&\frac{1}{2}\left(\left\|A^{-1}\right\| + \left\|B^{-1}\right\|\right)^{-1}\left(A^{-1} - B^{-1}\right)^{2} \\ &\leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\|A\|^{-1} + \|B\|^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^{2}.\end{aligned} \tag{2.4}$$

A continuous function $g : \mathcal{SA}_{I}(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_{I}(H)$, the class of selfadjoint operators on I, along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_{A}(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.5}$$

If the limit (2.5) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t) A + tB \mid t \in [0,1]\}.$$

We observe that $A, B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

We have the following gradient inequalities, see for instance :

Lemma 2. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then*

$$\nabla_B f(B-A) \geq f(B) - f(A) \geq \nabla_A f(B-A). \tag{2.6}$$

Let $T, S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = -T^{-1} S T^{-1} \tag{2.7}$$

for $T, S > 0$.

Using (2.7) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C) D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C) C^{-1}$$

that is equivalent to

$$D^{-1}(D-C) D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C) C^{-1} \tag{2.8}$$

for all $C, D > 0$.

If

$$m \leq D - C \leq M$$

for some constants m, M, then

$$mD^{-2} \leq D^{-1}(D-C) D^{-1}$$

and

$$C^{-1}(D-C) C^{-1} \leq MC^{-2}$$

and by (2.8) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.9}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2} \tag{2.10}$$

Corollary 2. *Assume that* $0 < \alpha \leq A \leq \beta$, $0 < \gamma \leq B \leq \delta$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$
\begin{aligned}
(2.11)\qquad 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4} \\
&\leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}.
\end{aligned}
$$

Proof. From (2.10) we have

$$0 < \frac{m}{\delta^2} \leq A^{-1} - B^{-1} \leq \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \leq \left(A^{-1}-B^{-1}\right)^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.11). □

Remark 2. *If the positive operators* A, B *are separated, namely* $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$ *for some constants* α, β, γ, δ, *then obviously* $0 < \gamma - \beta \leq B - A \leq \delta - \alpha$ *and by (2.11) for* $m = \gamma - \beta$ *and* $M = \delta - \alpha$, *we get*

$$
\begin{aligned}
(2.12)\qquad 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4} \leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \\
&\leq \frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}.
\end{aligned}
$$

If $0 < \|A\|\left\|B^{-1}\right\| < 1$, *then*

$$0 < \left\|A^{-1}\right\|^{-1} \leq A \leq \|A\| < \left\|B^{-1}\right\|^{-1} \leq B \leq \|B\|$$

and by (2.12) we get

$$
\begin{aligned}
(2.13)\qquad 0 &< \frac{1}{2}\left(\left\|A^{-1}\right\|+\left\|B^{-1}\right\|\right)^{-1}\frac{\left(\left\|B^{-1}\right\|^{-1}-\|A\|\right)^2}{\|B\|^4} \\
&\leq \frac{1}{2}\left(\left\|A^{-1}\right\|+\left\|B^{-1}\right\|\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \\
&\leq \frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\|A\|^{-1}+\|B\|^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(\|B\|-\left\|A^{-1}\right\|^{-1}\right)^2\left\|A^{-1}\right\|^4.
\end{aligned}
$$

We can present now our main results.

3. Main Results

We have:

Theorem 3. *For all $A,\ B>0$ we have*

$$\begin{aligned}
&\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=\frac{1}{2}\int_0^\infty\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1} \\
&\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)w(\lambda)\,d\mu(\lambda) \\
&\geq 0.
\end{aligned} \tag{3.1}$$

The function $\mathcal{D}(w,\mu)$ is an operator convex function on $(0,\infty)$

Proof. We have for all $A,\ B>0$

$$\begin{aligned}
&\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=\int_0^\infty w(\lambda)\left[\frac{(\lambda+A)^{-1}+(\lambda+B)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\right]d\mu(\lambda).
\end{aligned} \tag{3.2}$$

Since, by (2.1)

$$\begin{aligned}
&\frac{(\lambda+A)^{-1}+(\lambda+B)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1} \\
&=\frac{1}{2}\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1} \\
&\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right) \\
&\geq 0
\end{aligned}$$

for all $\lambda\geq 0$, then by (3.2) we obtain the representation (3.1).

Since $\mathcal{D}(w,\mu)$ is continuous in $\mathcal{B}(H)$ and satisfies Jensen's inequality (3.1), it follows that $\mathcal{D}(w,\mu)$ is an operator convex function on $(0,\infty)$. □

The case of operator monotone functions is as follows:

Corollary 3. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$. Then the function $[f(t)-f(0)]\,t^{-1}$ is operator convex on $(0,\infty)$. For all $A,\ B>0$ we have*

$$\begin{aligned}
&\frac{f(A)A^{-1}+f(B)B^{-1}}{2}-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1} \\
&\geq f(0)\left[\frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2}\right].
\end{aligned} \tag{3.3}$$

If $f(0)=0$, then $f(t)\,t^{-1}$ is operator convex on $(0,\infty)$ and

$$\frac{f(A)A^{-1}+f(B)B^{-1}}{2}\geq f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}$$

for all $A,\ B>0$.

Proof. From (1.1) we have

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{3.4}$$

for some μ, a positive measure on $(0,\infty)$, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. By utilising Theorem 3 and Lemma 1 we deduce the desired results. □

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. Then the function $\left[f(t)-f(0)-f'_{+}(0)t\right]t^{-2}$ is operator convex on $(0,\infty)$. For all A, $B>0$ we have*

$$\begin{aligned}\frac{f(A)A^{-2}+f(B)B^{-2}}{2}&-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2}\\&\geq f(0)\left[\frac{A^{-2}+B^{-2}}{2}-\left(\frac{A+B}{2}\right)^{-2}\right]\\&+f'_{+}(0)\left[\frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2}\right].\end{aligned} \tag{3.5}$$

If $f(0)=0$, then $\left[f(t)-f'_{+}(0)t\right]t^{-2}$ is operator convex on $(0,\infty)$ and

$$\begin{aligned}\frac{f(A)A^{-2}+f(B)B^{-2}}{2}&-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2}\\&\geq f'_{+}(0)\left[\frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2}\right]\end{aligned} \tag{3.6}$$

for all A, $B>0$.

Proof. From (1.3) we have

$$\left[f(t)-f(0)-f'_{+}(0)t\right]t^{-2}-c=\mathcal{D}(\ell,\mu)(t),$$

for some μ, a positive measure on $(0,\infty)$, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. By utilising Theorem 3 and Lemma 1 we deduce the desired results. □

When more assumptions are imposed on the operators A and B, then the following improvement and refinement of Jensen's inequality hold:

Theorem 4. *Assume that $0<\alpha\leq A\leq\beta$, $0<\gamma\leq B\leq\delta$ and $0<m\leq B-A\leq M$ for some constants α, β, γ, δ, m, M. Then*

$$\begin{aligned}0&<-\frac{m^2\gamma\alpha}{12(\alpha+\gamma)}\mathcal{D}'''(w,\mu)(\delta)\\&\leq\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\&\leq\frac{M^2}{2(\beta+\delta)}\left[-\mathcal{D}'(w,\mu)(\alpha)+\left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha)\right.\\&\left.-\frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha)\right].\end{aligned} \tag{3.7}$$

Proof. We have $0 < \alpha + \lambda \leq A + \lambda \leq \beta + \lambda$, $0 < \gamma + \lambda \leq B + \lambda \leq \delta + \lambda$ and $0 < m \leq B + \lambda - A - \lambda = B - A \leq M$ for all $\lambda \geq 0$. By (2.11) we get

$$0 < \frac{1}{2}\left(\frac{1}{\alpha+\lambda} + \frac{1}{\gamma+\lambda}\right)^{-1} \frac{m^2}{(\delta+\lambda)^4} \leq \frac{(A+\lambda)^{-1} + (B+\lambda)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \leq \frac{1}{2}\left(\frac{1}{\beta+\lambda} + \frac{1}{\delta+\lambda}\right)^{-1} \frac{M^2}{(\alpha+\lambda)^4}. \tag{3.8}$$

We have that

$$\left(\frac{1}{\beta+\lambda} + \frac{1}{\delta+\lambda}\right)^{-1} = \frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta+2\lambda} \leq \frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta} \tag{3.9}$$

and

$$\left(\frac{1}{\alpha+\lambda} + \frac{1}{\gamma+\lambda}\right)^{-1} = \frac{(\gamma+\lambda)(\alpha+\lambda)}{\alpha+\gamma+2\lambda} = g(\lambda).$$

We have

$$g'(\lambda) = \frac{(\alpha+\gamma+2\lambda)^2 - 2(\gamma+\lambda)(\alpha+\lambda)}{(\alpha+\gamma+2\lambda)^2} = \frac{(\alpha+\lambda)^2 + (\gamma+\lambda)^2}{(\alpha+\gamma+2\lambda)^2} > 0,$$

which shows that g is increasing on $[0,\infty)$.

Therefore

$$g(\lambda) \geq g(0) = \frac{\gamma\alpha}{\alpha+\gamma} \text{ for all } \lambda \geq 0. \tag{3.10}$$

By (3.8)-(3.10) we derive that

$$0 < \frac{1}{2}\frac{\gamma\alpha}{\alpha+\gamma}\frac{m^2}{(\delta+\lambda)^4} \leq \frac{(A+\lambda)^{-1} + (B+\lambda)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \leq \frac{1}{2}\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta}\frac{M^2}{(\alpha+\lambda)^4},$$

which implies that

$$0 < \frac{1}{2}m^2\frac{\gamma\alpha}{\alpha+\gamma}\int_0^\infty \frac{w(\lambda)}{(\delta+\lambda)^4}d\mu(\lambda) \leq \int_0^\infty \left[\frac{(A+\lambda)^{-1} + (B+\lambda)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1}\right] w(\lambda)\,d\mu(\lambda) \leq \frac{1}{2}\frac{M^2}{\beta+\delta}\int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4} w(\lambda)\,d\mu(\lambda). \tag{3.11}$$

We observe that, by the definition of $\mathcal{D}(w,\mu)(t)$, and the properties of the derivatives of integrals with a parameter, we have

$$\mathcal{D}'(w,\mu)(t) := -\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),$$

$$\mathcal{D}''(w,\mu)(t) := 2\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),$$

and

$$\mathcal{D}'''(w,\mu)(t) := -6\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^4}d\mu(\lambda),$$

which gives that

$$\int_0^\infty \frac{w(\lambda)}{(\lambda+\delta)^4}d\mu(\lambda) = -\frac{1}{6}\mathcal{D}'''(w,\mu)(\delta). \tag{3.12}$$

Also, we observe that

$$\begin{aligned}&\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}\\&=\frac{(\beta-\alpha+\lambda+\alpha)(\delta-\alpha+\lambda+\alpha)}{(\alpha+\lambda)^4}\\&=(\beta-\alpha)(\delta-\alpha)\frac{1}{(\alpha+\lambda)^4}+(\delta+\beta-2\alpha)\frac{1}{(\alpha+\lambda)^3}+\frac{1}{(\alpha+\lambda)^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}w(\lambda)\,d\mu(\lambda)\\&=\int_0^\infty \frac{w(\lambda)\,d\mu(\lambda)}{(\alpha+\lambda)^2}+(\delta+\beta-2\alpha)\int_0^\infty \frac{w(\lambda)\,d\mu(\lambda)}{(\alpha+\lambda)^3}\\&+(\beta-\alpha)(\delta-\alpha)\int_0^\infty \frac{w(\lambda)\,d\mu(\lambda)}{(\alpha+\lambda)^4}\\&=-\mathcal{D}'(w,\mu)(\alpha)+\left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha)\\&-\frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha).\end{aligned} \tag{3.13}$$

By making use of (3.11)-(3.13), we deduce (3.7). □

Corollary 5. *If the positive operators A, B are separated, namely $0<\alpha\le A\le \beta<\gamma\le B\le\delta$ for some constants α, β, γ, δ, then*

$$\begin{aligned}0&<-\frac{(\gamma-\beta)^2\gamma\alpha}{12(\alpha+\gamma)}\mathcal{D}'''(w,\mu)(\delta)\\&\le\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\&\le\frac{(\delta-\alpha)^2}{2(\beta+\delta)}\left[-\mathcal{D}'(w,\mu)(\alpha)+\left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha)\right.\\&\left.-\frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha)\right].\end{aligned} \tag{3.14}$$

We have:

Corollary 6. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function on* $[0,\infty)$ *with* $f(0)=0$, $0<\alpha\leq A$, $0<\gamma\leq B\leq\delta$ *and* $0<m\leq B-A$ *for some constants* α, γ, δ, m. *Then we have the refinement of Jensen's inequality*

$$0<-\frac{m^2\gamma\alpha}{12(\alpha+\gamma)}\left[\frac{f'''(\delta)\delta^3-3f''(\delta)\delta^2+6f'(\delta)\delta-6f(\delta)}{\delta^4}\right] \leq\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right). \tag{3.15}$$

Proof. From (3.4) for $f(0)=0$ we have

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)t-f(t)}{t^2},$$

$$\mathcal{D}''(\ell,\mu)(t)=\frac{f''(t)t^2-2f'(t)t+2f(t)}{t^3}$$

and

$$\mathcal{D}'''(\ell,\mu)(t)=\frac{f'''(t)t^3-3f''(t)t^2+6f'(t)t-6f(t)}{t^4}.$$

Employing the first part of (3.14) we derive (3.15). □

4. Some Examples

By employing the first inequality in Theorem 4, we derive (3.15).If $g(t)=t^{r-1}$ for $t>0$, $r\in(0,1)$, then

$$g'(t)=(r-1)t^{r-2},\ g''(t)=(r-1)(r-2)t^{r-3},$$

and

$$g'''(t)=(r-1)(r-2)(r-3)t^{r-4}.$$

From (1.6) we get

$$\mathcal{D}(w_r)(t)=\frac{\pi}{\sin(r\pi)}t^{r-1},\ t>0.$$

Then by (3.7) we get

$$\begin{aligned}0&<\frac{(1-r)(2-r)(3-r)m^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}}\\&\leq\frac{A^{r-1}+B^{r-1}}{2}-\left(\frac{A+B}{2}\right)^{r-1}\\&\leq\frac{M^2}{2(\beta+\delta)\alpha^{4-r}}\left[(1-r)\alpha^2+\left(\frac{\delta+\beta}{2}-\alpha\right)\alpha(1-r)(2-r)\right.\\&\left.+\frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r)\right]\end{aligned} \tag{4.1}$$

provided that $0<\alpha\leq A\leq\beta$, $0<\gamma\leq B\leq\delta$ and $0<m\leq B-A\leq M$ for some constants α, β, γ, δ, m, M.

If we take $r\to0+$ in (4.1), then we get

$$0<\frac{m^2\gamma\alpha}{2(\alpha+\gamma)\delta^4}\leq\frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1}\leq\frac{M^2\delta\beta}{2(\beta+\delta)\alpha^4} \tag{4.2}$$

which is the same as (2.11).

If $0 < \alpha \le A \le \beta < \gamma \le B \le \delta$ for some constants α, β, γ, δ, then

$$0 < \frac{(1-r)(2-r)(3-r)(\gamma-\beta)^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}} \le \frac{A^{r-1}+B^{r-1}}{2} - \left(\frac{A+B}{2}\right)^{r-1} \le \frac{(\delta-\alpha)^2}{2(\beta+\delta)\alpha^{4-r}}\left[(1-r)\alpha^2 + \left(\frac{\delta+\beta}{2}-\alpha\right)\alpha(1-r)(2-r) + \frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r)\right], \tag{4.3}$$

where $r \in (0,1)$.

If we take $r \to 0+$ in (4.3), then we get, see also (2.12),

$$0 < \frac{(\gamma-\beta)^2\gamma\alpha}{2(\alpha+\gamma)\delta^4} \le \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \le \frac{(\delta-\alpha)^2\delta\beta}{2(\beta+\delta)\alpha^4}. \tag{4.4}$$

We define the *upper incomplete Gamma function* as [12]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [13]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{4.5}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (4.5) we have

$$\mathcal{D}(w_{\cdot^{-a}e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{4.6}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (4.6) we get

$$\mathcal{D}(w_{e^{-\cdot}})(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{4.7}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{4.8}$$

Let $a = 1-n$, with n a natural number with $n \ge 0$, then by (4.6) we have

$$\mathcal{D}(w_{\cdot^{n-1}e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) = (n-1)!t^{n-1}e^t\Gamma(1-n,t). \tag{4.9}$$

If we define the generalized exponential integral [14] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [14, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we get

$$\begin{aligned} &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) \\ &= (n-1)!e^t E_n(t) \\ &= (n-1)!e^t\left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^t E_1(t) \end{aligned} \tag{4.10}$$

for $n \geq 2$ and $t > 0$.

For $n = 2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t) = \int_0^\infty \lambda e^{-\lambda}(t+\lambda)^{-1}d\lambda = 1 - t\exp(t)E_1(t) \tag{4.11}$$

for $t > 0$.

Proposition 1. *For all $a < 1$, the function $t^{-a}e^t\Gamma(a,t)$ is operator convex on $(0,\infty)$. In particular, $e^tE_n(t)$ is operator convex on $(0,\infty)$. As a consequence $e^tE_1(t)$ is operator convex and $te^tE_1(t)$ is operator concave on $(0,\infty)$.*

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned} \mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+a^2)}d\lambda \\ &= \frac{1}{(t^2+a^2)}\left(\frac{\pi t}{2a} - \ln t + \ln a\right) \end{aligned}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\begin{aligned} \mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+1)}d\lambda \\ &= \frac{1}{t^2+1}\left(\frac{\pi}{2}t - \ln t\right) \end{aligned}$$

for $t > 0$.

Proposition 2. *For all $a > 0$, the functions*

$$\frac{1}{(t^2+a^2)}\left(\frac{\pi t}{2a}-\ln t+\ln a\right)$$

are operator convex on $(0,\infty)$. In particular,

$$\frac{1}{t^2+1}\left(\frac{\pi}{2}t-\ln t\right)$$

is operator convex on $(0,\infty)$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [10] and [11].

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